Consciousness-Based Education and Mathematics, Part 1

Pure Mathematics in the Light of *Maharishi Vedic Science* and Maharishi Vedic Mathematics

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CONSCIOUSNESS-BASED EDUCATION AND MATHEMATICS

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Higher education faces a complex set of challenges today. We are seeing resources diminish at the same time we are hearing calls for greater access and affordability. Demands for greater transparency and accountability are being sounded by both the general public and the government. Government is exerting increasing controls in this long-independent area.

These challenges, however, are merely financial and political, and they are hardly limited to colleges and universities. The fundamental challenges are educational and center around the students themselves. Challenges include high levels of stress, pervasive substance abuse (particularly binge drinking), lack of preparedness for college-level work, and mental and emotional disabilities. In most of these areas, the problem is serious and worsening. Though colleges and universities are striving to address these challenges, few would claim we are turning the tide.

An encouraging trend is the increasing focus in higher education nationwide on promoting student learning. Yet these laudable efforts do not take into account the powerful forces working in opposition. It is well known that learning is inhibited by stress, sleep deprivation, alcohol, and poor diet—and these are among the most conspicuous features of the college student experience.

Something new is required. Education needs a reliable means of developing students directly from within. We need a systematic method for cultivating their creative intelligence, their capacity to learn, and their natural humanity. All education aims at these goals, of course—but the approach thus far has been from the outside in, and the results have been haphazard at best.

Consciousness-Based education was established to address this need. It integrates the best practices of education and places beneath them a proper foundation—direct development of the student from inside out.

The outcomes of Consciousness-Based education have been unprecedented and scientifically verified. These outcomes include significant
growth of intelligence, creativity, learning ability, field independence, ego development, and moral maturity, among others. These results are remarkable because many of these values typically plateau in adolescence—but Consciousness-Based education promotes this growth in students of all ages, developing potentials that otherwise would have remained unexpressed.

Beyond this rich cognitive growth, Consciousness-Based education significantly reduces student stress, boosts self-esteem, improves health, reduces substance use, and enhances interpersonal relationships. All of this comes together to create exceptional learning environments. This approach even measurably improves the quality of life in the surrounding society.

Consciousness-Based education was founded by Maharishi Mahesh Yogi, the world authority on the science of consciousness. First pioneered at Maharishi University of Management in Fairfield, Iowa, Consciousness-Based education is being adopted by schools, colleges, and universities around the world. It is easily integrated into any school, without any change in mission or curriculum.

Consciousness-Based education recognizes that student learning depends fundamentally on students’ levels of consciousness or alertness. The more alert and awake the student, the more successful and satisfying the learning.

Consciousness-Based education consists of three components:

• a practical technology for directly developing students’ potential from within,
• a theoretical understanding of consciousness that gives rise to a unifying framework for knowledge, enabling students to easily grasp the fundamental principles of any discipline and to connect these principles to their own personal growth, and
• a set of classroom practices, arising from this understanding, that also help promote effective teaching and learning.
The Transcendental Meditation Program

At the heart of Consciousness-Based education is the practice of the Transcendental Meditation technique. The technique was brought to light by Maharishi Mahesh Yogi from the Vedic tradition of India, the world’s most ancient continuous tradition of knowledge. It is practiced for 20 minutes twice daily, once in the morning and once in the afternoon, while sitting comfortably with eyes closed. It is simple, natural, and effortless—so simple, in fact, that ten-year-old children can learn and practice it. It has been learned by more than six million people worldwide, of all ages, religions, and cultures.

The Transcendental Meditation technique differs from other procedures of meditation and relaxation in its effortlessness. It involves no concentration or control of the mind. Neither is it a religion, philosophy, or lifestyle. It involves no new codes of behavior, attitudes, or beliefs, not even the belief it will work.

The Transcendental Meditation program is the most extensively validated program of personal development in the world. It has been the subject of more than 600 scientific research studies, conducted at more than 250 universities and research institutions in more than 30 countries worldwide. These studies have been published in more than 150 scientific and scholarly journals in a broad range of fields, including Science, Scientific American, American Journal of Physiology, International Journal of Neuroscience, Memory and Cognition, Social Indicators Research, Intelligence, Journal of Mind and Behavior, Education, Journal of Moral Education, Journal of Personality and Social Psychology, Business and Health, British Journal of Educational Psychology, Journal of Human Stress, Lancet, Physiology and Behavior, and numerous others. No approach to education has as much empirical support as Consciousness-Based education.

This approach, moreover, has been successfully field-tested over the past 35 years in primary, secondary, and post-secondary schools all over the world, in developed and developing nations, in a wide variety of cultural settings—the United States, Latin America, Europe, Africa, India, and China.

The Transcendental Meditation technique enables one to “dive within.” During the practice, the mind settles inward, naturally and spontaneously, to a state of deep inner quiet, beyond thoughts and perceptions. One experiences consciousness in its pure, silent state, uncol-
ored by mental activity. In this state, consciousness is aware of itself alone, awake to its own unbounded nature.

The technique also gives profound rest, which dissolves accumulated stress and restores balanced functioning to mind and body.

This state of inner wakefulness coupled with deep rest represents a fourth major state of consciousness, distinct from the familiar states of waking, dreaming, and sleeping, known as Transcendental Consciousness.

In this restfully alert state, brain functioning becomes highly integrated and coherent. EEG studies show long-range spatial communication among all brain regions. This coherence is in sharp contrast to the more or less uncoordinated patterns typical of brain activity.

With regular practice, this integrated style of functioning carries over into daily activity. Research studies consistently show a high statistical correlation between brainwave coherence and intelligence, creativity, field independence, emotional stability, and other positive values. The greater one’s EEG coherence, in other words, the greater one’s development in these fundamental areas. At Maharishi University of Management, students even have the option of a Brain Integration Progress Report—an empirical measure of growth of EEG coherence between their first and last years at the University.

The brain is the governor of all human activity—and therefore personal growth and success in any field depend on the degree to which brain functioning is integrated. The increasingly integrated brain functioning that spontaneously results from Transcendental Meditation practice accounts for its multiplicity of benefits to mind, body, and behavior.

Every human being has the natural ability to transcend, to experience the boundless inner reality of life. Every human brain has the natural ability to function coherently. It requires only a simple technique.

**Theoretical component—**
**a unified framework for teaching and learning**

Scholars have long called for a way to unify the diverse branches of knowledge. Current global trends are making this need ever more apparent. The pace of progress is accelerating, the knowledge explosion continues unabated, and knowledge is becoming ever more specialized.
Academic disciplines offer a useful way of compartmentalizing knowledge for purposes of teaching, learning, research, and publication. But each academic discipline explores only one facet of our increasingly complex and interrelated world. The real world, however, is not compartmentalized—an elephant is not a trunk, a tusk, and a tail. Academic disciplines, consequently, are criticized as inadequate, in themselves, for understanding and addressing today’s challenging social problems.

Today, more than ever, we need a means of looking at issues comprehensively, holistically. We need a way of discovering and understanding the natural relationships among all the complex elements that compose the world, even among the complex elements that compose our own disciplines.

Various attempts to address this need have been made under the rubric of interdisciplinary studies—programs or processes that aim to synthesize the perspectives and promote connections among multiple disciplines. Some of these efforts have been criticized as superficial joinings of disciplinary knowledge. But the chief criticism of interdisciplinary studies—leveled even by its proponents—is that looking at an issue from multiple perspectives does not, in itself, enable one to find the common ground among contrasting viewpoints, to resolve conflicts, and to arrive at a coherent understanding.

The diverse academic disciplines can be properly unified at only one level—at their source. All academic disciplines are expressions of human consciousness—and if the fundamental principles of consciousness could be identified and understood, then one would gain a grasp of all human knowledge in a single stroke.

This brings us to the theoretical component of Consciousness-Based education. Consciousness-Based education does precisely this—and not as an abstract, theoretical construct but as the result of students’ direct experience of their own silent, pure consciousness. In this sense, practice of the Transcendental Meditation technique forms the laboratory component of Consciousness-Based education, where the theoretical predictions of Consciousness-Based education can be verified through direct personal experience.

This theoretical component offers a rich and deep yet easy-to-grasp intellectual understanding of consciousness—its nature and range, how
it may be cultivated, its potentials when fully developed. This theo-
retical component also identifies how the fundamental dynamics of
consciousness are found at work in every physical system and in every
academic discipline at every level.

With this knowledge as a foundation, teachers and students in all
disciplines enjoy a shared and comprehensive understanding of human
development and a set of deep principles common to all academic
disciplines—a unified framework for knowledge. With this unified
framework as a foundation, students can move from subject to sub-
ject, discipline to discipline, and readily understand the fundamental
principles of the discipline and recognize the principles the discipline
shares with the other disciplines they have studied. This approach
makes knowledge easy to grasp and personally relevant to the student.

**Pure consciousness and the unified field**

Consciousness has traditionally been understood as the continuous flux
of thoughts and perceptions that engages the mind. Thoughts and per-
ceptions, in turn, are widely understood to be merely the by-product of
the brain’s electrochemical functioning.

Maharishi has put forward a radically new understanding of human
consciousness. In Consciousness-Based education, the foundation and
source of all mental activity is understood to be pure consciousness, the
most silent, creative, and blissful level of the mind—the field of one’s
total inner intelligence, one’s innermost Self. (This unbounded value of
the Self is written with an uppercase “S” to distinguish it from the ordi-
nary, localized self we typically experience.) Direct experience of this
inner field of consciousness awakens it, enlivens its intrinsic properties
of creativity and intelligence. Regular experience of pure conscious-
ness through the Transcendental Meditation technique leads to rapid
growth of one’s potential, to the development of higher states of human
consciousness—to *enlightenment*.

But consciousness is more, even, than this.

Throughout the 20th century, leading physicists conjectured upon
the relation between mind and matter, between consciousness and the
physical world; many expressed the conviction that mind is, somehow,
the essential ingredient of the universe. But Maharishi goes further.
He has asserted that mind and matter have a common source, and that
this source is pure consciousness. Consciousness in its pure, silent state is identical with the most fundamental level of nature’s functioning, the unified field of natural law that has been identified and described by quantum theoretical physicists over the past several decades. Everyone has the potential to experience this field in the simplest form of his or her own awareness. Considerable theoretical evidence, and even empirical evidence, has been put forward in support of this position.

Maharishi has developed these ideas in two bodies of knowledge, the first known as the Science of Creative Intelligence, the second as Maharishi Vedic Science and Technology. The Science of Creative Intelligence examines the nature and range of consciousness and presents a model of human development that includes seven states of consciousness altogether, including four higher states beyond the familiar states of waking, dreaming, and sleeping. These higher states, which develop naturally and spontaneously with Transcendental Meditation practice, bring expanded values of experience of one’s self and the surrounding world. Each represents a progressive stage of enlightenment. Maharishi Vedic Science and Technology examines the dynamics of pure consciousness in fine detail. It reveals the fundamental principles of consciousness that may then be identified in every field of knowledge and every natural system.

Most important for teaching and learning, these sciences reveal how every branch of knowledge emerges from the field of pure consciousness and how this field is actually the Self of every student.

Strategies for promoting teaching and learning
Consciousness-Based education also includes a battery of educational strategies that promote effective teaching and learning. Foremost among these is the precept that parts are always connected to wholes and that learning is most effective when learners are able to connect parts to wholes. In Consciousness-Based education, the parts of knowledge are always connected to the wholeness of knowledge, and the wholeness of knowledge is connected to the Self of the student.

One means of doing this is through Unified Field Charts. These wall charts, developed by the faculty at Maharishi University of Management and used in every class, do three things: (1) They show all the branches of the discipline at a glance. (2) They show how the discipline
emerges from the field of pure consciousness, the unified field of natural law at the basis of the universe. (3) They show that this field is the Self of the student, which the student experiences during practice of the Transcendental Meditation technique.

In this way students can always see the relation between what they are studying and the discipline as a whole, and they can see the discipline as an expression of their own pure consciousness. Again, this is more than an intellectual formulation—it is the growing reality of students’ experience as they develop higher states of consciousness.

Another strategy is *Main Point Charts*. Developed by the faculty for each lesson and posted on the classroom walls, these charts summarize in a few sentences the main points of the lesson and their relationship to the underlying principles of consciousness. In this way students always have the lesson as a whole in front of them, available at a glance.

**The next paradigm shift**

If higher education is fundamentally about student learning and growth, then Consciousness-Based education represents a major paradigm shift in the history of education. To understand this change, it is useful to reflect on the encouraging paradigm shift that has already been taking place in education over the past several decades.

This shift involves a move from what many call an *instruction paradigm* to a *learning paradigm*. In the instruction paradigm, the mission of colleges and universities is to provide instruction; this is accomplished through a transfer of knowledge from teacher to student. In the learning paradigm, the mission is to produce student learning; this mission is achieved by guiding students in the discovery and construction of knowledge.

This shift is a vitally important advance in education, leading to more successful outcomes and more rewarding experiences for students and teachers alike. But a further paradigm shift remains, and we can understand it by examining a fundamental feature of human experience.

Maharishi observes that every human experience consists of three fundamental components: a knower, a known, and a process of knowing linking knower and known. We may also use the terms experiencer, object of experience, and process of experiencing, or observer, observed, and process of observation.
This three-fold structure of experience is nowhere more evident than in schools: The knowers are the students, the known is the knowledge to be learned, and the process of knowing is what the full range of teaching and learning strategies seek to promote.

Understanding this three-fold structure helps us understand the paradigm shifts that are taking place.

The instruction paradigm places emphasis on the known. It focuses on the information students are to absorb and the skills they are to learn. In this paradigm, the instructor’s role is to identify what students need to know and deliver it to them.

The learning paradigm emphasizes the process of knowing. It recognizes that students must be actively involved in the learning process, that knowledge is something individuals create and construct for themselves, that students have differing learning styles and differing interests that must be taken into account. In this paradigm, the instructor’s role is to create learning environments and experiences that promote the process of learning.

The Consciousness-Based paradigm embraces the known and the process of knowing but places primary emphasis on the knower—on developing the knower’s potential for learning from within. The following diagram shows the respective emphases of each approach:
But the learning paradigm does not so much abandon the instruction paradigm as enlarge it, so that it includes the process of knowing as well as the known. And the Consciousness-Based approach completes the enlargement to include the knower:

Consciousness-Based education, in summary, is a theory and practice grounded in a systematic science and technology of consciousness, making available the complete experience, systematic development, and comprehensive understanding of the full range of human consciousness. More than 30 years’ experience and extensive scientific research confirm the success of this approach and its applicability to any educational institution.
About this book series

This series of twelve volumes is the result of a unique faculty-wide project that began with the founding of Maharishi University of Management in 1971 and continues to this day. Each volume in the series examines a particular academic discipline in the light of our Consciousness-Based approach to education.

Each volume includes:

• an introductory paper introducing the Consciousness-Based understanding of the discipline,
• a Unified Field Chart, if available for publication, for the discipline—a chart that conceptually maps all the branches of the discipline and illustrates how the discipline emerges from the field of pure consciousness and how that field is the Self of every individual. Thus, these charts connect the “parts” of knowledge to the “wholeness” of knowledge and the wholeness of knowledge to the Self of the student;
• subsequent papers that show how this understanding may be applied in various branches of the discipline,
• some examples of student work exploring how the Consciousness-Based approach enhances learning in the discipline, and
• an appendix describing Maharishi Vedic Science and Technologies of Consciousness in detail.
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We welcome inquiries and further contributions to this series.

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Mathematics begins with wonder. Throughout the ages, mankind has innocently asked, and attempted to answer, endless questions about numbers, lines, geometric figures, and other kinds of mathematical objects that seem somehow to be “given” to us for exploration with our mind’s eye. The heart of mathematics is the process of asking such questions, discovering answers, representing both questions and answers in the rigorous language of mathematics, and demonstrating the correctness of these answers through deductive reasoning based on a fundamental set of axioms. One of the remarkable consequences of this play of consciousness is that the patterns of orderliness discovered in this purely abstract way can be used to model the physical world in great detail, even to the extent of allowing scientists to predict nature’s behavior and to develop extraordinary technologies that transform the way life on earth is lived.

The activity of mathematics is by nature a self-referral activity: the mathematician calls upon faculties of his own consciousness to discover answers to questions that arise about mathematical forms and relationships, which are nothing other than forms and relationships of that very same consciousness. Therefore, it is perhaps no surprise that the structures and relationships that have emerged within mathematics, as well as the mathematical method itself, reflect the principles and dynamics of the structure of consciousness, as described by Maharishi Vedic Science.

Discovery, elaboration, and application of these connections, on the basis of the direct experience of pure consciousness through regular practice of the Transcendental Meditation and TM-Sidhi programs, is the heart of Consciousness-Based Education in mathematics. Seeing how mathematical structures and relationships reflect the nature and characteristics of one’s own consciousness, both student and researcher of mathematics find the technical topics of mathematics to be easier to grasp and fundamentally more relevant. Then, the heightened clarity of awareness that develops from practice of the Transcendental Medita-
tion and TM-Sidhi programs permits the student to enjoy an enhanced creativity with concepts that tend to be abstract and complex at times. The goal of this Consciousness-Based approach is that the very process of learning and doing mathematics stirs and awakens the “Cosmic Mathematician” deep within each student, resulting not only in great mathematical accomplishment, but also in the ability to engage the orderly mathematical intelligence of nature itself to create a fulfilling and mistake-free life.

Some examples of mathematical notions for which deep parallels to principles of Maharishi Vedic Science have been discovered—parallels that will be elaborated in this volume—include the following:

- the structure of the real line, in which each point is related to the line as a whole through the principle of nested intervals, an analogue to the *Akshara* principle of Maharishi Vedic Science by which points arise in the collapse of the unbounded continuum, yet retain the memory (*Smriti*) of their source;
- mathematical quantification of change in analysis, whereby one mathematically computes the unmanifest impulse of change at a point of a function;
- the concept of symmetry, by which one locates nonchange in the midst of change; this fundamental and pervasive idea rests at the core of geometry and was a key factor in the development of the modern-day theory of groups;
- central to the mathematical method itself, the process of abstraction, whereby deeper levels of mathematical knowledge are seen to unify diversity; this move toward abstraction from concrete particulars appears throughout mathematics, from applied areas of mathematics, in which, for example, a single differential equation can capture, in a single symbolic expression, the entire range of behavior of a law of nature, to the full abstraction of all mathematical theories, culminating in category theory, which provides a unification of all of mathematics in terms of fundamental universal constructions.

One area of mathematics that exhibits Vedic principles especially clearly is the area of mathematical foundations—particularly set theory,
which is the standard foundational theory for all of modern mathematics. Set theory consists of a small collection of axioms formulated in the formal language of sets, from which all the theorems about sets, and, consequently, all of mathematics, may be derived. The universe of sets that is built up sequentially through the use of these axioms contains in principle all mathematical objects, represented as sets. In this way, set theory naturally plays the role of the unified field of mathematics, providing a unified basis for the diverse expressions that collectively constitute the field of mathematics.

A testament to the depth of the knowledge provided by mathematical foundations is that it is able to precisely describe its own boundaries, its own limitations. An important example is found in a famous theorem due to Kurt Gödel, the Incompleteness Theorem, which states that no foundational theory, including set theory, can establish by way of logical deduction from its axioms every statement that is true in the mathematical universe. His result points to a fundamental limitation in the very method of mathematics, which relies on logical derivations from axioms to discover mathematical truths. Another example is concerned with the mathematical formulation of the concept of the Infinite. While it is known that there is an endless variety of different sizes of infinity in mathematics, a phenomenon that modern set theory has been unable to account for is the appearance within mathematics of notions of infinity so large that (by another theorem of Gödel) they cannot be proven to exist; these infinities are known as large cardinals. A desirable solution to this well-known problem would be to expand the axioms of set theory in some natural way in order to account for these enormous infinities, but mathematical intuition so far has not been clear enough to see how to do this.

Maharishi Vedic Science, especially Maharishi Vedic Mathematics—the mathematics of self-referral consciousness (Maharishi Mahesh Yogi, 1994:339)—offers an approach to knowledge that sheds light on these and other limitative results in foundations. For example, the mismatch between truth and provability, demonstrated by the Incompleteness Theorem, points to a fundamental principle of Maharishi Vedic Mathematics: complete knowledge is not available at the level of the intellect alone. For example, the Bhagavad-Gita, a portion of Vedic literature, declares (Maharishi Mahesh Yogi, 1969:3.42)
That which is beyond even the intellect is he... referring to the nature of the Self, the field of pure consciousness. (This unbounded value of the Self is written with an uppercase “S” to distinguish it from the ordinary, localized self we typically experience.) By contrast, according to Maharishi Vedic Mathematics, for a fully awake consciousness, all knowledge is available instantly, in a single stroke. Comparing modern mathematics to his Vedic Mathematics, Maharishi explains, “Modern Mathematics is the field of steps, whereas Vedic Mathematics is the field of pure intelligence that gets what it wants instantly without steps” (Maharishi Mahesh Yogi, 1994:389). The logical, step-based approach to arrive at mathematical truths in modern mathematics is therefore enhanced by the no-step approach to gaining knowledge through Maharishi Vedic Mathematics.

Likewise, by offering knowledge of the nature of wholeness—Brahman—Maharishi Vedic Mathematics offers a deep understanding of the true nature of the Infinite, an intuition that has already proven useful in moving toward a resolution of the problem of large cardinals. Maharishi Vedic Mathematics reveals the ultimate ground of all numbers, finite or infinite, to be what Maharishi has termed the Absolute Number; indeed, he explains that this Absolute Number is itself the self-referral dynamics of wholeness, the basis for unfoldment of all numbers and mathematical structures in such a way that each individual expression remains connected to its source in wholeness (Maharishi Mahesh Yogi, 1994:380-381, 615-621). He further explains: “This [the fact that everything in the objective world is the expression of wholeness] presents to us the need for an Absolute Number in the field of Mathematics, a number that can help us to account for the infinite number of wholenesses within the universe—a number that will help us to account for the theme of creation and evolution in terms of wholeness” (Maharishi Mahesh Yogi, 1994:611). Two of the articles in this volume explore a possible solution to the problem of large cardinals that makes use of the principles and dynamics of Maharishi’s Absolute Number to supply the heretofore missing intuition.

In these and other ways, Maharishi Vedic Mathematics offers a key to bringing to fulfillment the structuring impulses underlying modern mathematics. By overcoming the inherent limitation of knowledge on the basis of deductive logic alone, Maharishi Vedic Mathematics,
through its Consciousness-Based approach to knowledge, opens the door to complete knowledge, and not just on the level of concepts, but as a living reality, leading to a life and a society that embody the full benefit of nature’s mathematics: a mistake-free life lived in waves of success and fulfillment.

This two-part set on mathematics, as part of the Consciousness-Based education series of volumes, organizes in a single publication some of the most significant articles written by the Maharishi University of Management Department of Mathematics over the past 30 years. These articles probe into some of the deepest areas of modern mathematics to show how the basis of mathematics is itself structured in the qualities, principles, and dynamics of pure intelligence. They also explore Maharishi Vedic Mathematics, showing how this aspect of Maharishi Vedic Science brings fulfillment to the trends and aspirations of modern mathematics. Other articles bring to light Consciousness-Based methods of teaching mathematics, to enliven the Cosmic Mathematician deep within every student. And still another section of articles makes use of Maharishi Vedic Science as a set of principles that are used to refine and bring to fulfillment in a technical way the field of mathematics itself.

Part 1: Pure Mathematics in the Light of Maharishi Vedic Science and Maharishi Vedic Mathematics locates principles of Maharishi Vedic Science in the various fields of pure mathematics, including their foundation in set theory, and elaborates on the role of Maharishi Vedic Mathematics in bringing fulfillment to the field of mathematics.

Part 2: Applications of Maharishi Vedic Science to Mathematics Education and Mathematics Research includes articles that discuss applications of Maharishi Consciousness-Based approach to the field of mathematics education, and other articles in which Maharishi Vedic Science is used as a tool for deeper research into mathematics itself.

SUBJECT HEADINGS FOR PART 1

Section I: Maharishi Vedic Science and Foundations of Mathematics begins with Michael Weinless’s article “The Samhita of Sets,” which explores the deep connections between modern mathematics, especially modern set theory, and Maharishi Vedic Science. The author discusses the sense in
which the “absolute infinite” notions of the universe $V$ of sets and the class $\Omega$ of all ordinals are analogous to the transcendental wholeness of Maharishi Vedic Science, and ways in which the three-in-one structure of pure consciousness finds expression in these expansive mathematical models. Venturing into a study of alternative foundations, the article finds that topos theory, playing the role of Vedanta, successfully integrates intuitionistic mathematics into the classical set-theoretic foundation. The article also explains how recent results on the Axiom of Determinacy illuminate the connection between Vedanta and Jyotish.

Weinless’s article is followed by the “Unified Field Chart for Mathematics,” developed by the Maharishi University of Management Department of Mathematics. This chart locates the source of mathematics within the unified field of natural law, which is seen to be at once the fundamental field of existence, as described by modern quantum field theory, and the source of thought, experienced directly during the practice of the Transcendental Meditation program. The emergence of mathematical knowledge from this field starts from the abstract level of foundations and unfolds through pure and applied fields, into mathematical applications for the development of technology and for the well-being of society. The chart provides a snapshot view of the whole field of mathematical knowledge from its source to its expressions in its most applied values.

As a natural commentary on the Unified Field Chart, the next article in the volume is an excerpt from the Heaven on Earth book, originally published in 1989. Highlighting themes of unfoldment from Maharishi Vedic Science, the article surveys the entire range of mathematics from its source in pure intelligence, through foundational theories and pure mathematics, to all the applied areas of mathematics.

The “Richo Akshare Chart for Mathematics,” which follows next, elaborates, from the perspective of eight different mathematical disciplines, the role of the Richo Akshare verse of Rk Veda (1.164.39) in the fabric of mathematics. Using the lines of the verse as a template for discussion, each discipline locates the significance of the collapse of infinity to a point, and the emergence of the fundamental impulses of intelligence from this collapse. In addition, each discipline locates within its own structure a deep and vital aspect of mathematical intelligence, without which a proper understanding of the discipline would
be impossible, but knowing which, the depths of the discipline become easily accessible.

Further elaborating this chart with respect to set theory, Michael Weinless’s article “Mathematical Foundations in the Light of the Richo Akshare Verse,” revisits some of the themes presented in the first article of the volume, “The Samhita of Sets,” now appreciated from the point of view of this pivotal verse from Rk Veda.

The final article in Part I, Paul Corazza’s “The Wholeness Axiom,” takes up the challenge raised in Weinless’s “Samhita of Sets,” to use Maharishi Vedic Science as a means to provide the necessary intuition to solve the “problem of large cardinals.” To that end, the article introduces and motivates a new axiom, to be added to the standard set theory axioms, which asserts, in the language of set theory, that “wholeness moves within itself, knows itself, remains unchanged by its own transformation, and remains ever present at each point in creation.” From this axiom, virtually all large cardinals are shown to be derivable.

Section II:
Modern Mathematics in the Light of Maharishi Vedic Mathematics addresses the vision of possibilities in raising modern mathematics to its supreme level through the knowledge and technologies of Maharishi Vedic Mathematics. John Price’s article, “Maharishi’s Absolute Number: The Mathematical Theory and Technology of Everything,” demonstrates how, by providing direct access to the dynamics of pure intelligence, beyond the senses and even beyond the intellect, Maharishi Vedic Mathematics and Maharishi’s Absolute Number provide a natural completion of the knowledge of modern mathematics and go on to offer profound solutions to problems ordinarily considered to be outside the scope of mathematics, such as providing invincible defense for any nation. Catherine Gorini’s article, “Maharishi Vedic Mathematics: The Fulfillment of Modern Mathematics,” takes up a similar theme, arguing that modern mathematics’ innate drive toward unification through abstraction and toward a complete understanding of natural law through techniques of mathematical modeling find fulfillment in Maharishi Vedic Mathematics, where nature’s intelligence and the intelligence of the mathematician come together in the dynamic source of all knowledge, the
unified field of natural law, that field of intelligence that transcends the intellect and governs the entire manifest universe.

**Section III:**
**Self-Referral Dynamics and Mathematics** is concerned with those mathematical theories and structures that reflect the self-referral dynamics of pure intelligence, as described by Maharishi Vedic Science. Michael Weinless’s article, “Self Referral in the Foundations of Mathematics,” surveys self-referral expressions of knowledge in the foundations of mathematics by examining themes from the theory of non-wellfounded sets, topos theory, impredicative definitions in analysis, the $\lambda$-calculus, and denotational semantics.

**Section IV:**
**Geometry, Symmetry, and Consciousness** brings to light the natural way in which fundamental notions in geometry can be seen to display the characteristics of pure consciousness. In Catherine Gorini’s article, “Consciousness: The Last Frontier of Geometry,” she argues that the deeper themes explored in the field of geometry, such as continuity, higher dimensions, infinity, symmetry, and the homogeneity of space, reflect qualities of the very creator of those concepts, the intelligence of the geometer himself. She concludes that the “last frontier” of the field of geometry can be nothing other than the field of consciousness. In “Symmetry: A Link Between Mathematics and Life,” the last article of Part I, Gorini shows how the important geometric notion of symmetry can be effectively communicated to students of mathematics by referring to parallel themes found in artistic and cultural contexts, and in particular in the context of the Vedic literature.

**Subject Headings for Part 2**

**Section V:**
**Consciousness-Based Mathematics Education** begins the second volume with articles that discuss the educational value of the Consciousness-Based approach to teaching mathematics. Articles address not only teaching techniques and styles, but also in some cases present insights that have proven effective for instructional purposes in the classroom. One article is Catherine Gorini’s “How Maharishi Vedic
Science Answers the Questions of the Unreasonable Effectiveness of Mathematics in the Sciences,” which addresses the long-held question in the scientific community: Why is mathematics, a subjective creation of the human intellect, so effective in the sciences, which study the objective, physical world? The answer she proposes is based on the observation by Maharishi Vedic Science that the subjective and objective fields of knowledge have their common basis in consciousness.

Anne Dow, “A Unified Approach to Developing Intuition in Mathematics,” raises the issue, well-known to mathematics educators, that the crucial step in understanding and doing mathematics, involving a mixture of intuition and analysis, is actually the aspect of mathematics that is hardest to impart to students. Using Maharishi’s model of the thinking process, she proposes that the experience of transcending thought, through the Transcendental Meditation program, directly addresses this need.

“Preparing the Student to Succeed at Calculus,” also by Anne Dow, points out that modern approaches in teaching calculus have successfully addressed only two of the three fundamental components of gaining knowledge: the content of the subject matter (the known) and the process of knowing. These approaches have failed to address the need to expand the container of knowledge, the knower—the third component of gaining knowledge. This becomes necessary, she argues, when the student is asked to grasp deeper ideas in the subject, such as the concept of a “limit” in calculus. She proposes that regular practice of the Maharishi Transcendental Meditation technique addresses this need, and cites her own teaching experience at Maharishi University of Management (formerly Maharishi International University), together with published scientific research, to support this claim.

Catherine Gorini’s article “Using the Study of Consciousness to Teach Calculus” also points to the importance of addressing the need to expand the consciousness of the knower, the student of calculus, through the practice of the Transcendental Meditation program, in order to optimize his or her educational experience. She also demonstrates the use of principles from Maharishi Vedic Science to illuminate challenging concepts in the subject (the continuum, the limit, the derivative, the integral); these principles, she argues, provide much easier and deeper access to these mathematical notions because of the
student’s growing experience of transcending and of deeper layers of consciousness.

**Section VI: Maharishi Vedic Science as a Research Tool for Modern Mathematics** contains articles in which Maharishi Vedic Science is used to advance or more deeply elucidate the foundations of modern mathematics. Paul Corazza’s, “Vedic Wholeness and the Mathematical Universe: Maharishi Vedic Science as a Tool for Research in the Foundations of Mathematics” offers a first look at the Wholeness Axiom (a subject that is developed further and from a different perspective in the author’s other article on this topic in Part I), using as a starting point the following question: Which qualities and dynamical principles of wholeness, described by Maharishi Vedic Science, are clearly present and lively in the universe of sets, as described by modern set theory, and which ones seem to be absent? The article proposes that a proper answer to this question leads to the introduction of new dynamics in the mathematical universe, expressible in the form of a new set-theoretic axiom: the Wholeness Axiom. The article shows that the axiom succeeds not only to “awaken” in the set-theoretic universe qualities and dynamics of wholeness that had apparently been absent before, but also results in an elegant solution to the problem of large cardinals.

Exploring an alternative foundation based on arrows instead of sets, Michael Weinless’s final article in Part 2, “Categories and Toposes: Dynamism at the Foundation of Mathematics,” gives a short course on a highly successful foundation based on category theory, called topos theory. A topos is a category exhibiting such strong closure properties that it is almost a model of set theory, a universe of sets, but not quite. Because its closure properties have a geometric feel, toposes provide natural models for deep results in areas of geometry such as sheaf theory and algebraic geometry. The generality of the concept of a topos makes it possible to define toposes that are genuinely models of set theory, modeling various set theoretic axioms (for example, topos theory provides an alternative view of set-theoretic forcing for producing models). At the same time, because the internal logic of a topos is intuitionistic, topos theory provides the richest known source of intuitionistic theories. The versatility of topos theory leads the author to suggest that toposes play the role of Vedanta in mathematical foundations, synthesizing
diverse, even incompatible, foundational theories. The article gives a systematic development of this emerging field and uses Maharishi Vedic Science to shed light on its underlying principles.

References
Section I

Maharishi Vedic Science
and Foundations of Mathematics
The Samhita of Sets:

*Maharishi Vedic Science* and the Foundations of Mathematics

Michael Weinless, Ph.D.
ABOUT THE AUTHOR

Michael Weinless, Ph.D., received his B.S. from M.I.T. in 1964 and his Ph.D. in mathematics from M.I.T. in 1968. He went on to Harvard University where he held the positions of Benjamin Pierce Lecturer and Assistant Professor of Mathematics from 1968 to 1971. In 1972, Dr. Weinless became one of the founding faculty members of Maharishi International University (renamed Maharishi University of Management in 1995), where he pioneered the development of a unified-field-based mathematics curriculum, integrating principles of the Science of Creative Intelligence and Maharishi Vedic Science with the traditional mathematical content of the courses. Dr. Weinless was chairman of the Department of Mathematics from 1972 to 1990.
This paper examines the relationship between foundational areas of modern mathematics and Maharishi Mahesh Yogi’s formulation of his Vedic Science. A number of parallels between the two disciplines are systematically developed. It is argued that the set-theoretic concept of the “absolute infinite,” as expressed both in the universe of sets $V$ and the absolute ordinal $\mathcal{W}$, provides a natural mathematical expression of the transcendental reality of pure consciousness, as described in Maharishi Vedic Science.

Three different themes of analysis of the absolute infinite in terms of Rishi (knower), Devata (process of knowing), and Chhandas (known) are presented, and these are shown to correspond to the Rik, Sama, and Atharva Samhitas of Maharishi Vedic Science. Several aspects of mathematical intuition underlying the axiomatization of set theory are analyzed, and on this basis the relevance of Maharishi Vedic Science to the description of the creative process at the source of set theory is discussed.

In particular, it is argued that Maharishi Vedic Science can provide needed motivation for “large” large cardinal axioms. The foundational viewpoints of intuitionism, category theory, and topos theory are described; it is shown how topos theory integrates intuitionistic mathematics into the set-theoretic foundation and thereby plays the role of Vedanta in the unified structure of modern mathematics.

In this context, the sheaf semantics of a topos are shown to provide a mathematical expression of the self-referral structure of pure knowledge, as described by Maharishi Vedic Science. Recent developments in definability theory concerning the Axiom of Determinacy are shown to illuminate the relationship between Jyotish and Vedanta in Maharishi Vedic Science.

The direction of development of modern mathematics is discussed, and it is argued that the state of enlightenment, as described in Maharishi Vedic Science, can be identified as the goal and fulfillment of the study of mathematics. The paper concludes by suggesting several promising directions for future research into the relationship of mathematics to Maharishi Vedic Science.

Introduction

In this paper we shall present the first steps of a systematic exploration of the relationship between modern mathematics and Maharishi Vedic Science. We shall examine in particular a number of parallels between the most central themes in foundational areas of modern
mathematics and the principles of Maharishi Vedic Science describing
the nature of an underlying field of intelligence and its internal dynam-
ics. On this basis, we shall see that Maharishi Vedic Science provides a
unified theoretical and experiential framework in which the whole dis-
cipline of modern mathematics can be naturally embedded. Maharishi
Vedic Science will thereby offer significant new levels of both under-
standing and experience to the study of mathematics.

One theme we shall develop is the way in which Maharishi Vedic
Science completes the range of mathematics to include both the source
and the goal of all mathematical knowledge. This will be achieved,
firstly, by tracing the source of mathematical knowledge beyond the
axioms of set theory back to the dynamics of intelligence at the source
of the axioms. Maharishi Vedic Science in this way will provide a new
understanding of the creative process at the source of the sequential
unfoldment of mathematical knowledge. Secondly, Maharishi Vedic
Science will supplement the mathematical knowledge of the infinite
with the direct experience of a holistic value of intelligence that can
be equated with the “absolutely infinite” wholeness of set theory, the
infinite value of the universe of sets. Maharishi has explained how, in
the fully developed state of human life, this experience is permanently
established in awareness and serves as a foundation for all thought,
speech, and action. In examining the structure of this most evolved
state of human consciousness, we shall locate the fulfillment of the cen-
tral evolutionary themes in the development of modern mathematics.

In examining the relationship of mathematics to Maharishi Vedic
Science, we shall identify a number of ways in which Maharishi Vedic
Science can enrich both the study of mathematics and the research
activities of the working mathematician. This will involve, first of all,
the way in which Maharishi Vedic Science provides a link between the
abstract knowledge of mathematics and one’s direct experience.

Because of its purely abstract nature, mathematical knowledge gen-
erally seems unrelated to one’s own concrete experience of life, except
perhaps in a very remote way, for example, through its application in
physical science. For many, this makes mathematics difficult to under-
stand and appreciate.

Through the development of consciousness available in Maharishi
Vedic Science, one gains access to a level of experience in which much
of the abstract structural content of modern mathematics is seen to be
directly mirrored. Because of the strong parallelism between the prin-
ciples of Maharishi Vedic Science describing this experiential reality
and the deepest principles of modern mathematics, the abstract prin-
ciples of mathematics can be appreciated in a much more intimate way:
they provide a kind of commentary, from a very precise, analytic view-
point, on the structure and dynamics of the most fundamental level of
one’s own intelligence. The abstract knowledge of mathematics thereby
becomes directly relevant to one’s own experience.

We shall also discuss in this paper the way in which Maharishi Vedic
Science can be applied to the theory of large cardinals, thereby contrib-
uting to the growth of mathematical knowledge towards completeness.
This will involve utilizing the principles of Maharishi Vedic Science to
motivate new axioms that offer progressively more complete and holis-
tic mathematical knowledge of the infinite. Furthermore, the practical
component of Maharishi Vedic Science, the Maharishi Technology of
the Unified Field, will directly develop the subjective faculty of mathe-
matical intuition that is essential for this direction of mathematical
progress. More generally, the development of creativity and intelligence
brought about by the practice of the Maharishi Technology of the Uni-
fied Field will provide the basis for success at all levels of mathematical
study and research.

**What Is Maharishi Vedic Science?**

Maharishi Vedic Science is a science of consciousness formulated by
Maharishi Mahesh Yogi. Maharishi’s exposition of his Vedic Science
has its roots in the ancient Vedic tradition of India. His contribution is
unique in that it brings to light an entirely new approach to the structure
and meaning of the ancient Vedic texts. It is beyond the scope of this
paper to provide an introduction to Maharishi Vedic Science; the reader
is referred to Chandler (1987) and Maharishi Mahesh Yogi (1986). For
a vision of the sequential steps through which Maharishi’s exposition of
Vedic Science has matured, the reader is referred to Maharishi Mahesh
Yogi (1966, 1969, 1976, 1978). Here we shall simply review the main
principles of Maharishi Vedic Science that are germane to our analy-
sis of the relationship of Maharishi Vedic Science to the discipline of
mathematics.
Maharishi Vedic Science is a complete science in that it takes into account all the constituents of knowledge: the knower, the known, and the link between them, the process of knowing. This contrasts sharply with the “objective” approach of Western science, which focuses exclusively on the object of knowledge to the exclusion of both the knower and the process of knowing. Maharishi Vedic Science supplements the objective approach of modern science with a technology of consciousness, the Maharishi Technology of the Unified Field, through which one directly experiences, in a systematic way, progressively more refined levels of mental activity, culminating in the experience of the simplest state of awareness, a state of silent wakefulness. This state of awareness, called Transcendental Consciousness, represents a fourth major state of consciousness, distinct from the commonly experienced waking, dreaming, and deep sleep states.

The experience of Transcendental Consciousness lies at the very heart of Maharishi Vedic Science. The striking feature of this experience is that it has a totally unified nature, yet at the same time contains the seed of diversity. Maharishi describes it as the experience of an unbounded, undifferentiated ocean of consciousness, absolutely silent yet awake within itself. The wakefulness of Transcendental Consciousness is the expression of its own intelligent nature. This makes the discriminative quality of the intellect lively within it. The primordial activity of this most fundamental level of intellect is the discrimination of three distinct values within the unified state of Transcendental Consciousness. These three values are called Rishi, Devata, and Chhandas, and they represent respectively the values of the knower, the process of knowing, and the known.

Maharishi describes Transcendental Consciousness as a state in which consciousness knows itself, and itself alone. In this state, the knower is one’s own consciousness, what is known is one’s own consciousness, and the process of knowing, which links the knower to the known, is also one’s own consciousness. Knower, known, and process of knowing are one and the same. Thus Transcendental Consciousness contains within itself the three-fold structure of knowledge, but this structure of knowledge has a self-referral nature: the knower knows himself. Maharishi describes this self-referral structure of knowledge as the state of “pure knowledge.” The unity of the three values of Rishi,
Devata, and Chhandas in the structure of pure knowledge is described by the term Samhita, meaning “togetherness” or “connectedness.”

In the ultimate analysis, Maharishi emphasizes, the three values of Rishi, Devata, and Chhandas are just an intellectual conception, created by the discriminative value of the intellect. What is ultimately real is the undifferentiated, unified field of Transcendental Consciousness. This ultimate field of reality displays not only the value of pure intelligence, but also the value of pure existence. For this reason, it is often referred to as the field of Being. Since it is the most fundamental state of one’s own consciousness, it is also referred to as the Self. (This unbounded value of the Self is written with an uppercase “S” to distinguish it from the ordinary, localized self we typically experience.)

According to Maharishi Vedic Science, the unbounded field of consciousness experienced in Transcendental Consciousness underlies, in fact, all of creation. In Maharishi’s analysis of the mechanics of creation, all levels of both subjective and objective creation are described as unfolding, in a precise, sequential manner, from the interaction of the three values of Rishi, Devata, and Chhandas in the structure of pure knowledge. This description is strikingly parallel to the description of nature presented in the recently developed unified field theories of physics. According to these theories, all of the fields and particles in nature sequentially derive from a totally unified field of natural law on the basis of the self-interacting dynamics of that field.

The strong parallel between the description of nature in Maharishi Vedic Science and contemporary theoretical physics has led several prominent physicists to suggest that the unified field of physics and the field of Transcendental Consciousness are one and the same. A systematic discussion of this question, including a presentation of both theoretical and practical evidence for their identity, can be found in Hagelin (1987). Maharishi himself has asserted that these two fields should be identical, and he has borrowed the language of physics in describing the field of pure consciousness as the unified field of natural law.

The fact that Maharishi Vedic Science describes all of nature as sequentially unfolding from the field of pure intelligence on the basis of the creative activity of the intellect suggests a profound intimacy between Maharishi Vedic Science and the discipline of mathematics, which is so directly grounded in the totally abstract functioning of the
intellect. The natural first step in the study of the relationship between mathematics and Maharishi Vedic Science is to locate the Samhita value in mathematics and identify its three-in-one structure. This will be our point of departure.

We shall see how the supreme expression of infinity in mathematics, a level of infinity that we shall call the absolute infinite, gives mathematical expression to the transcendental, unbounded value of the field of pure consciousness. This value of absolute infinity is to be distinguished from the different levels of infinity displayed by the infinite sets that are the objects of ordinary mathematical study. The absolute infinite is the absolutely unbounded transcendental wholeness at the source of modern mathematics. We shall identify its role at the source of set theory, the unified foundational theory of modern mathematics.

**Organization of this Paper**

This paper is in two parts. Part 1, “The Absolute Infinite,” systematically explores the set-theoretic concept of the absolute infinite as a mathematical expression of the three-in-one structure of the Samhita of Maharishi Vedic Science. Part 2, “The Structure of Mathematical Knowledge,” examines the organization, validation, and sequential unfoldment of the knowledge of modern mathematics. The principles of Maharishi Vedic Science are applied to illuminate this structure of knowledge and further to identify its source and goal in the structure of pure knowledge, that is, in the three-in-one structure of the Samhita.

There are two fundamental expressions of the absolute infinite in set theory: the universe of sets, $V$, and the absolute ordinal number, $\Omega$. In Section 1 of Part 1, we shall examine the concept of the universe of sets, and we shall see how this concept provides the basis for the intellectual analysis of the absolute infinite in terms of a three-fold structure: sets (as wholes), membership relation, and sets (as points), corresponding to the values of Rishi, Devata, and Chhandas, respectively. In this context the Samhita value corresponds to the absolutely infinite value of $V$ itself. In Section 3 we shall examine the concept of the absolute ordinal and shall see how this concept presents a second mathematical expression of the three-in-one structure of the Samhita: ordinals, order relation, and ordinals, where the Samhita corresponds to the absolutely infinite value of $\Omega$. 
The relationship between these two mathematical expressions of the absolute infinite is provided by the iterative mechanics of set formation, considered in Sections 2 and 4. Examination of this iterative process will identify a third fundamental expression of the three-in-one structure of the absolute infinite in which the Rishi value is expressed by the ordinals, the Devata value by the power-set operation, and the Chhandas value by the partial universes. Here the power-set operation provides an infinitely dynamic expression of the discriminative value of intelligence, through which the sequential structure of the ordinals gives rise to all possible sets.

To establish the equivalence of the two mathematical expressions of the absolute infinite, \( V \) and \( \Omega \), a second fundamental expression of the discriminative value of intelligence is required that is capable of transforming sets into ordinals. This is provided by the Axiom of Choice, examined in Section 5. We shall see in particular how the global Axiom of Choice establishes the equality of the two fundamental expressions of the absolute infinite at the foundation of set theory. On this basis we shall identify the three different three-in-one structures of the absolute infinite with the Rik Samhita, Sama Samhita, and Atharva Samhita. This will conclude Part 1 of the paper.

Section 6 of Part 2 examines the creative process at the source of the sequential unfoldment of mathematical knowledge provided by axiomatic set theory. In this section we have endeavored to identify the primordial principles underlying the axioms themselves; these principles most directly express the creative process at the foundation of set theory. This will provide a mathematical parallel to the theme of Maharishi’s *Apaurusheya Bhashya*, or uncreated commentary, of Rik Veda.

Section 7 examines the way in which set theory provides a unified foundation for the diverse abstract theories of modern mathematics. It is noted how the set-theoretic construction of the real number continuum provides a mathematical expression of the continuous nature of the Samhita, and on this basis makes possible the mathematical quantification of the laws of nature that sequentially emerge from the self-interacting dynamics of the Samhita.

Section 8 examines how set theory provides a natural foundation for the field of mathematical logic, the study of the structure and function of the symbolic language of mathematics. In this way set theory
is found to describe not only the mechanics through which its own symbolic language sequentially unfolds, but also the mechanics of transformation through which the organizing power contained in the structure of its language becomes concretely expressed. In this context, the technique of forcing will be seen to provide a mathematical parallel to the technology of the Yagyas in Maharishi Vedic Science.

Section 9 examines the foundational viewpoints of category theory and intuitionistic mathematics and describes the way in which each is integrated into the set-theoretic foundation on the basis of the theory of large cardinals and topos theory, respectively. The application of topos theory in this context provides a striking mathematical expression of self-referral, whereby the objects of a topos represent simultaneously stages of knowing (values of the knower) and sets (values of the known).

Section 10 discusses a new direction in the theory of large cardinals in which the holistic knowledge of the infinite provided by large cardinal axioms can give rise to great computational power. This provides a mathematical parallel to the relationship between Jyotish and Vedanta in Maharishi Vedic Science.

Section 11 examines the direction of development of modern mathematics and identifies the state of enlightenment, as described by Maharishi Vedic Science, as the goal and fulfillment of the study of mathematics.

The final section suggests several promising directions in the study of the relationship of Maharishi Vedic Science to mathematics.

This paper is largely a further development of ideas presented in Weinless (1983). Because of the obvious limitations of space, it has not been possible to elaborate many of the fundamental mathematical concepts and constructions. For an introduction to modern set theory, the reader is referred to Hrbacek and Jech (1984), Wang (1981, pp. 119–155), and Rucker (1982, pp. 238–286). A comprehensive account of the relevant aspects of set theory at an advanced level is contained in Jech (1978) and Kanamori and Magidor (1978). For an elementary introduction to category theory and topos theory emphasizing foundational aspects the reader is referred to Weinless (2011, Vol. 5, Pt. 2, in this series). Other elementary treatments are contained in Arbib and Manes (1975) and Goldblatt (1984); for a more complete development of category theory and topos theory at an advanced level see Mac...
Lane (1971), Johnstone (1977), Makkai and Reyes (1977), and Scedrov (1984). A good introduction to mathematical logic, including intuitionistic logic, is provided by Van Dalen (1983); elementary accounts of Gödel’s incompleteness theorems are contained in Nagel and Newman (1958) and Rucker (1982, pp. 287–317). A general discussion of philosophical issues in the foundations of mathematics is contained in Wang (1974), Benacerraf and Putnam (1964), and Davis and Hersh (1981). Additional references will be given where appropriate.

Part I: The Absolute Infinite

1. The Universe of Sets
Set theory is the “unified field theory” of modern mathematics. It provides a unified foundation for all the diverse abstract theories of modern mathematics based upon a single structural concept: the iterative concept of a set. This concept makes possible the direct intellectual analysis of the nature of the infinite; the success of set theory as a unified foundational theory of mathematics has been a result of its deep penetration into the nature of the infinite.

We propose in this paper that the ultimate foundation of set theory is the infinite reality of the Self, that field of reality transcending the intellect. This reality is experienced when the individual awareness, freed from attention to any localized values, gains and appreciates its own unbounded status. This is the supreme, absolute value of infinity; it is expressed in set theory in the holistic nature of the universe of sets, the ultimate mathematical wholeness resulting from the synthesis of all possible sets. One does not usually equate subjective experience with precise mathematical concepts such as infinity. We shall see, however, as we systematically explore the relationship of Maharishi Vedic Science to set theory, that such an equation is natural, and probably even necessary, for a consistent understanding of the nature of abstract mathematical knowledge.

In this section we examine the way in which the universe of sets indeed presents the supreme value of infinity lying beyond the intellect. This is intimately connected with the way in which the universe of sets expresses the self-referral value of the knower-known relationship. On this basis we can identify, in the universe of sets, the mathematical
expression of the three-in-one structure of the Samhita of Maharishi Vedic Science.

Set theory was founded by the great German mathematician Georg Cantor in the latter part of the nineteenth century. We shall begin by examining the general concept of a set as defined by Cantor in the very first sentence of his monumental treatise of 1895 and 1897:

By a “set” we shall understand any collection into a whole $M$ of definite, distinct objects (which will be called the “elements” of $M$) of our intuition or our thought. (Wang, 1974, p. 188)

The elements of a set are “objects of our intuition or our thought.” They are thus objects of knowledge and therefore correspond to the value of Chhandas, the known, in Maharishi Vedic Science. A set results from the synthesis of a collection of such objects into a whole. Thus a set is a reality that exists in the field of consciousness, in which different objects of perception are synthesized into a wholeness: they are intellectually conceived as a whole. The synthetic functioning of intelligence that creates wholes is associated in Maharishi Vedic Science with the value of Rishi. Thus, in a natural way, a set as a whole is found to express the value of Rishi, the elements of the set the value of Chhandas, and the relationship of the set to its elements, or membership relation, the value of Devata.

For the next step of analysis we must probe more deeply into the nature of the Chhandas, the elements of a set. One of the striking features of modern set theory is its great conceptual elegance: it deals with only one fundamental object, a set, and one fundamental relationship, the membership relation. This means that every element of a set must itself be a set. This is a deep concept at the heart of set theory: elements of sets are sets.

A set thus has two sides to its nature: (1) its value as a whole, in relation to its elements, and (2) its value as a point, as a single element of another set. Therefore, when we equate the membership relation with the Devata value, both the Rishi and Chhandas values become equated with sets, but the values differ: a set as a whole expresses the Rishi value, whereas a set, as a point, expresses the Chhandas value.

We have identified a correspondence between the values of Rishi, Devata, and Chhandas in Maharishi Vedic Science and the fundamental structural concepts of set theory: the concepts of set and mem-
bership relation. Further, the membership relation naturally expresses the knower-known relationship in the field of consciousness. We next inquire whether set theory can locate the self-referral value of this relationship.

In a general way, the concept of a set is implicitly self-referral, because the elements of a set are themselves sets; the Chhandas values of one set are the Rishi values of some other sets. It is natural to inquire whether this self-referral value can be made completely explicit by directly equating the Rishi and Chhandas values, that is, by having a set be an element of itself.

The obvious candidate for a set expressing this self-referral value of the membership relation is the set of all sets. This set must certainly be an element of itself. It was discovered, however, at the turn of the century that the self-referral concept of a “set of all sets” leads to logical contradictions and therefore is not mathematically tenable.

One way to see how the concept of a “set of all sets” leads to a contradiction is the following simple argument, put forward by Bertrand Russell (1902/1967),

Russell’s Paradox: Suppose there were a set of all sets, \( U \). Then \( U \) would be an element of itself: \( U \in U \) (the notation “\( \in \)” means “is an element of”). Consider the subset \( R \) of \( U \) consisting of all those sets that are not elements of themselves. We then ask: would the set \( R \) be an element of itself? Is \( R \in R \)? By the construction of \( R \), it is easily seen that \( R \in R \) if and only if \( R \not\in R \). This means that the statement \( R \in R \) would be true if and only if it were false.

The “resolution” of Russell’s paradox in modern set theory is that certain collections of sets, such as the collection of all sets, are too “big” to be sets. These collections are called proper classes. In particular, the wholeness of set theory, the collection of all possible sets, is a proper class called the universe of sets, \( V \). Because \( V \) is not a set, it is not an element of itself. Hence \( V \) does not express the explicit value of self-referral \( "V \in V" \). Nevertheless, a deep principle of set theory, the Reflection Principle, expresses the self-referral nature of \( V \). The Reflection Principle describes the way in which the relation of \( V \) to its elements “approximates,” in a logically consistent way, the inconsistent self-referral property \( "V \in V" \). The Reflection Principle asserts that any conceivable structural property of \( V \) must be reflected in some set: if \( V \) has some property \( P \), then there
must exist some set \( A \) having property \( P \). Since \( A \) is a set and \( V \) is the universe of sets we have \( A \in V \). Thus every conceivable structural property of \( V \) is reflected in an element of itself. This means that \( V \), although not containing itself as an element, does contain elements reflecting each of its conceivable properties. In this way the Reflection Principle expresses the self-referral nature of the wholeness of mathematics, the universe of sets, in a logically consistent way.

(The reader is warned that we are referring in this article to the most general formulation of the Reflection Principle and not to the familiar restricted formulation that is provable in Zermelo-Fraenkel set theory. A discussion of the general formulation of this principle and its connection with the axiomatization of set theory can be found in Reinhardt (1974); Rucker (1982, pp. 238–286); and Wang (1974, p. 189).)

The Reflection Principle expresses the self-referral nature of the universe of sets. At the same time, it upholds the “indescribable,” transcendent nature of the universe of sets. The Reflection Principle is just the assertion that the universe of sets is structurally undefinable: no structural property can ever capture the uniqueness of the universe of sets. Any structural property of \( V \) will necessarily be reflected in some set, which cannot be \( V \) itself because \( V \) is not a set. Hence no structural property can capture the holistic value of \( V \).

Since the Reflection Principle applies to any intellectually conceivable property, it follows that the holistic, unbounded nature of the universe of sets must transcend the intellect. The universe of sets thus gives mathematical expression to a holistic value of infinity that transcends the intellect. We shall refer to this infinite value of the universe of sets as the absolute infinite.

In Maharishi Vedic Science, the unbounded field of pure consciousness, experienced as the state of Transcendental Consciousness, is described as transcending the intellect:

\[ \text{yo buddheh paratas tu sah} \]

That which is beyond even the intellect is he.

(Bhagavad-Gita 3.42)

(Translations of Sanskrit passages in this paper are based on Maharishi’s translations.) This is the infinite value of the Samhita. Because the
absolutely infinite value of the universe of sets likewise transcends the intellect, we shall view the universe of sets as a mathematical expression of the Samhita of Maharishi Vedic Science.

At first glance, equating the absolutely infinite value of the universe of sets with the value of the Samhita may seem unjustified, because the universe of sets is naively conceived as a totality of distinct elements (all possible sets), whereas the Samhita, the transcendental reality of pure consciousness, is a single, undifferentiated state. One can argue, however, that the transcendental value of wholeness that characterizes the universe of sets should not be considered pluralistic in its nature. This involves a theme of analysis that identifies the wholeness of the universe of sets with the primordial, undifferentiated mathematical wholeness from which all differentiated values emerge.

This analysis considers the way knowledge of the localized values of specific sets can be sequentially unfolded from the universe of sets by means of the Reflection Principle, which in turn expresses the self-referral nature of the universe of sets. The sets are seen to sequentially emerge as intellectual constructs on the basis of the self-referral dynamics of intelligence inherent in the absolutely infinite nature of the universe of sets. Details of this analysis will be presented in Section 6.

One aspect of the self-referral nature of the sequential process of generating sets is that the sets are conceived as lying within $V$. To recreate the holistic value of $V$ from the isolated, fragmented values of sets requires, however, a supreme step of synthesis that creates a wholeness of a fundamentally different character than the value of the parts. This irreducible connectedness of the value of wholeness of the universe of sets is the mathematical expression of the connectedness of the Samhita of Maharishi Vedic Science.

Maharishi has described the way the supreme state of enlightenment, Brahman Consciousness, is structured through a process in which all diversity is synthesized into the wholeness of the Self, the Samhita. He has further pointed out that this is the underlying theme of the Brahma Sutra:

\[
tattu samanvayat
\]

*But that itself [Brahman Consciousness] is from synthesis.*

(1.1.4)
The range of diversity being synthesized extends to the farthest reaches of space and time:

\[ \textit{dure drisham grihapatim} \]

\textit{Far in the distance is seen the owner of the house [the Self].} 
(Rik Veda, 7.1.1)

In the following sections we shall examine the sequential emergence of sets from the point value of the null set, giving rise to expanding values of infinity that express greater and greater values of synthesis. What is located “far in the distance” is the ultimate holistic value of infinity, arising through the synthesis of all possible sets to structure the transcendental wholeness of set theory, the universe of sets. This ultimate value of infinity is then found to be none other than the absolutely infinite value of the Self, located at the source of the sequential emergence of set theory.

The point is that when one considers greater and greater values of infinity, the value of synthesis becomes more and more predominant, giving rise to wholenesses expressing progressively more profound values of integration and connectedness. The ultimate expression of this theme of synthesis is the absolutely infinite wholeness of the universe of sets, and this value of wholeness reflects most completely the unbounded, connected, totally unified character of the Samhita of Maharishi Vedic Science.

Cantor, in a letter of 1908 (cited in Dauben, 1979, p. 290), viewed the absolutely infinite wholeness of the universe of sets as a fitting mathematical expression for the single, transcendental wholeness of life, beyond the grasp of the intellect.

What surpasses all that is finite and transfinite is no “Genus”; it is the single, completely individual unity in which everything is included, and which includes the “Absolute,” incomprehensible to the human understanding.

Having tentatively satisfied ourselves with the reasonableness of viewing the universe of sets as a mathematical expression of the Samhita of Maharishi Vedic Science, we now consider its three-in-one structure.
Earlier in this section we saw how the values of Rishi, Devata, and Chhandas could naturally be identified in set theory with sets (as wholes), membership relation, and sets (as points), respectively. When we identify the Samhita value with the absolutely infinite value of the universe of sets, we find two different expressions of the three-in-one structure of the universe of sets in terms of these three values:

1. When we think of the structure of $V$ as a totality of sets, related to one another by the membership relation, then the structure of $V$ is naturally appreciated in terms of sets (as wholes), membership relation, and sets (as points), corresponding to the values of Rishi, Devata, and Chhandas discussed above.

2. The universe of sets $V$, as the synthesis of all possible sets into a whole, expresses the transcendental value of the Rishi element, the value of synthesis that structures wholes. This presents a second theme of analysis in which the Rishi value is expressed by $V$, the Devata value by the membership relation, and the Chhandas value by the different sets that reflect properties of $V$. This theme of analysis brings out, through the Reflection Principle, the expression of the Devata value as self-referral.

The second theme of analysis requires some clarification, because $V$ appears as both Samhita and Rishi. Since the Reflection Principle is concerned with the conceivable properties of $V$, the role of $V$ as Rishi is to be understood in the context of the totality of its conceivable properties. These constitute what we might call the describable nature of $V$. We equate the Samhita value with the indescribable nature of $V$, the holistic value of $V$ transcending the intellect, which makes $V$ structurally undefinable. We analyzed above the way in which the Reflection Principle upholds the indescribable nature of $V$; in the present context this shows how the relation between Rishi and Chhandas upholds the transcendental value of the Samhita.

The two “natures” of $V$ have an interesting parallel in Maharishi Vedic Science. In the Bhagavad-Gita, Lord Krishna, representing the Absolute, defines his two “natures”: 

```
Earth, water, fire, wind, space, mind, intellect, and ego: this is the eightfold division of my nature.  
(Bhagavad-Gita, 7.4)

Apareyam itas tvanyam prakiritam 
viddhi me param jivabhutam

This is my lower nature.  
Know my other, higher, transcendental nature, the Self.  
(Bhagavad-Gita, 7.5)

The Samhita has thus, Maharishi explains, two aspects to its nature: its transcendental, indivisible, totally unified nature (paraprakriti) and its expressed, eight-fold, creative nature (aparaprakriti), the latter being expressed in the sequential emergence of creation from the self-interacting dynamics of the Samhita. These two natures of the Samhita correspond to the two natures of the universe of sets at the foundation of set theory.

To continue our analysis of the universe of sets, we must consider its hierarchical structure, expanding from a point to infinity. This will be the subject of the following sections.

2. The Mechanics of Set Formation
In this section we shall probe further into the nature of the universe of sets, using a viewpoint complementary to that of Section 1. This will locate the infinite dynamism of intelligence within the mathematical structure of the absolute infinite. These two set theoretic viewpoints will correspond to two complementary “viewpoints” lively within the structure of the Samhita. We shall consider this description in some detail because of its relevance to the foundations of set theory.

The two complementary viewpoints have their basis in two complementary aspects of the Samhita brought to light by Maharishi: the
value of infinity and the value of a point within that infinity. On the one hand, the Samhita has the structure of an unbounded continuum of consciousness, which expresses the value of infinity. On the other hand, the intelligent nature of this ocean of consciousness makes the discriminative value of the intellect lively within it, which then identifies a point value within that continuum, a point that expresses the extreme value of nothingness. The liveliness of these two complementary values within the structure of intelligence gives rise to two complementary viewpoints: infinity “looking” at the point, and the point “looking” at infinity.

The bidirectional relationship between infinity and the point in turn gives rise to two simultaneous dynamic processes within the structure of the Samhita: infinity collapsing to a point, and the point expanding to infinity. Maharishi has described these two simultaneous processes of contraction and expansion as giving rise to a “hum” of “infinite frequency” lying at the as-yet-unmanifest source of creation. The infinite dynamism of this pulsation between infinity and the point is the primordial expression of the self-interacting dynamics of the Samhita, from which all of creation sequentially unfolds. (We shall discuss later in this section the relationship of this description to the analysis of the Samhita presented in Section 1.)

Maharishi comments that the collapse of infinity to a point is expressed in the first syllable of Rik Veda, AK. (Note that according to rules of Sanskrit phonology AK becomes AG in the first word of Rik Veda.) This collapse Maharishi refers to as the phenomenon of Akshara. The sound A expresses the value of infinity, and the stop K expresses the value of a point. The syllable AK thus presents the self-interacting dynamics of the Samhita. From the point value of K, the whole structure of the Rik Veda sequentially unfolds. The sequential structure of the Rik Veda gives expression, on the level of sound, to the hierarchical structure of natural law sequentially unfolding from the self-interacting dynamics of the Samhita.

This sequential process continues in the emergence of creation from the structure of the laws of nature. The process of expansion culminates in the ultimate synthesis of all diversity in the infinite wholeness of the Self, expressed by A. This sequential process thus provides a second expression of the theme of expansion of the point to infinity.
Because of the way the Veda sequentially emerges from the point value of K, Maharishi has explained that one can locate the structure of the Samhita as well as its infinite self-interacting dynamics within the point, K. For this reason Maharishi has described K as the expression of the “fullness of emptiness.”

From the above description, it is apparent that the concept of “point” used in Maharishi Vedic Science in the context of the Akshara is rather different from the ordinary mathematical concept of a point as something indivisible that exists but has no structure or liveliness within it. In Maharishi Vedic Science, the point K has a rich structure, expressing a number of contrasting qualities. In Nyaya, the Vitarka Samgraha 3 enumerates four fundamental values associated with the point:

1. pradbvamsabhava: annihilation, or convergence to the point;
2. atyantabhava: emptiness or nothingness;
3. anyonyabhava: infinite dynamism;
4. pragabhava: creation, or expansion from the point.

The value of convergence (1) is expressed in the Akshara: the collapse of A to K. The value of emptiness (2) is expressed in the content of K as “fullness of emptiness.” If we think of the empty set or null set, $\emptyset = 0 = \{}$, as the expression of the extreme point value in set theory, then the content of the empty set, nothing, expresses this value of the point. The value of expansion (4) is expressed in the expansion of K to A, point to infinity, as well as the sequential emergence of the Richas (verses) of the Rik Veda from K. This theme is expressed in set theory in the sequential mechanics of generating sets from the null set, which we shall consider in detail later in this section. The value of infinite dynamism (3) is expressed in the way the total value of the Samhita is lively within K. Maharishi has described the emergence of the Richas of the Veda from the Akshara as a process that is sequential and yet simultaneous; the simultaneous emergence of the totality of the Veda from K is the expression of the infinite dynamism lively within the point.

In set theory, the infinite dynamism of the point is expressed in the emergence of all possible sets from the null set. This also is a sequential process that must be viewed as simultaneous: if it were in any sense
sequential in time, then the infinite processes required to create large infinite sets could never be completed.

It perhaps sounds strange to a mathematician to attribute infinite dynamism to the null set or, for that matter, any dynamism to the null set. If we think, however, of the null set as a structure existing in consciousness, then the null set expresses that value of consciousness in which the Rishi value is lively (it is a wholeness, a set) and yet there is no object of perception (there are no elements). This is precisely the structure of Transcendental Consciousness, in which consciousness is fully awake within itself and yet there is no object of perception. Transcendental Consciousness contains within its own structure the infinite dynamism of the Samhita at the source of the sequential emergence of the Richas of the Veda. In this sense the null set can be said to contain within its own structure the infinite dynamism of intelligence that sequentially unfolds all sets from the point value of the null set.

Before proceeding further, we must comment on the relationship of the Akshara to the three-in-one structure of the Samhita. We have described the self-interacting dynamics of the Samhita in two different ways, one in terms of the knower-known relationship in the structure of pure knowledge and the other in terms of the Akshara. What is the relationship between these two? They are to be understood as the descriptions of the self-interacting dynamics of the Samhita from two complementary intellectual viewpoints.

Maharishi has explained that in the different texts that constitute the Vedic literature, a number of contrasting intellectual approaches to reality are presented. Vedanta presents the grand synthesis of all the viewpoints in the wholeness of the Samhita. For the final awakening of the holistic value of consciousness, reality must be simultaneously appreciated from these contrasting and complementary viewpoints.

In our own analysis of the Samhita of set theory, we shall systematically develop two fundamental viewpoints of Maharishi Vedic Science: that of the three-in-one structure of pure knowledge, and that of the Akshara. We have found it necessary to develop both viewpoints in order to provide the first glimpses of a consistent and comprehensive picture of the unified totality of mathematical knowledge from the perspective of Maharishi Vedic Science.
In Section 6 we shall identify several different mathematical expressions of the *Akshara* that are intimately connected with the creative process at the source of the sequential unfoldment of mathematical knowledge. Perhaps the most fundamental of these is the Reflection Principle itself, which we shall now discuss.

The Reflection Principle expresses the viewpoint of “infinity looking at the point within it.” The universe of sets $\mathcal{V}$ is the ultimate expression of infinity in mathematics. The Reflection Principle locates within $\mathcal{V}$ specific elements, that is, specific point values, that reflect the properties of $\mathcal{V}$. This corresponds to the way $\mathcal{K}$ contains within itself the reflection of the structure of $\mathcal{A}$, the wholeness of the Samhita.

Asserting that a set reflecting properties of $\mathcal{V}$ should in any sense correspond to a point undoubtedly sounds strange, particularly to a mathematician familiar with set theory. After all, the deepest applications of the Reflection Principle are in connection with the theory of large cardinals (discussed in Section 6). In this context, the relevant sets are extremely huge infinite sets expressing much of the holistic value of $\mathcal{V}$. How can these be reasonably viewed as expressing the value of a point?

When we consider a large cardinal from the ordinary perspective of the mathematician, then it certainly appears quite huge. When we consider it, however, in relation to $\mathcal{V}$, then its hugeness shrinks to insignificance. No matter how large, any set is insignificantly small in relation to the absolutely infinite size of $\mathcal{V}$. It may reflect some very great quality of hugeness belonging to the “describable” nature of $\mathcal{V}$, but in relation to the absolutely infinite “indescribable” nature of $\mathcal{V}$, the size of any set must be minuscule. We have described how the phenomenon of *Akshara* emerges from the viewpoint of $\mathcal{A}$ looking at $\mathcal{K}$: infinity looking at the point. The relevant viewpoint is thus the viewpoint of $\mathcal{A}$, which corresponds in set theory to the viewpoint of $\mathcal{K}$. From the viewpoint of $\mathcal{V}$, any set must certainly express the point value of $\mathcal{K}$.

What happens when $\mathcal{V}$ looks at the empty set, $\emptyset$ or 0, the ultimate point value in set theory? It sees a reflection of at least some of its properties, for example, the property of being equal to itself: $\mathcal{V} = \mathcal{V}$ and $0 = 0$. We shall see in Section 6 that this reflection of $\mathcal{V}$ in the point value of 0 (in the parallel context of the ordinals) can in a sense be viewed as the primordial expression of the Reflection Principle at the source of the sequential process of creating sets.
We have examined the way the Reflection Principle expresses the collapse of infinity to a point. The converse process of the point expanding to infinity can be located in the iterative mechanics of set formation, the sequential process whereby all possible sets sequentially unfold in expanding layers from the point value of the null set. This sequential process culminates in the synthesis of all possible sets into the transcendent wholeness of the universe of sets, the mathematical expression of the fullness of A.

In the remainder of this section we shall consider the iterative process of set formation in some detail, and we shall locate within it the mathematical expression of the infinite self-referral dynamism of the Samhita. (A general discussion of the iterative concept of set is contained in Wang, 1974, pp. 181–223.)

We begin by considering the nature of the null set itself. The null set is the first expression of the synthetic functioning of intelligence that creates sets. The field of intelligence, in a self-sufficient way, synthesizes nothingness into a wholeness, a set. This is the mathematical display of the self-referral activity of the field of pure intelligence, the Samhita: functioning totally within itself, it creates something out of nothing, on the basis of its abstract conceptual ability.

Once the null set has been created as a conceptual entity, then the process of creation naturally continues with the creation of a set, 1, containing the null set, 0, as its single element; 1 = {0} = {{}}. In this we see the Rishi value of the first set being transformed into the Chhandas value of the second set. When the Rishi value of the first set, 0 = { }, becomes appreciated as an object of perception, then a new Chhandas value, 0 = { } and a new Rishi value, 1 = {0} = {{}}, emerge.

From the two sets 0 and 1, two new sets can be created: the two-element set 2 = {0, 1} = {{}, {{}}}, as well as the set {1} = {[0]} = {{{}}, yield a total of four sets. This process of creating sets can naturally be continued, without end.

The sequential continuation of this creative process, based solely on the conceptual activity of the abstract field of intelligence, gives rise to all possible sets; these in turn represent all the diverse values of mathematical structure and relationship studied in modern mathematics (see Section 7). This presents a striking parallel to the theme of the mechanics of creation in Maharishi Vedic Science, whereby all
diversity sequentially unfolds from the point value of K on the basis of the discriminative value of the intellect; all of the diversity of creation, Maharishi points out, is ultimately just an intellectual conception!

To continue the examination of the iterative process of set formation, we must examine in some detail the two fundamental concepts underlying this process: the power-set operation, expressing the dynamics of intelligence, and the ordinal numbers, expressing the value of synthesis. We consider the power-set operation in the remainder of this section and the ordinal numbers in Section 3.

The power-set of a set \( S \) is the set consisting of all possible subsets of \( S \). Fundamental to the concept of the power set is the concept of an arbitrary subset of a set \( S \). The fundamental intuition regarding this concept in modern set theory is that an arbitrary subset is combinatorially determined; that is, the formation of an arbitrary subset of \( S \) involves freely choosing, for each element of \( S \), whether or not to include it in the subset. (This is to be contrasted with the logical notion of a subclass, where the criterion for membership in the subclass must be determined by some property. For a discussion of the relationship between these two notions, see Maddy, 1983.)

We illustrate the concept of a subset by means of an example. Consider the set \( \mathbb{N} \) of natural numbers: \( \mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots \} \). The set \( \mathbb{N} \) is an example of an infinite set: there is no largest natural number. Now, a finite subset of \( \mathbb{N} \) can be described by simply listing all the elements of the subset, for example, \( \{1, 3, 8\} \). However, we cannot list all the elements of an infinite subset of \( \mathbb{N} \). To concretely describe such a subset we must take recourse to some rule giving a well-defined criterion for membership in the subset, for example, the “set of all even natural numbers,” the “set of all odd natural numbers,” or the “set of all natural numbers that are perfect squares.”

The concept of an arbitrary, combinatorially determined subset, however, is that in principle a subset can be formed by freely choosing, for each natural number, whether or not to include it in the subset, without taking recourse to any particular rule. Thus a subset would be formed, in general, on the basis of an infinite number of independent choices, one for each natural number. Furthermore, this infinity of choices must be completed to form the subset. Therefore, this infinity of choices must logically be viewed as simultaneous.
The concept of an arbitrary subset of an infinite set is therefore inseparable from the concept of an intellect capable of an infinity of simultaneous choices. This is certainly a capability transcending the localized sequential functioning of the human intellect. Yet in the continuation of our discussion of set theory we shall see how the very existence of the ascending values of infinity in set theory is fundamentally dependent on the dynamics of such a field of intelligence. This raises the question of whether such a field of intelligence exists; the answer to this question must certainly play a central role in elucidating the foundations of set theory.

Maharishi Vedic Science makes an important and relevant comment on this issue. Maharishi has identified an infinitely dynamic aspect of intellect belonging to the Samhita. This is expressed in the way the total sequential structure of the Veda emerges simultaneously from the self-interacting dynamics of the Samhita. A further expression of it is found in the way the intelligent aspect of the Samhita simultaneously organizes and manages the almost infinite diversity of creation. This holistic value of intelligence, in its role as the organizer of creation, is called by Maharishi cosmic intelligence; this level of intellect is capable, at the very least, of many simultaneous choices.

The description of cosmic intelligence in Maharishi Vedic Science does not explicitly consider the possibility of making infinitely many choices simultaneously, which is the requirement for the foundation of set theory. Nevertheless, it seems natural, and even compelling, to attribute this capability to the field of cosmic intelligence. This field of intelligence is, after all, unbounded and infinite in its character in an ultimate, absolute sense. To envision its capability as restricted to any finite expression of activity seems arbitrary and unnatural. The very ease with which the human mind can conceive of a level of intelligence capable of doing infinitely many things simultaneously, as one does all the time when working within set theory, attests to the naturalness of this concept and makes it seem dogmatic to reject it as an impossibility. In the remainder of this paper we shall therefore adopt the viewpoint that the field of cosmic intelligence is the transcendental field of intelligence at the basis of the infinitary constructions of set theory.

The most fundamental expression of this transcendental field of intelligence in the sequential mechanics of set formation is provided by
the power-set operation. The power set of a set \( S \) is the set consisting of all possible subsets of \( S \); the power-set operation is the mathematical transformation from a set \( S \) to its power set \( P(S) \). The power-set operation applied to an infinite set \( S \) expresses the simultaneous formation of all possible subsets of \( S \) and their synthesis into a wholeness. It is the power-set operation that sequentially unfolds each succeeding level of the set-theory universe in the iterative mechanics of set formation. Based upon the above analysis of the concept of an arbitrary subset, we see that this aspect of the sequential process of generating sets displays the discriminative functioning of intellect belonging to the field of cosmic intelligence.

The viewpoint that infinite sets are a reality structured on the basis of the dynamics of cosmic intelligence corresponds to Cantor’s own vision of the significance of his mathematics of the infinite. In a letter to Hermite (cited in Dauben, 1979, p. 228), Cantor remarked that the infinite totalities of set theory “exist in the highest degree of reality as eternal ideas in the Divine Intellect.” We can equate the “Divine Intellect” with that infinitely expanded value of intellect belonging to the field of cosmic intelligence, which indeed underlies the sequential mechanics of generating sets.

From the perspective of Maharishi Vedic Science, we can assert that ultimately nothing exists, except for the transcendental reality of the Self—all else is an intellectual conception. Set theory itself provides a mathematical commentary on this theme by explicitly showing how all levels of mathematical existence can be sequentially unfolded on the basis of the self-referral dynamics of the intellect. Infinite sets can be said to exist to the extent that abstract concepts are real.

There is a long-standing debate about the reality of abstract mathematical objects (see for example Wang, 1974; Benacerraf & Putnam, 1964; and Davis & Hersh, 1981). The foundational viewpoint called objectivism or Platonism holds that abstract mathematical objects (such as infinite sets) are real; this viewpoint was forcibly advocated by the great logician and mathematician Kurt Gödel, who in fact attributed all his major discoveries to his objectivist attitude (see Section 8).

Certainly, mathematical objects do not have the apparent concreteness of physical objects, which are perceived through the gross senses. Modern physics, has taught us, however, that physical reality is
in essence very different than it appears to us naively in the “classical”
world; ultimately all physical existence is just an excitation of an under-
lying self-interacting quantum field, a unified field that can be iden-
tified with the unmanifest, unified reality of pure consciousness, the
Samhita of Maharishi Vedic Science (Hagelin, 1987). Physical reality,
like mathematical reality, has its ultimate basis in the self-interacting
dynamics of cosmic intelligence; in the case of mathematics, however, it
is much simpler to trace the sequential steps through which creation of
the diverse expressions of localized values of existence takes place.

Recent developments in set theory offer additional support to the
viewpoint that there is a well-defined mathematical reality described
by the principles of mathematics. This concerns the striking way all
the important large cardinal concepts thus far formulated are linearly
ordered in terms of their strength, giving rise to a well-defined direc-
tion of expansion of mathematical knowledge towards completeness (see
Sections 6 and 11). If the axioms of set theory were truly arbitrary, one
would expect instead to find competing, mutually incompatible formu-
lations of the first principles of set theory. It appears rather that there is
an intuition of a well-defined underlying mathematical reality (the uni-
verse of sets) that is being more and more completely unfolded through
the formulation of new, more powerful axioms.

In the light of the most recent developments in modern science and
modern mathematics, together with the revival of the holistic knowl-
dge of life provided by Maharishi Vedic Science, we feel the only rea-
sonable attitude is to view abstract mathematical reality as being at least
as real as anything else, with the understanding that ultimately noth-
ing is real except for the transcendental reality of the Self. This founda-
tional viewpoint might be called Vedic objectivism.

We shall continue now with our examination of the sequential
mechanics of set formation. We have thus far identified the role of the
discriminative value of the intellect, as expressed in the power-set opera-
tion. In Maharishi Vedic Science, the discriminative value of the intel-
lect is associated with the Devata value, whose primordial expression is
found in the discrimination between the knower and the known in the
structure of the Samhita. A second essential component of the iterative
mechanics of set formation is the even more profound expression of the
value of synthesis, the Rishi element. It is certainly true that the power-
set operation already expresses a value of synthesis in that it unites all possible subsets into a single wholeness. Yet a further expression of synthesis is required, transcending the scope of the power-set operation.

We can appreciate the need for this further step of synthesis by considering the initial stages of the mechanics of generating sets from the null set. We begin with the null set, 0. This is the partial universe at stage 0, designated $V_0$. We then apply the power-set operation to obtain the partial universe at stage 1, $V_1 = P(V_0)$. Applying the power-set operation again and again we obtain $V_2 = P(V_1)$, $V_3 = P(V_2)$, and so on. However, at every stage in this sequential process we obtain only finite sets. To obtain an infinite set, this unending sequential process must itself be transcended. This step of transcendence is achieved through a step of synthesis, in which all these sequentially generated partial universes $V_n$, for $n = 0, 1, 2, \ldots$, are united into a single set, the union of the $V_n$. The resulting set is designated $V_\omega$ and is called the partial universe at stage $\omega$. Unlike the $V_n$, $V_\omega$ is an infinite set.

From $V_\omega$, the next partial universe $V_{\omega+1}$ is generated by applying once more the power-set operation: $V_{\omega+1} = P(V_\omega)$. This is the first stage at which the power-set operation is applied to an infinite set; this application of the power-set operation is the first expression of the infinite dynamism of cosmic intelligence.

The sequential process continues, applying the power-set operation over and over again: $V_{\omega+2} = P(V_{\omega+1})$, $V_{\omega+3} = P(V_{\omega+2})$, and so forth. A second step of synthesis then unites all the sequentially emerging partial universes, $V_{\omega+1}$, $V_{\omega+2}$, $V_{\omega+3}$, . . . in a single partial universe, designated $V_{\omega+\omega}$. The partial universe $V_{\omega+\omega}$ is already large enough to contain all the ordinary theories of mathematics.

Set theory does not stop here, however. The iterative mechanics continues into the unfathomable levels of the infinite; in this process more and more profound expressions of synthesis are encountered, which generate partial universes expressing more and more completely the holistic nature of the absolutely infinite reality of the universe of sets. These stages of synthesis reflect the transfinite sequential organization of the ordinal numbers, considered in the following section.
3. The Ordinals

The universe of sets is organized in hierarchical layers that sequentially unfold from the point value of the null set. The ordinal numbers provide the ladder for this sequential process. The ordinal numbers begin with the familiar natural numbers: 0, 1, 2, 3, . . . . Beyond these lie the transfinite or infinite ordinal numbers, which extend the sequential number concept to the most expanded values of the infinite. In this section we shall examine the ordinal number concept, and we shall see how the absolute ordinal Ω provides a second fundamental mathematical expression of the three-in-one structure of the Samhita.

To understand the meaning of the ordinal numbers, we must consider first the concept of a well-ordering.

If $S$ is any set, an order relation on $S$ is a relation between elements of $S$, which we shall designate by “<”, that satisfies the laws:

1. **Trichotomy**: for all $a$ and $b$ in $S$ exactly one of the following three alternatives holds: $a = b$, $a < b$, or $b < a$;
2. **Transitivity**: if $a < b$ and $b < c$ then $a < c$.

An order relation on $S$ is a well-ordering if it satisfies the additional property:

3. **Well-ordering**: every non-empty subset of $S$ has a least element, that is, if $B$ is a subset of $S$ and $B \neq \emptyset$, then there exists an element $r$ in $B$ such that $r < s$ for all other elements $s$ in $B$.

An example of a well-ordering is the familiar order relation “less than” on the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$; criterion (3) simply says that every non-empty set of natural numbers contains a smallest number. The “less than” relation on the set of integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$, however, is not a well-ordering; the subset of negative integers $\{\ldots, -3, -2, -1\}$, for example, does not contain a smallest number. Likewise, the set of positive rational numbers is not well-ordered; the subset $\{1/2, 1/3, 1/4, \ldots \}$, for example, does not contain a smallest number.

The **ordinal numbers** describe the different possible patterns of well-ordering. The finite ordinal numbers are the same as the natural num-
The ordinal number 5, for example, describes the pattern of well-ordering defined by five objects arranged in a sequence: 

\[ a < b < c < d < e. \]

Beyond the finite ordinal numbers are the infinite or transfinite ordinal numbers, which describe the different possible types of infinitely long well-orderings. The smallest of the infinite ordinal numbers is the ordinal number \( \omega \), which describes the pattern of well-ordering consisting of a single infinite sequence; \( \omega \) describes, for example, the well-ordered structure of the infinite sequence of natural numbers 0, 1, 2, 3, . . .

The ordinal numbers display a striking value of self-referral: the ordinal numbers, which describe the different possible types of well-orderings, are themselves well-ordered. This has its basis in the fact that two well-orderings can always be compared in a natural way; either they will be of the same “length,” or else one will be “longer” than the other. Thus there is a natural criterion of comparison between ordinal numbers; if \( \alpha \) and \( \beta \) are different ordinals, then either the order type \( \alpha \) is longer than the order type \( \beta \), or vice versa. In the first case we define \( \beta \) to be less than \( \alpha \), in symbols \( \beta < \alpha \), and in the second case \( \alpha \) to be less than \( \beta \), in symbols \( \alpha < \beta \). This relation between ordinal numbers is then found to satisfy all the axioms for a well-ordering.

Once we introduce the order relation between ordinals, it is found that each ordinal \( \alpha \) describes the order type of the set of all ordinals preceding \( \alpha \). For example, the ordinal 5 describes the pattern of well-ordering defined by five elements in a sequence; this is precisely the pattern of well-ordering of the set of ordinals less than 5: \{0, 1, 2, 3, 4\}. Likewise, the first infinite ordinal describes the order type of the set of all finite ordinals \{0, 1, 2, 3, . . .\}, which is precisely the set of all ordinals less than \( \omega \). The next ordinal after \( \omega \), denoted \( \omega + 1 \), describes the well-ordered structure of the set of preceding ordinals \{0, 1, 2, 3, . . . , \( \omega \)\}, which consists of a single infinite sequence, followed by one more element.

We see from this how the concept associated with a particular ordinal number is simply the “memory” of the sequential organization of the totality of the preceding ordinals. This allows us to identify the three-fold structure of Rishi, Devata, and Chhandas in the transfinite sequence of ordinal numbers as follows.
The Samhita of Sets

An ordinal, in its most abstract value, is the concept of a particular abstract pattern of well-ordering. This abstract concept, which emphasizes the value of synthesis in the structure of a well-ordering, we associate with the Rishi value. At the same time, the ordinals are point values in the well-ordered transfinite sequence of the ordinals. This point value of the ordinals we associate with the value of Chhandas. In this context, the Devata value is simply the order relation between ordinals. This is because an ordinal $\alpha$, as an abstract concept, simply presents the memory of the ordinals less than $\alpha$, as point values, in their sequential structure. Thus the ordinals preceding $\alpha$ naturally represent the objects of knowledge synthesized in the structure of knowledge presented by the ordinal $\alpha$.

Having identified the Rishi, Devata, and Chhandas values in the context of the ordinals, the Samhita value is naturally associated with the absolute ordinal $\Omega$, the ultimate ordinal number describing the well-ordered structure of the entire transfinite sequence of ordinals. The absolute ordinal $\Omega$ represents a single point value following the whole sequence of ordinals: for every ordinal $\alpha$, we have $\alpha < \Omega$. The absolute ordinal $\Omega$ is thus not an “ordinary” ordinal. If it were, we would have the impossible relationship $\Omega < \Omega$. The relationship $\Omega < \Omega$ should be satisfied by the naive self-referral concept of “the ordinal describing the well-ordering of all the ordinals (including itself),” just as the relationship $V \in V$ should be satisfied by the naive self-referral concept of a “set of all sets (including itself).” But both of these naive self-referral concepts lead to logical paradoxes. Just as the totality of all sets structures a transcendental wholeness that can no longer be regarded as a set, the well-ordered sequence of the totality of the ordinals defines an absolute ordinal that transcends the conceptual boundaries of the ordinary concept of an ordinal number.

The transcendental, self-referral nature of the universe of sets is expressed by the Reflection Principle. The Reflection Principle likewise applies to the absolute ordinal $\Omega$: any conceivable property of $\Omega$ must be reflected in some ordinal $\alpha < \Omega$. This establishes the self-referral nature of $\Omega$, as well as its status transcending the intellect.

The analysis of the universe of sets in terms of sets, membership relation, and sets, considered in Section 1, thus has a direct correspondence in the analysis of the absolute infinite ordinal $\Omega$ in terms of ordinals,
order relation, and ordinals. In modern set theory, in fact, the ordinals are identified with specific sets, in such a way that each ordinal is simply the set of all preceding ordinals: $0 = \emptyset = \text{the empty set}$, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, \ldots, $\omega = \{0, 1, 2, \ldots\}$, and so on. With the ordinals thus represented by sets, the order relation between ordinals becomes identical to the membership relation between the corresponding sets: $\alpha < \beta$ if and only if $\alpha \in \beta$. There is thus a profound intimacy between these two mathematical expressions of the three-in-one structure of the absolute infinite.

There is one striking difference, however: the ordinals present a value of perfect coherence lacking in the universe of sets. Any two ordinals will begin in exactly the same way and will differ only in their length. If the length of $\beta$ exceeds the length of $\alpha$ we will have $\alpha < \beta$ and also $\alpha \in \beta$. For any two ordinals, one will always be just an initial segment of the other, one of the Chhandas values that the other ordinal comprehends in its value as Rishi. This perfect coherence between the ordinals is not available between sets in general.

The apparent lack of coherence within the universe of sets is the result of the diversifying value of the power-set operation. The infinitely diverse structural values contained within the universe of sets sequentially emerge through the interplay of the perfectly coherent self-referral structure of the ordinals and the diversifying tendency of the power-set operation, an expression of the discriminative functioning of the intellect. The interplay of these two values gives rise to the sequential unfoldment of the hierarchical levels of the set theory universe through the self-referral mechanics of transfinite recursion; this we shall consider in the following section.

4. Recursive Generation of Sets
In this section we shall complete the description, begun in Section 2, of the iterative mechanics of set formation on the basis of the well-ordered transfinite sequence of ordinal numbers; this will identify a third fundamental expression of the three-in-one structure of Rishi, Devata, and Chhandas at the foundation of set theory.

We examined in Section 2 the initial stages of the iterative mechanics of set formation. Through the repeated application of the power-set operation, starting from the null set, the partial universes
$V_0$, $V_1$, $V_2$, \ldots were sequentially unfolded. This infinite sequence of partial universes was then synthesized into the partial universe $V_\omega$. Again sequentially applying the power-set operation, the partial universes $V_{\omega+1}$, $V_{\omega+2}$, \ldots were unfolded, whose synthesis gave rise to the partial universe $V_{\omega+\omega}$. The subscripts labeling these partial universes, 0, 1, 2, \ldots, $\omega$, $\omega+1$, $\omega+2$, \ldots, $\omega+\omega$ are just the ordinal numbers up to $\omega+\omega$. In general, the ordinal numbers provide the ladder for the iterative mechanics of set formation; for each ordinal number $\alpha$, there is a partial universe $V_\alpha$ created at stage $\alpha$. The rules governing the dynamics of this iterative process are extremely simple; there are just two rules, based upon the distinction between a successor ordinal and a limit ordinal.

An ordinal that has an immediate predecessor is called a successor ordinal. A successor ordinal $\alpha$ is the first ordinal after some other ordinal $\beta$; we write $\alpha = \beta + 1$. For example, 3 is a successor ordinal because it has the immediate predecessor 2, and we write $3 = 2 + 1$. The ordinal $\omega$, however, is not a successor ordinal. We can see this as follows: if $\alpha$ is any ordinal less than $\omega$, then $\alpha$ must be a natural number, because $\omega$ is by definition the first ordinal greater than all the natural numbers. But if $\alpha$ is a natural number, then $\alpha + 1$ also will be a natural number, and therefore $\omega$ cannot equal $\alpha + 1$. Thus $\omega$ is not a successor ordinal.

An ordinal that is not a successor is called a limit ordinal. The ordinal $\omega$ is an example of a limit ordinal; likewise $\omega + \omega$ is a limit ordinal.

We can now state the two rules governing the iterative mechanics of set formation:

1. If $\alpha$ is a successor ordinal and $\alpha = \beta + 1$, then $V_\alpha = P(V_\beta)$.

2. If $\alpha$ is a limit ordinal then $V_\alpha = \bigcup\{V_\beta : \beta < \alpha\}$.

Rule (1) says that if $\alpha$ is a successor ordinal, then $V_\alpha$ is obtained by forming the power set of the partial universe $V_\beta$ corresponding to the immediate predecessor $\beta$ of $\alpha$. Rule (2) says that if $\alpha$ is a limit ordinal, then the partial universe $V_\alpha$ is obtained by forming the union of the partial universes $V_\beta$ obtained at all stages $\beta$ preceding $\alpha$, that is, $V_\alpha$ in this case is defined to be the synthesis of all the partial universes obtained at earlier stages.
The two rules given above define the partial universe $V_\alpha$ in terms of the “memory” of the partial universes $V_\beta$ for the ordinals $\beta$ preceding $\alpha$. In the case of a successor ordinal all that is required is the memory of the immediately preceding stage; in the case of a limit ordinal, the total memory of all preceding stages is required. The two rules together describe the complete mechanics of transformation of the memory of the preceding stages into the present stage.

It can be shown that the above two rules determine a unique partial universe $V_\alpha$ for each ordinal number $\alpha$. This is a consequence of the well-ordered nature of the transfinite sequence of ordinals. If the rules did not determine a unique partial universe for each ordinal, then there would have to be a first ordinal $\alpha$ for which they failed; but this means that $V_\beta$ would be well-defined for each $\beta < \alpha$ and on this basis rules (1) and (2) would then unambiguously determine $V_\alpha$.

Rules (1) and (2) together provide the complete definition of the concept of a “partial universe.” The appearance of partial universes on the right-hand sides of (1) and (2) gives this definition a self-referral character: the concept of a partial universe is defined in terms of the concept of a partial universe. The defining formulas (1) and (2) are said to be recursive—they “curve back” on themselves. It is because of this self-referral nature of the formulas that these two compact expressions can be expanded to sequentially unfold the totality of partial universes that collectively structure the absolutely infinite wholeness of the universe of sets.

The sequential generation of sets by recursion exemplifies the theme of sequential creation described by Lord Krishna in the Bhagavad-Gita.

\[ \text{prakritim svam avashtabhyas visrjami punah punah} \]
\[ \text{Curving back on myself, I create again and again.} \]

(9.8)

Maharishi comments that in the sequential emergence of the Richas of the Veda from the Akshara, each successive stage of transformation is upheld by the self-interacting dynamics of the Samhita, infinitely lively within the Self, which unfolds each successive expression on the basis of the “memory” of what has come before.
When we examine the iterative mechanics of set formation, three different expressions of this self-referral theme in Maharishi Vedic Science can be located:

1. The recursive nature of the basic rules governing the process, whereby the process refers to itself.
2. The infinite dynamism of cosmic intelligence invoked at each stage of application of the power-set operation, whereby each new step makes reference to the dynamics of intelligence at the level of the Samhita, the level of the Self.
3. The continual reference to the ordinals, which provide the background for the process, whereby the process refers to the absolutely infinite structure of the sequence of ordinals, a mathematical expression of the absolutely infinite reality of the Self.

We can now identify a third fundamental expression of the values of Rishi, Devata, and Chhandas at the foundation of set theory, in which the Rishi value corresponds to the ordinals, the Devata value the power-set operation, and the Chhandas value the partial universes $V_a$. The Samhita corresponds to the common absolutely infinite value of $\Omega$ and $V$. (We shall justify the equality of the infinite “sizes” of $\Omega$ and $V$ in Section 5.)

*The ordinals as Rishi.* The ordinals provide the background for the sequential process of generating sets and as such express another primary value of the Rishi, the witnessing value. “Witnessing” refers to the role of Rishi as the silent, uninvolved background onto which the content of experience is projected.

As analyzed above, the iterative mechanics unfold each level on the basis of the memory of what has come before; the concept associated with each ordinal is simply the memory of the sequential organization of the preceding ordinals. The ordinals thus express a fine aspect of the value of memory: the sequential organization of memory independent of the content of memory. This fine aspect of memory is naturally associated with the witnessing value of the Rishi.

The limit ordinals furthermore express the values of synthesis required to create partial universes reflecting more and more completely the
absolutely infinite wholeness of the universe of sets. We shall discuss this theme in Section 6 in connection with the theory of large cardinals. The expression of synthesis in the transfinite sequence of the ordinals also is naturally associated with the Rishi element.

The power-set operation as Devata. We discussed in Section 2 the way in which the power-set operation presents the dynamics of cosmic intelligence, a field of intelligence capable of an infinity of simultaneous choices. This dynamics of cosmic intelligence, which is applied over and over again in the iterative process of generating sets, naturally expresses the Devata value.

The partial universes as Chhandas. The partial universes themselves present all possible sets, all the different possible objects of mathematical knowledge studied in the diverse theories of modern mathematics. These naturally express the Chhandas value.

The absolute infinite as Samhita. The Samhita value is again the absolute infinite, expressed as the transcendental value of the Rishi element in the absolute ordinal $\Omega$, and as the transcendental value of the Chhandas element in the universe of sets $V$. The relationship between these two expressions of absolute infinity, in the context of the iterative mechanics of set formation, is simply: $V = V_\Omega$. The “partial universe” at stage $\Omega$ is the synthesis of all the partial universes at stages $\alpha < \Omega$ and as such is simply the universe of sets itself.

The relation $V = V_\Omega$ is not yet enough to equate the “absolutely infinite” value of $V$ with the “absolutely infinite” value of $\Omega$. It reveals how $V$ is created on the basis of $\Omega$, but it is conceivable that the infinite “size” of $V$ could be greater than the “length” of the infinite sequence of ordinals. The diversifying value of the power-set operation could give rise to a greater value of infinity. To be able to precisely compare these two mathematical expressions of absolute infinity, it is necessary to consider a second fundamental mathematical expression of the infinite dynamics of cosmic intelligence, complementing the diversifying value of the power-set operation. This is the Axiom of Choice, which can transform the “disorder” implicit in the structure of arbitrary sets into
the perfectly orderly, perfectly coherent structure of the ordinals. This we shall consider in the following section.

5. The Axiom of Choice

We have examined the way in which the power-set operation presents the dynamics of cosmic intelligence, a field of intelligence capable of an infinity of simultaneous choices. This dynamics of intelligence, based upon the discriminative value of the intellect, is responsible for the creation of the infinitely diverse expressions of mathematical reality from the perfectly coherent structure of the ordinal numbers.

At the foundation of set theory is a second, complementary expression of the dynamics of cosmic intelligence that has the ability to well-order any set. This is the Axiom of Choice. The Axiom of Choice can be applied to give any set the perfectly orderly structure of an ordinal number. We shall see in this section that the Axiom of Choice, by establishing \( \Omega \) to be the “measure” of \( V \), reveals \( \Omega \) and \( V \) to be two expressions of a common absolutely infinite value, the infinite value of the Samhita of mathematics. We shall begin by considering the formulation of the Axiom of Choice.

The Axiom of Choice describes the formation of a set on the basis of an infinity of simultaneous choices. It was first explicitly introduced as a principle of set theory by Zermelo in 1904. The precise formulation of the Axiom of Choice is the following:

*Axiom of Choice.* Let \( S \) be a set, each of whose elements is a non-empty set, and such that the elements of \( S \) are pairwise disjoint, that is, no two elements of \( S \) have any elements in common. Then there exists a set \( C \) containing exactly one element of every element of \( S \).

The idea of the Axiom of Choice is that if we are given a collection of sets \( S \), then we can form a set \( C \) by “choosing” one element from each set in the collection. In case the original collection of sets \( S \) is infinite, then the formation of the set \( C \) typically involves an infinity of choices, which must be completed to form the set \( C \). The Axiom of Choice thereby directly describes the dynamics of a level of intelligence capable of an infinity of simultaneous choices.

The profundity of the Axiom of Choice is demonstrated by its consequences. One such consequence is the Banach-Tarski paradox. This
asserts that a sphere can be decomposed into four pieces, which can then be reassembled to form two spheres identical to the original, and this process involves no stretching or distortion of the pieces. These pieces are “unmeasurable” sets, which cannot be visualized, but they can be created on the basis of the Axiom of Choice. It is known that in principle such unmeasurable sets can never be created by “constructive” methods, those methods grounded in the localized, sequential functioning of the human intellect.

The Axiom of Choice underlies extremely powerful “nonconstructive” methods that are prominent in almost all areas of modern mathematics; from the perspective of this paper these nonconstructive methods are directly grounded in the infinite dynamism of cosmic intelligence. These nonconstructive methods bring to light mathematical possibilities that are “impossible” from the localized, sequential perspective of constructive methods.

One very important consequence of the Axiom of Choice is the well-ordering theorem (Zermelo, 1904): the Axiom of Choice can be applied to arrange the elements of any set in a well-ordered sequence. This makes it possible to use a subclass of the ordinals, the cardinal numbers, to measure the sizes of sets.

The basic idea is simple: once the elements of a set are well-ordered, then the pattern of well-ordering will be described by an ordinal number $\alpha$, which can then be used to represent the size of the original set. There is one complication however: if the original set is infinite, then there will be different possible ways to well-order its elements, giving rise to well-ordered sequences of different lengths, which will correspond to different ordinals. To measure the size of the set in an unambiguous way, we therefore use the ordinal describing the shortest possible well-ordering of the set. The ordinals that arise in this way are called the cardinal numbers.

The cardinals are a subclass of the ordinals. The finite cardinals are the same as the finite ordinals: $0, 1, 2, 3, \ldots$. The first infinite cardinal is $\omega$, the first infinite ordinal. There is a big gap after $\omega$, however, until one reaches the second infinite cardinal, $\omega_1$, which is much greater than $\omega + \omega$, for example.
The Axiom of Choice in this way establishes that the ordinals not only represent all possible types of well-ordering but also contain all possible infinite sizes of sets.

The concept of the infinite cardinals was introduced by Georg Cantor, the founder of set theory. Cantor used the Hebrew letter \( \aleph \) (aleph) to represent the infinite cardinal numbers: \( \aleph_0 \) represents the smallest infinite cardinal, \( \aleph_1 \) represents the second infinite cardinal, and so on. Cantor showed that the power-set operation has the following remarkable property: for any set \( S \), the power-set \( P(S) \) always has greater cardinality than \( S \). In symbols, \( |P(S)| > |S| \). (The notation \( |S| \) is used to designate the cardinality of the set \( S \), that is, the cardinal number measuring the size of the set \( S \).)

Since the set of natural numbers has cardinality \( \aleph_0 \), in symbols \( |N| = \aleph_0 \), this showed that the power set \( P(N) \) has cardinality greater than \( \aleph_0 \), in symbols \( |P(N)| > \aleph_0 \); \( P(N) \) is called an uncountably infinite set. This discovery highlighted the profundity of the dynamics of cosmic intelligence expressed in the power-set operation, which could create from one infinity a greater infinity.

How much greater is the infinite size of \( P(N) \) than that of \( N \)? Cantor conjectured that it should be the next biggest infinite cardinality, \( \aleph_1 \). This famous conjecture of Cantor’s that \( |P(N)| = \aleph_1 \), is called the Continuum Hypothesis. The name derives from the fact that the power set \( P(N) \) can be shown to have the same cardinality as the set of points on a continuous line (see Section 7). However, neither Cantor nor anyone since has been able to either prove or refute the continuum hypothesis, and the precise cardinality of \( P(N) \) is still unknown. It has been shown that this question cannot be resolved on the basis of the standard Zermelo-Fraenkel axioms of set theory; recent developments in the theory of large cardinals indicate, however, that a final solution may be on the horizon, based upon the developing knowledge of the most unbounded and holistic expressions of the infinite in set theory (see Section 11).

The cardinal numbers provide a way to systematically measure the “sizes” of the partial universes that sequentially unfold through the iterative mechanics of set formation. For any partial universe \( V_\alpha \), there will be a unique cardinal \( \beta \) measuring its size: \( |V_\alpha| = \beta \). What happens when we consider the universe of sets \( V \) itself? If one assumes a powerful form
of the Axiom of Choice, called the global Axiom of Choice, then it follows that the cardinality of the universe of sets is $\Omega$: $|\mathcal{V}| = \Omega$. This means that the absolute ordinal $\Omega$ is at the same time the absolute cardinal measuring the totality of all possible sets.

The global Axiom of Choice simultaneously chooses an element of all possible sets in the set theory universe. This axiom can be applied to well-order, in one stroke, the entire universe of sets, and on this basis establish the absolute infinite ordinal $\Omega$ as the measure of the totality of all possible sets. Thus the ordering ability of cosmic intelligence expressed in the global Axiom of Choice establishes the “equality” of the absolutely infinite value of $\Omega$, the "absolute number," with the absolutely infinite value of $\mathcal{V}$, the “totality of everything.” This common value of absolute infinity we identify as the Samhita value of set theory and take this to be the mathematical expression of the holistic value of the Samhita of Maharishi Vedic Science.

In terms of our identification of Rishi, Devata, and Chhandas with the ordinals, the power-set operation, and partial universes, the Devata value was identified with the power-set operation. It transforms Rishi into Chhandas, that is, ordinals into partial universes. The Axiom of Choice does the opposite. Through its expression in the well-ordering theorem, it transforms Chhandas into Rishi, that is, sets into ordinals. This allows one to view sets as ultimately equivalent to ordinals, and thereby brings out that the Rishi value is everywhere present in the diversified field of the Chhandas: all values of mathematical structure can be viewed as just relationships between ordinals. The power-set operation brings forth the expression of all mathematical possibilities from the ordinals; the Axiom of Choice then locates the totality of these possibilities within the structure of the ordinals themselves. The two expressions of the dynamics of cosmic intelligence presented by the power-set operation and the Axiom of Choice together uphold the Samhita value of the absolute infinite, expressed both in the absolute ordinal, $\Omega$, the transcendent value of the Rishi element, and the universe of sets, $\mathcal{V}$, the transcendent value of the Chhandas element.

In our preceding discussions we have identified three different mathematical expressions of the three-in-one structure of the absolute infinite. How can we account for the non-uniqueness of this three-in-one structure at the foundation of set theory? We should first observe
that in relating Maharishi Vedic Science to the different academic disciplines different expressions of the three-in-one structure of the Samhita are found at the basis of the different disciplines, corresponding to the different avenues of intellectual inquiry reflected in these disciplines. The universality of the three-in-one structure of the Samhita just means that it should be locatable through any channel of systematic inquiry into the foundation of a particular area of knowledge. This applies not only to the different academic disciplines, but also to the different viewpoints present in any single discipline and within any particular theory of the discipline. Thus one may legitimately locate several different equally fundamental expressions of the three-in-one structure of the Samhita.

When we consider the specific relationship between the three different three-in-one structures discussed above, we find a striking parallel to Maharishi’s description of the emergence of the Sama, Yajur, and Atharva Samhitas from the Rik Samhita. This we shall now consider.

We discussed in Section 2 the way the syllable AK gives expression to the self-interacting dynamics of the Samhita in terms of the phenomenon of Akshara: collapse of infinity to a point. This is the first syllable of Rik Veda. Maharishi has explained that the sequential flow of the Richas, or verses, of Rik Veda is just the expression, on the level of sound, of the hierarchical structure of the laws of nature sequentially unfolding from the self-interacting dynamics of the Samhita, the unified field of natural law. Maharishi has explained further that every aspect of the Vedic literature emerges, in a systematic way, from the internal dynamics of the Samhita. Maharishi has recently elaborated the detailed mechanics of transformation that sequentially give rise to each of the specific parts of the Vedic literature, and he has presented it in the form of a play called the Veda Lila (play of the Veda).

The theme of the Veda Lila is that the totality of the Vedic literature emerges from the Samhita according to the principle:

\[ Vritti sarupya itaratra \]

\[ What you see, you become. \]

(\textit{Yoga Sutra} 1.4)
Maharishi has explained that this principle describes the dynamics of consciousness involved in every level of experience. In the process of experience, the awareness of the knower becomes identified with the object of knowledge; it is this transformation of the value of the knower into the value of the known that structures the experience.

Maharishi has further explained that the most fundamental expression of this principle is found in the internal dynamics of the Samhita. Because Rishi, Devata, and Chhandas are ultimately nothing other than pure consciousness, each can separately assume the role of knower, or “seer.” This gives rise to three different values of relationship between Rishi, Devata, and Chhandas and the Samhita: Rishi “seeing” Samhita, Devata seeing Samhita, and Chhandas seeing Samhita. Each of these three relationships gives rise to its own characteristic mechanics of transformation; from these transformations the Sama, Yajur, and Atharva Vedas respectively emerge. For example, when Rishi sees Samhita, Rishi becomes Samhita, and in the process of transformation the Sama Veda emerges.

The text of Rik Veda is also called the Rik Samhita; this is because it is the direct expression of the self-referral structure of natural law within the Samhita. Likewise, the texts of Sama, Yajur, and Atharva Vedas are called the Sama, Yajur, and Atharva Samhitas. These three Vedas also give expression to the self-referral structure of the Samhita but from three different viewpoints: the viewpoints of Rishi, Devata, and Chhandas, respectively, seeing the Samhita. These three viewpoints give rise to three values of transformation: Rishi becoming Samhita, Devata becoming Samhita, and Chhandas becoming Samhita.

The relationship between the Rik, Sama, and Atharva Samhitas portrayed in the Veda Lila corresponds precisely to the relationship between the three different three-in-one structures of the absolute infinite we have described in this paper. Consider the identification of Rishi, Devata, and Chhandas with ordinals, power-set operation, and sets. We saw that \( \Omega \) represents the transcendental value of the Rishi element (the absolute ordinal) and that \( V \) represents the transcendental value of the Chhandas element (the absolutely infinite set). The triple (ordinals, order relation, ordinals) thus presents the three-in-one structure of the absolute infinite from the point of view of Rishi, and the triple (sets, membership relation, sets) presents the three-in-one structure
of the absolute infinite from the point of view of Chhandas. If the triple (ordinals, power-set operation, sets) is taken to correspond to the three-in-one structure of the Rik Samhita, then the triple (ordinals, order relation, ordinals) should correspond to the three-in-one structure of the Sama Samhita, since this three-in-one structure emerges when the Rik Samhita, the absolute infinite, is seen from the viewpoint of Rishi. The Samhita value of Sama is then the absolute ordinal Ω; Ω presents the expansion of the ordinal number concept to embrace the absolute infinite, and as such represents the transformation of Rishi (ordinals) into Samhita (absolute infinite). Similarly, the triple (sets, membership relation, sets) should correspond to the three-in-one structure of the Atharva Samhita, with the Samhita value corresponding to the universe of sets, \( V \). The Samhita value of Rik is then the common absolutely infinite value of Ω and \( V \).

The three different three-in-one structures of the absolute infinite that we have described thus correspond naturally to the three-in-one structures of the Rik, Sama, and Atharva Samhitas, as described in Maharishi Vedic Science. The determination of the three-in-one structure of the absolute infinite corresponding to the Yajur Samhita remains an open question. One possibility is to identify the Yajur Samhita with the global Axiom of Choice; this axiom defines a transformation from the Sama Samhita, Ω, to the Atharva Samhita, \( V \), that establishes a one-to-one correspondence between these two expressions of the absolute infinite. It is not at all clear, however, how to find an appropriate three-in-one structure consistent with our identification of Devata as the power-set operation. The author feels strongly that it should be possible to find a natural and compelling solution to this puzzle and welcomes any suggestions in this direction.

**Part II: The Structure of Mathematical Knowledge**

**6. Axiomatic Set Theory**

In this section we shall examine the nature of the creative process at the source of the sequential unfoldment of the knowledge of set theory; we shall relate this creative process to Maharishi’s description of the creative process at the source of the sequential unfoldment of the Richas of the Veda.
We saw in Section 2 the way in which the first syllable of Rik Veda, AK, expresses the self-interacting dynamics of the Samhita at the source of the sequential emergence of the Richas of the Veda, which present the hierarchical structure of the laws of nature. Maharishi has emphasized that the structure of natural law expressed in the Richas of the Veda is something eternal and non-changing. This structure of natural law belongs to the transcendental field of the Samhita and can be directly cognized there when awareness is firmly grounded in the experience of Transcendental Consciousness.

The Samhita is the field of pure knowledge. The Richas of the Veda, which are lively there, naturally contain the three-fold structure of knowledge: each Richa contains its own value of Rishi, Devata, and Chhandas. Maharishi refers to the Richas of the Veda as the expressions of pure knowledge.

The knowledge contained in the Richas of the Veda is the knowledge of the laws of nature they represent. This level of knowledge is something much deeper than the ordinary level of intellectual understanding of natural law: it is a self-referral level of knowledge in which the laws of nature know themselves.

When one considers the Richas of the Veda as expressions of knowledge, the sequential progression of the Richas presents a sequential unfolding of knowledge described by Maharishi as the Apaurush-eya Bhashya (uncreated commentary) of the Rik Veda. According to Maharishi’s Apaurusheya Bhashya, the first syllable AK of the Rik Veda is the expression of total knowledge in its most compact form. The rest of the Rik Veda, and in fact all of the Vedic literature, then provides a sequence of packages of knowledge, each elaborating and commenting on the previous package.

The principle that the first syllable AK contains the totality of knowledge is expressed in the following verse of Rik Veda, as analyzed by Maharishi:

\[
\text{Richo Akshare parame vyoman}
\]
\[
\text{yasmin Deva adhi vishve nishedub}
\]
\[
\text{yastanna veda kim richa karishyati}
\]
\[
\text{ya ittadvidus ta ime samasate}
\]
The verses of the Veda exist in the collapse of fullness (the kshara of A) in the transcendental field, in which reside all the Devas, the impulses of Creative Intelligence, the Laws of Nature responsible for the whole manifest universe. He whose awareness is not open to this field, what can the verses accomplish for him? Those who know this level of reality are established in evenness, wholeness of life.

(1.164.39)

We shall now see that the theme of the Apaurusheya Bhashya is precisely the theme of sequential unfoldment of knowledge found in the axiomatic structure of modern mathematics.

Modern mathematics is structured in the form of axiomatic theories. An axiomatic theory is structured according to the theme of sequential unfoldment of knowledge. One begins with a collection of principles called the axioms of the theory. The axioms present the total knowledge of the theory in its most compact form. From the axioms one then sequentially unfolds the theorems of the theory on the basis of the invariable principles of logical inference.

Because of the universal validity of the principles of logical inference, the correctness of each theorem unfolding in sequence is assured, provided the axioms are themselves valid. The theorems of the theory provide a sequential elaboration and commentary on the totality of knowledge contained, in seed form, in the axioms; this precisely parallels the theme of Maharishi’s Apaurusheya Bhashya of Rik Veda, whereby the Richas of the Veda, the expressions of pure knowledge, provide a sequential elaboration and commentary on the totality of knowledge contained, in seed form, in the first syllable of Rik Veda, AK.

The diverse axiomatic theories of modern mathematics have a unified foundation in axiomatic set theory. Set theory was first axiomatized by Zermelo in 1908. The standard formulation of axiomatic set theory today uses an extension of Zermelo’s axioms called the Zermelo-Fraenkel axioms; from these axioms, all the theorems of axiomatic set theory sequentially unfold. These theorems in turn provide the set-theoretic foundation for modern mathematics. The Zermelo-Fraenkel
axioms thereby contain, in seed form, the totality of almost all known mathematics. (The exception consists of those mathematical developments directly grounded in the theory of large cardinals, discussed later.)

In this section we shall examine the creative process at the source of the axioms of set theory, from which the diverse principles of mathematics sequentially unfold. We shall see the relevance of fundamental principles of Maharishi Vedic Science in elucidating this creative process. We shall see in particular the central role played by the phenomenon of Akshara, collapse of infinity to a point, which is the expression of the creative process at the source of the sequential emergence of the Richas of the Veda. We shall begin by examining the nine Zermelo-Fraenkel axioms.

1. **Axiom of the Null Set.** There exists a set containing no elements.
2. **Axiom of Extensionality.** Two sets that contain precisely the same elements must be equal.
3. **Axiom of Pairs.** For any sets, A and B, one can form the set \{A, B\} containing precisely A and B as its elements.
4. **Axiom of Unions.** For any set A, one can form the union set \(\cup A\) consisting of all elements of elements of A.
5. **Power-Set Axiom.** For any set A, there exists a power set \(P(A)\) consisting of all subsets of A.
6. **Axiom of Infinity.** There exists an infinite set containing all the natural numbers: 0, 1, 2, 3, . . .
7. **Axiom of Replacement.** If we have a set S, and we replace each element of S by a new element according to some rule, the totality of the new elements will themselves form a set. (In the precise formulation of Zermelo-Fraenkel set theory, this principle of replacement is expressed not by a single axiom but by an axiom scheme consisting of an infinite number of axioms, one for each possible rule for replacing the elements.)
8. **Axiom of Foundation (Axiom of Regularity).** There can be no infinite sequence of sets \(A_1, A_2, A_3, A_4, \ldots\) such that each is an element of the preceding set: \(\ldots \in A_4 \in A_3 \in A_2 \in A_1\).
9. **Axiom of Choice.** Let S be a set whose elements are non-empty and pairwise disjoint. Then there exists a set T containing exactly one element of each element of S.
The axiom scheme of separation is sometimes included among the Zermelo-Fraenkel axioms: for any set $S$ and any property $P$, there exists a set $T$ consisting precisely of those elements of $S$ having property $P$. This axiom can be shown to be a consequence of the axiom of replacement and therefore need not be taken as fundamental.

In the context of the set-theoretic foundation, all the principles of modern mathematics are validated on the basis of the axioms of set theory. On what basis are the axioms themselves validated? The axioms have their origin in the creative process at the foundation of set theory. According to the viewpoint of “Vedic objectivism” that we have adopted in this paper, the ultimate validation of the axioms is by the subjective faculty called mathematical intuition, which directly recognizes the correctness of certain fundamental principles governing the iterative concept of a set.

In this section we shall analyze several aspects of mathematical intuition at the basis of the set-theory axioms. This will explain the relevance of the principles of Maharishi Vedic Science in describing the creative process underlying the axioms. In a sense, this will provide a “justification” of the axioms in terms of more fundamental principles of Maharishi Vedic Science. It should be borne in mind, however, that in the axiomatic development of set theory the axioms themselves are never “proved” but rather taken as the starting point for the logical development of the theory.

A number of different aspects of mathematical intuition underlie the Zermelo-Fraenkel axioms. The deepest are those regarding the nature of the infinite, which we shall now discuss.

Fundamental to both the Power-Set Axiom and the Axiom of Choice is the intuition of an arbitrary, combinatorially determined subset of an infinite set. The Axiom of Choice creates a subset on the basis of an infinite number of independent choices; the power-set operation synthesizes into a single wholeness all possible subsets of an infinite set. Both describe the dynamics of a field of intelligence capable of performing an infinite number of independent, simultaneous choices.

One aspect of intuition involved in these two axioms is that of a level of intellect capable of an infinite number of independent choices. We have seen how this level of intellect is associated with the field of cosmic intelligence in Maharishi Vedic Science. This mathematical intuition of
the nature of cosmic intelligence is therefore intimately connected with both the Axiom of Choice and the Power-Set Axiom.

We inquire further: What is the intuitive justification for the step of synthesis involved in the formation of the power set of an infinite set? Suppose we start with the infinite set $\mathbb{N}$ of natural numbers. This set, although infinite, is nevertheless rather concrete; we can for example systematically assign finite symbolic names to all its elements, such as is provided by ordinary decimal notation, for example, the numerals 5 and 138. However, the power set $P(\mathbb{N})$ is uncountably infinite, which implies that there is no way to systematically assign finite symbolic names to all of its elements. It seems that the power set $P(\mathbb{N})$ cannot be grasped as a whole in the same concrete way that $\mathbb{N}$ can be, and therefore some justification for the step of synthesis that creates the set $P(\mathbb{N})$ is required.

This justification is provided by the theory of the continuum. The conceptual analysis of a continuous line in terms of discrete points leads to a natural correspondence between all possible points on a line and all possible subsets of $\mathbb{N}$. As these points are synthesized into a single wholeness in the concept of the continuous line, the synthesis of the corresponding subsets of $\mathbb{N}$ into a single wholeness is thereby justified.

The identification of the points on a line with the subsets of $\mathbb{N}$ is based upon the principle of nested intervals, which is itself derived from the conceptual analysis of the sequential collapse of the continuum to its own points (see Section 7). Thus this mathematical expression of the phenomenon of Akshara underlies the step of synthesis involved in the formation of the power set $P(\mathbb{N})$. Furthermore, it is seen to be intimately connected to the intuition of an arbitrary, combinatorially determined subset of $\mathbb{N}$ through the identification of points on a line with infinite sequences of choices (for example, infinite sequences of 0s and 1s), which create a corresponding subset of $\mathbb{N}$. This in turn is intimately connected with the countable Axiom of Choice, which describes the formation of a set on the basis of a countably infinite sequence of choices.

We can say therefore that the analysis of the phenomenon of Akshara, in the context of the sequential collapse of a line to a point, lies at the foundation of the concept of the power set $P(\mathbb{N})$ and also is intimately related to the countable Axiom of Choice. The power-set operation
applied to an arbitrary set, as well as the general form of the Axiom of Choice, are then “justified” on the basis of the belief that these operations should be generalizable to sets of any infinite size. This step of generalization gives mathematical expression to the dynamics of a level of intelligence capable of doing uncountably many things simultaneously.

On the other hand, if one wants to maintain the reasonably concrete connection with the sequential functioning of human intelligence, one might wish to take a more “constructive” approach to the development of set theory beyond the level of $P(\mathbb{N})$. Such an approach is provided, for example, by the development of mathematics in the “Axiom of Determinacy” world $L(\mathbb{R})$; here things can be quite concretely managed from the level of human intelligence, based on the way everything sequentially unfolds, in a constructive way, from the set of real numbers $\mathbb{R} = P(\mathbb{N})$. Of course, the fundamental laws governing $L(\mathbb{R})$ are themselves ultimately derived from the knowledge of the holistic non-constructive “Axiom of Choice” world, $V$. We shall discuss the relationship of $L(\mathbb{R})$ to $V$ in the context of Maharishi Vedic Science in Section 10.

We consider next the Axiom of Foundation. The Axiom of Foundation asserts that there are no infinite descending chains $\ldots \in A_4 \in A_3 \in A_2 \in A_1$. This means that every descending chain must terminate in the null set after a finite number of steps: if one never reached the null set, then one could continue on and on, giving rise to an infinite descending chain. We could just as well start with $V$; since every element of $V$ is an ordinary set, every chain starting from $V$ must terminate in the null set after a finite number of steps.

This means that if we start with $V$, the expression of absolute infinity, and allow attention to sequentially converge inwards to experience elements of elements of elements of $\ldots$, following any possible path, then this process always terminates in the point value of the null set. The axiom of foundation is thus directly formulated in terms of the phenomenon of Akshara, collapse of infinity to a point, and establishes the null set as a kind of omnipresent reality in the universe of sets. This corresponds to the understanding in Maharishi Vedic Science that the point value of $K$ is an omnipresent reality in the unbounded totality of natural law, $A$.

The significance of the Axiom of Foundation in modern set theory is that it establishes the completeness of the iterative mechanics of set
formation: it implies that all possible sets can be sequentially unfolded from the null set. The axiom of foundation thereby establishes the connectedness of the universe of sets, the expression of A, to the null set, the expression of K.

We consider next the Axiom of Infinity and the Axiom of Replacement. These two axioms can be justified on the basis of the Reflection Principle. We saw in Section 2 that the Reflection Principle is a natural mathematical expression of the phenomenon of \textit{Akshara}. The Reflection Principle expresses a deep intuition regarding the nature of the infinite: the intuitive understanding that the grand synthesis of all expressed mathematical possibilities should give rise to a value of wholeness that transcends the intellect in that it cannot be grasped by any intellectually conceivable property. This corresponds in Maharishi Vedic Science to the theme of structuring Brahman Consciousness through synthesis, as discussed in Section 1.

We shall consider first the way in which the Reflection Principle can be used to justify the Axiom of Infinity. The Axiom of Infinity asserts that there exists an infinite set containing the totality of the natural numbers: 0, 1, 2, \ldots. The set of natural numbers \{0, 1, 2, \ldots\} is constructed at stage \(\omega\) in the iterative process of generating sets. The fact that there exists an infinite ordinal \(\omega\) can therefore be used to “justify” the Axiom of Infinity. The fact that there exists an infinite ordinal \(\omega\) can in turn be justified on the basis of the Reflection Principle: the absolute ordinal \(\Omega\) has the property of being “a limit ordinal greater than 0,” and this property is reflected first in \(\omega\).

In fact, virtually everything known about the ordinals can be traced ultimately to the Reflection Principle. For example, for any ordinal \(\alpha\), \(\Omega\) has the property of being “greater than \(\alpha\).” By reflection, this implies there must exist an ordinal \(\beta\) having the property of being “greater than \(\alpha\),” and by well-ordering there must exist a smallest such \(\beta\), which we designate \(\alpha + 1\). Hence the property of every ordinal having a successor can be understood as having its basis in the Reflection Principle.

Even the existence of the first ordinal \(0\) can be justified on the basis of the Reflection Principle: \(\Omega\) exists, and therefore there must exist ordinals less than \(\Omega\) (because, for example, the property \(\Omega = \Omega\) must be reflected, so there must exist an ordinal \(\alpha\) such that \(\alpha = \alpha\)); the smallest such ordinal is the first ordinal, 0.
Putting together the existence of the first ordinal 0, with the property that every ordinal \( \alpha \) has a successor, \( \alpha + 1 \), we see that \( \Omega \) has the property of being greater than 0 and not being a successor. By the Reflection Principle there must be a first ordinal that reflects this property, a smallest limit ordinal greater than 0. This is the first infinite ordinal, \( \omega \). In this way the existence of the infinite ordinal \( \omega \) can be justified on the basis of the Reflection Principle.

The Axiom of Replacement likewise can be justified on the basis of the Reflection Principle. One such argument would be the following. Suppose we are given a set \( A \) and a rule that associates each element \( x \) of \( A \) with some new set \( y(x) \), and suppose the set \( y(x) \) is first created at level \( \alpha(x) \), so \( y(x) \in V_{\alpha(x)} \) (and \( y(x) \) is therefore an element of all succeeding partial universes). The absolute ordinal \( \Omega \) then has the property of being “greater than \( \alpha(x) \) for all \( x \in A \).” This property is a “conceivable” property and therefore must be reflected; therefore there exists an ordinal \( \beta \) such that \( \alpha(x) < \beta \) for all \( x \in A \). Thus \( y(x) \in V_\beta \) for all \( x \in A \), and therefore the replacement set, \( \{ y(x) \mid x \in A \} \), is created at level \( \beta + 1 \) as a subset of \( V_\beta \).

Although we have given arguments to “justify” certain of the Zermelo-Fraenkel axioms, we should bear in mind that these axioms are not “proved” in the axiomatic development of set theory; rather, they are taken to be the first principles on the basis of which all the theorems of set theory are then sequentially derived. The purpose of our discussion of these axioms has been to identify the basic intuitions regarding the nature of the infinite that underlie their acceptance as “first principles” of mathematics. We have seen that these intuitions are closely connected to fundamental principles of Maharishi Vedic Science, particularly the description of the self-interacting dynamics of the Samhita in terms of the phenomenon of Akshara.

We have seen how the Reflection Principle can be applied to justify certain of the Zermelo-Fraenkel axioms. The deepest applications of the Reflection Principle to the axiomatization of set theory are found in large cardinal axioms, additional axioms extending the framework of axiomatic set theory to express progressively more complete and holistic values of the infinite.

The need for such additional axioms is a consequence of the Incompleteness Theorem (see Section 8). Although the standard Zermelo-Fraenkel axioms are adequate for the development of all “ordinary”
mathematical theories, they are intrinsically incomplete; the Incompleteness Theorem shows in fact that any fixed axiomatization of set theory must be incomplete. The natural tendency to grow in the direction of complete, holistic knowledge has given rise to an open-ended expansion of axiomatic set theory, in which a progression of more and more powerful new axioms have been added. These axioms, called large cardinal axioms or higher axioms of infinity, assert the existence of large infinite cardinal numbers displaying extraordinary properties not predictable on the basis of the Zermelo-Fraenkel axioms.

Because the large cardinal axioms transcend the Zermelo-Fraenkel axioms, their justification must take direct recourse to the deepest levels of mathematical intuition regarding the nature of the infinite. It is here that the Reflection Principle plays a vital role in providing the fundamental principle of validation for these new axioms. The fundamental idea is that if one “reflects” on the absolute cardinal $\Omega$, and determines that it should have some property $P$, then one is justified in adding an axiom asserting that “there exists a cardinal $\alpha$ having property $P$.” Likewise, if one reflects on the universe of sets $V$, and determines that it should have some property $P$, then one is justified in adding an axiom asserting that “there exists a cardinal $\alpha$ such that the partial universe $V_\alpha$ has property $P$.” The application of the Reflection Principle to justify new axioms in this way is extremely delicate and subtle, and makes a direct appeal to very deep levels of intuition regarding the nature of the infinite. (For a discussion of the role of the Reflection Principle in justifying large cardinal axioms, see Reinhardt, 1974; Rucker, 1982, pp. 273–286; and Wang, 1974, p. 189.)

The study of the theory of large cardinals has brought to light a progression of more and more powerful large cardinal properties: inaccessibility, hyperinaccessibility, Mahlo property, indescribability, partition properties, measurability, supercompactness, extendibility, and hugeness, to mention the most important. Each successive property completely dwarfs the preceding ones, giving a whole new level of mathematical expression to the unboundedness of the absolute infinite.

The “smaller” of the large cardinals—the inaccessibles, the hyperinaccessibles, the Mahlos, and the indescribables—all have a clear justification based upon the Reflection Principle. Reinhardt (1974) shows how a reflection argument can be given to justify the existence
of extendible cardinals, and thereby also the measurable and super-compact cardinals. Reinhardt’s analysis is fascinating but involves some questionable assumptions involving the possibility of “extending” the universe. Beyond the extendibles are the huge cardinals, for which there is no known justification based upon reflection arguments (and doubt among set theorists whether there will ever be one!).

On what basis can one justify the postulation of the “large” large cardinals? If the Reflection Principle does not suffice, then perhaps one should look for an expression of the self-interacting dynamics of the Samhita that is in some sense even deeper than the Reflection Principle. A clue to the nature of such a new principle is presented by a common theme displayed by the large cardinal concepts from measurability upwards: the theme of elementary embeddings of the universe of sets (see Jech, 1978; Kanamori & Magidor, 1978).

An elementary embedding is a highly coherent type of mathematical transformation playing a fundamental role in model theory. An elementary embedding is a transformation \( T \) from a mathematical structure \( A \) to a mathematical structure \( B \) such that structural properties are preserved in the following very strong sense: if \( x \) is any element of \( A \), and if \( T \) transforms \( x \) into the element \( y \) of \( B \), then the properties that \( x \) enjoys in \( A \) are identical to the properties that \( y \) enjoys in \( B \). (“Elementary” means that the properties being considered are expressible in first-order logic.)

One extreme expression of the concept of an elementary embedding would be an elementary embedding of the universe of sets into itself. Maharishi has spoken of the self-interacting dynamics of the Samhita in terms of wholeness moving within itself in a perfectly orderly, coherent way. The concept of an elementary embedding of \( V \) into itself provides a natural mathematical expression of this theme of Maharishi Vedic Science. Such an embedding would therefore provide an ideal mathematical expression of the self-interacting dynamics of the Samhita. It was discovered, however, by Kunen (1971) that there are no elementary embeddings from \( V \) into \( V \), except for the trivial identity embedding, where every set is simply transformed into itself.

Although the ideal of an elementary embedding of \( V \) into itself cannot be mathematically realized, there can nevertheless exist mathematical transformations that “approximate” an elementary embedding of \( V \)
into itself. These transformations are closely connected with the different large cardinal concepts from measurability upwards.

One type of approximation to an elementary embedding of $V$ into $V$ is an elementary embedding of some partial universe $V_\alpha$ into another partial universe $V_\beta$. The existence of such embeddings is directly connected to the notion of an extendible cardinal.

Another type of approximation is an elementary embedding from $V$ into a subclass $M$ of $K$, where $M$ is a “subuniverse” of $V$ satisfying all the Zermelo-Fraenkel axioms. Here the class $M$ contains all the ordinals, but is “skinnier” than the true universe of sets $V$. (Technically, $M$ is required to be a transitive model of Zermelo-Fraenkel set theory.) The existence of such an embedding is equivalent to the existence of a measurable cardinal. By imposing additional requirements expressing the richness of the structure of $M$, one is led to the concepts of supercompactness and hugeness.

In all these cases the connection of the large cardinal with the elementary embedding is simply that the associated large cardinal $\alpha$ is the first cardinal moved by the embedding; every cardinal less than $\alpha$ will be left unchanged, but $\alpha$ will be transformed into a greater cardinal. Thus the internal dynamics of the universe of sets, expressed in the elementary embedding, singles out a particular point value $\alpha$, the first cardinal moved. The point value $\alpha$ is then found to be a large cardinal profoundly reflecting the structural properties of the absolute cardinal $\Omega$. That is, even though the existence of $\alpha$ cannot be inferred from the Reflection Principle, the large cardinal $\alpha$ is found to reflect the properties of $\Omega$, and this reflection becomes progressively more complete for the more powerful large cardinal concepts. For example, supercompact cardinals reflect all the so-called $\Sigma_2$ formulas, extendible cardinals reflect all formulas of the larger class $\Sigma_3$, and stationarily superhuge cardinals reflect all formulas expressible in the symbolic language of Zermelo-Fraenkel set theory. However, from the point of view of the absolute cardinal $\Omega$, these “large” large cardinals are still merely point values. This is the expression of the phenomenon of *Akshara* in the context of the “large” large cardinals: the collapse of $A$ to a point $K$ reflecting the wholeness of $A$.

We have described how the “large” large cardinals are closely connected with elementary embeddings approximating the ideal of an
elementary embedding of \( V \) into itself. This situation presents an interesting parallel to our discussion of the Reflection Principle in Section 1, where we saw the Reflection Principle to be the logically consistent “approximation” to the inconsistent self-referral property: “\( V \) is an element of itself.” Here we are dealing with approximations to the inconsistent property: “There exists a non-trivial elementary embedding of \( V \) into \( V \).”

In view of the above observation, we suggest that perhaps the appropriate principle to justify the “large” large cardinal axioms is the principle that “there should exist arbitrarily close approximations to an elementary embedding of \( V \) into \( V \).” The justification for such a principle could come directly from Maharishi Vedic Science through its own analysis of the internal self-referral dynamics of the Samhita. We shall discuss this idea further in Section 11.

In considering both the Zermelo-Fraenkel axioms and large cardinal axioms, we have located the expression of fundamental principles of Maharishi Vedic Science at the source of the sequential unfoldment of mathematical knowledge presented in axiomatic set theory. In the following sections, we shall see how set theory provides a foundation for all of modern mathematics; we shall see particularly how the holistic knowledge of the infinite provided by the theory of large cardinals has made possible the final stages of synthesis of the diverse streams of mathematical knowledge in the transcendental wholeness of the universe of sets.

7. The Set-Theoretic Foundation

In the preceding sections we have identified the ultimate foundation of set theory in the self-referral structure of the absolute infinite, the mathematical expression of the three-in-one structure of the Samhita of Maharishi Vedic Science. In this section and the following sections, we shall further develop this theme by examining the way all the diverse theories and viewpoints found in modern mathematics have a unified foundation in the Samhita of set theory.

Modern mathematics is organized into abstract theories, each describing some particular type of structure. The fundamental concept at the basis of this organization of knowledge is the concept of a mathematical structure or set with structure: a set of distinct point values on which
various operations and relations are defined. An example of a set with structure is the familiar system of natural numbers, in which the set of elements is the set of natural numbers, \( \mathbb{N} = \{0, 1, 2, 3, \ldots \} \), and on which the operations of addition and multiplication and the order relation “<” are defined. The knowledge of the properties of mathematical structures is organized into abstract theories, where each theory describes a category of structures satisfying a common set of axioms.

Set theory provides a unified foundation for all the diverse abstract theories of modern mathematics by showing how all the different structural relationships can be sequentially unfolded from the membership relation, the primordial relation of set theory. This sequential process involves a sequence of definitions, in which all the different structural concepts are defined in terms of the membership relation. Thus one defines, in sequence, the concepts of ordered pair, cartesian product, relation, function, operation, and so on. This sequence of definitions presents a sequence of viewpoints that allows one to locate all conceivable values of mathematical structure and relationship in the unified reality of the set theory universe, which is structured solely in terms of the membership relation.

We have analyzed earlier the way in which the membership relation, the primordial relation of set theory, is a mathematical expression of the knower-known relationship in the field of consciousness. The knower-known relationship is identified in Maharishi Vedic Science as the primordial value of relationship in creation; from the self-referral value of the knower-known relationship in the structure of pure knowledge, the Samhita, all values of relationship sequentially unfold. Ultimately, all these values of relationship are seen to be just intellectual viewpoints. The set-theoretic foundation provides a profound commentary on this theme of Maharishi Vedic Science. It shows how all values of mathematical relationship sequentially emerge from the primordial mathematical expression of the knower-known relationship, the membership relation; it shows how all are ultimately just conceptual viewpoints that locate all conceivable values of mathematical relationship in the Samhita of mathematics, the universe of sets.

One aspect of the set-theoretic foundation of particular significance is the way in which continuous geometrical structure is described.
Here the theory of infinite sets plays an indispensable role in analyzing the continuous reality of geometry in terms of discrete point values.

At the heart of this development is the set-theoretic construction of the real number system, a number system representing all points on a continuous line. This provides the set-theoretic foundation not only for the study of geometry, but also for the field of analysis, which provides the mathematical description of continuous change. It is the field of analysis that is most widely applied in modern science to mathematically quantify the laws of nature.

Scientists have often commented on the “unreasonable effectiveness” of mathematics in the sciences: the effectiveness of the purely abstract creations of the intellect in describing so exactly the orderliness of the physical world (see for example the classic essay of Wigner, 1967). What is perhaps most striking is the way the mathematics of the infinite, set theory, is required to describe the laws governing finite observable quantities. Certainly the concrete system of rational numbers is adequate to represent any result of a physical measurement. Yet the abstract, uncountably infinite reality of the real number system, grounded in the abstract dynamics of cosmic intelligence described by set theory, is required for the development of a mathematical theory capable of quantifying the underlying universal laws of nature.

Maharishi Vedic Science illuminates the deep connection between the mathematics of the infinite and the structure of the laws of nature. Maharishi Vedic Science locates the primordial expression of the laws of nature in the structure of the Samhita itself: the Richas of the Veda are the laws of nature. The laws of nature have their origin in the Akshara, the internal, self-interacting dynamics of the Samhita. Furthermore it is a fundamental theme of Maharishi Vedic Science that the Samhita has the structure of a continuum; $A$ is a continuum. The mathematical structure of the laws of nature should most naturally have its foundation in the mathematical structure of the Akshara within the Samhita, the intellectually conceived collapse of the continuum of $A$ to a point within it, $K$.

The simplest mathematical expression of the phenomenon of Akshara that does justice to the continuous nature of $A$ is the intellectually conceived collapse of a continuous line to its own points. This is precisely
the theme of analysis at the origin of the mathematical theory of the real number continuum.

If we represent real numbers by infinite decimals, each infinite decimal is seen to directly present the infinite sequential collapse of the continuous line to a specific point within it. For example, the infinite decimal 2.7354... presents the sequence of smaller and smaller closed intervals [2, 3], [2.7, 2.8], [2.73, 2.74], ..., converging to a point. The completeness property of the real number system, which expresses its continuity, can be directly formulated in terms of this theme of Akshara as the principle of nested intervals: A sequence of closed intervals, each contained within the previous one, whose lengths approach zero, must converge to a unique point. This precise characterization of the continuous quality of the line lies at the basis of the set-theoretic construction of the real numbers.

The set-theoretic construction of the real number system is thus seen to have its roots in the field of consciousness, in the phenomenon of Akshara, expressed as the intellectually conceived collapse of the continuum to the points contained within it.

The mathematical continuum, as a mathematical expression of the continuum of Samhita, reflects the fundamental values of the Samhita. The value of self-referral is expressed in the impredicative nature of the completeness principle (Wang, 1974, pp. 77–80, 123–129). An impredicative definition is a definition of an object by reference to a totality that already contains that object. Such a self-referral value of definition is implicit in the precise formulation of the completeness principle, expressing the continuous quality of the line. It is a consequence of this self-referral quality of continuity that the continuum, as a set of points, turns out to be a vast, uncountably infinite set—a transcendental mathematical wholeness. It is most natural and appropriate that this mathematical expression of the transcendental continuum of the Samhita, grounded in the intellectual analysis of the Akshara, should provide the basis for the mathematical description of the laws of nature since the laws themselves sequentially unfold from the self-interacting dynamics of the Samhita.

In this section we have examined the way in which set theory provides a foundation for the diverse abstract theories of modern mathematics. We considered in particular the way the abstract theory of
infinite sets provides the foundation for the theory of the continuum, which in turn provides the basis for all areas of mathematics dealing with continuous structure and change, including the mathematical formulation of modern physics. In the following sections we shall examine the different foundational approaches prominent in modern mathematics, and we shall see how set theory provides a natural foundation for each of these approaches, even those directly contradicting the set theoretic approach.

8. Mathematical Logic
Mathematical logic is the study of the structure and function of the symbolic language of mathematics. At the heart of this study is the formalization of mathematical theories. This involves creating an exact symbolic language for expressing the principles of mathematics, as well as providing a precise description of the fundamental rules of logical inference as mechanical rules of transformation of these symbolic expressions. If the symbolic expressions constituting the axioms of an axiomatic theory are themselves specified by a finite number of mechanical rules, then in principle all the theorems of the theory can be sequentially unfolded in a purely mechanical way; such a structure of knowledge is called a formal system.

Zermelo-Fraenkel set theory is an example of a formal system. The axioms, as symbolic mathematical formulas, can be generated from a small number of mechanical rules for manipulating symbols. The theorems of Zermelo-Fraenkel set theory can then be unfolded sequentially from the axioms by means of another collection of rules for manipulating symbolic expressions, these latter rules describing the fundamental principles of logical inference. Thus in principle the whole content of Zermelo-Fraenkel set theory can be sequentially unfolded by a suitably programmed computer!

With the formalization of mathematical theories during the early decades of this century, there emerged a foundational viewpoint called formalism, which maintained that mathematics was ultimately reducible to a symbolic game in which symbolic expressions were manipulated according to well-defined rules. According to the formalists, the “real” part of mathematics consisted of the symbolic expressions themselves, and the abstract infinite set-theoretic reality described by the symbols...
was ultimately a fiction. The criterion of correctness of a mathematical theory was just that it be consistent; that is, it should not lead to symbolic contradictions, symbolic formulas of the form “\(P\) and not \(P\).

The founder and leading proponent of the formalist school was the great German mathematician David Hilbert (see Reid, 1970; Hilbert, 1926). Hilbert’s formalist program was intended to provide a foundation for modern mathematics demonstrably free of the paradoxes of the infinite, such as Russell’s paradox. This was to be achieved by establishing the consistency of the axiomatic development of mathematics. It was furthermore required that consistency be demonstrated in a constructive way, that is, the proof of consistency should be based upon the concrete, finitary type of reasoning acceptable to the intuitionists (see Section 9).

Hilbert’s program was dealt its deathblow with Kurt Gödel’s discovery of his famous incompleteness theorems in 1931. Gödel’s Second Incompleteness Theorem showed that the consistency of a sufficiently rich formal system could never be established by mathematical arguments that could be formalized within the system: a formal system could never prove its own consistency (unless it was inconsistent, in which case it could prove anything!). This meant that the concrete, constructive types of metamathematical arguments acceptable to the formalists could never establish the consistency of axiomatic set theory; any argument acceptable to the formalists could easily be formalized in the much more powerful framework of axiomatic set theory, and therefore such an argument could never establish the consistency of axiomatic set theory.

The proof of the incompleteness theorems rested upon a self-referral theme introduced by Gödel called the *arithmetization of metamathematics*. This involved a coding process whereby statements about symbolic formulas and their logical relationships were translated into statements about numbers and their relationships to one another; metamathematical statements were translated into arithmetical statements. These arithmetical statements could then be expressed in the symbolic language of the theory being described, provided that theory was powerful enough to develop ordinary arithmetic. In this way statements about the logical structure of a mathematical theory could be expressed within the
symbolic language of the theory itself: the theory could talk about its own structure, a striking expression of self-referral.

Gödel’s remarkable proof of the incompleteness theorems was based upon construction of an arithmetical formula in the symbolic language of a theory $T$ that expressed the metamathematical assertion: “I am not a theorem of the theory $T$.” That is, the formula asserted that the very sequence of symbols constituting the formula could not be unfolded from the symbolic axioms of the theory by using the rules of logical inference. Once this self-referral formula was constructed, the incompleteness theorems were easily inferred; on the assumption that the system gave a valid description of arithmetic, this formula could be shown to be true and hence unprovable in the theory $T$, showing that the theory $T$ was incomplete. This was the First Incompleteness Theorem.

A more refined analysis showed that the arithmetical formula expressing the metamathematical assertion “the theory $T$ is consistent” could not be a theorem of the theory $T$. This established the Second Incompleteness Theorem: the consistency of a theory $T$ can never be proved within the theory $T$ (assuming that $T$ is consistent). The kind of self-referral argument involved in the proof of the incompleteness theorems could never have been discovered by the formalists because of their commitment to ignoring the meaning of the symbols when studying the structure of the symbolic language.

Wang (1974, pp. 7–13) contains a fascinating account of Gödel’s own observations about the line of thinking that led to the discovery of the incompleteness theorems as well as his other monumental discoveries. Gödel attributed all of these to his unique “epistemological attitude,” in which he recognized as the primary reality of mathematics the abstract infinite reality of the concepts, rather than the more concrete finite reality of the symbolic language. In Gödel’s own analysis of his discovery of the incompleteness theorems, he points out that the proof rested upon the distinction between the transcendental notion of “truth” and the concrete notion of “proveability in a particular formal system”; the proof of the incompleteness theorems was based upon constructing a mathematical formula that could be shown to be true but at the same time unprovable in a particular formal system. It was this distinction between truth and proveability that had been lost in the formalist
approach to the foundations of mathematics, because it lacked a transcendental notion of truth.

Gödel’s arithmetization of metamathematics shows how any axiomatic theory adequate for the development of arithmetic can describe the mechanics of its own sequential unfoldment. This means that in the theme of sequential unfoldment of knowledge of such an axiomatic theory, at some stage one can find the description of the mechanics of transformation governing the entire sequential process itself. This theme has a parallel in Maharishi Vedic Science in the Apaurusheya Bhashya of Rik Veda discussed in Section 6 above. According to Maharishi’s description of the Apaurusheya Bhashya, the mechanics of transformation contained in the gaps between the successively emerging expressions of knowledge are themselves elaborated in the Vedic text. In the case of an axiomatic theory, these mechanics of transformation are the rules of logical inference; these rules can be described by mathematical formulas of the theory through the arithmetization of metamathematics (provided the theory is rich enough to express the basic operations and relations of arithmetic).

While this self-referral theme is already expressed in elementary number theory, the axiomatic theory of arithmetic, it finds its most complete expression in axiomatic set theory. Here the mathematical description of the structure and function of the language goes far beyond the mere description of the mechanics of sequential unfoldment of the expressions of knowledge, the theorems of the theory, to encompass the much more profound mechanics of transformation through which the models of the theory can be unfolded from the structure of knowledge. This further development provides a mathematical commentary on the relationship between knowledge and organizing power brought to light in Maharishi Vedic Science.

We have discussed Maharishi’s description of the Richas of the Veda as the structure of pure knowledge, the unmanifest structure of the laws of nature in the transcendental field of pure consciousness. In the theme of the Apaurusheya Bhashya, the sequential emergence of the Richas of the Veda continues with the emergence of creation itself from the structure of the laws of nature. The Richas of the Veda thus have a creative aspect through which they bring forth the diverse expressions of the laws of nature in phenomenal creation. Maharishi uses the term
“organizing power” to describe this creative potential latent within the structure of pure knowledge, the Richas of the Veda. The expression of the organizing power of a particular law of nature is found in those phenomena that are governed by that law.

Maharishi speaks of the Richas of the Veda as the expressions of the language of nature. This language has a meaning, but this level of meaning differs from the ordinary level of meaning of language that is confined to intellectual understanding. The meaning of the language of nature is the organizing power latent within it, and this meaning spontaneously manifests itself in the diverse phenomena of creation.

Ultimately, Maharishi explains, all of creation is just the expression of the organizing power contained in the structure of pure knowledge. In the Vedic literature, the Brahmanas present the mechanics of transformation through which the organizing power contained in the structure of pure knowledge, the Richas of the Veda, becomes concretely unfolded.

In mathematics, the symbolic structure of an axiomatic theory presents the structure of “pure knowledge”: symbolic expressions of knowledge considered on the purely abstract level of their logical relationships to one another, without considering the interpretation or meaning of the symbols. The models of the theory then present concrete expressions of the organizing power contained in the theory. A model of a theory is, by definition, a concrete interpretation of the symbolic language of the theory in the context of a particular mathematical structure such that all the axioms become true statements about the structure. This means simply that the structure is governed by the laws expressed in the axioms. The relationship between a theory and its models is thus a mathematical presentation of the relationship between the structure of pure knowledge and the concrete expressions of its organizing power.

The study of the relationship between theories and models is the theme of model theory. At the heart of model theory is Gödel’s (1930) Completeness Theorem, whose discovery rested upon a new insight into the mechanics of transformation of the structure of mathematical knowledge into the expressions of its organizing power, corresponding to the theme of the Brahmanas in Maharishi Vedic Science.

The Completeness Theorem asserts that any consistent axiomatic theory must have a model: if the axioms of the theory do not lead to
logical contradictions, then there must exist some mathematical structure satisfying the axioms of the theory. Gödel’s proof of the Completeness Theorem showed how a model can be created from the symbolic language in which the axioms are expressed. This transformation of the symbolic formulas into the model involves a nonconstructive process, directly grounded in the infinitary methods of set theory. Gödel thereby introduced a second great innovation into the study of the foundations of mathematics: a nonconstructive approach to metamathematics, in which the abstract transformations of set theory are freely applied to the symbolic formulas.

Gödel attributed his discovery of the Completeness Theorem to his epistemological attitude, just as he did his discovery of the incompleteness theorems (see Wang, 1974). The concept that the abstract transformations of set theory could be legitimately applied to the concrete reality of the symbolic formulas was grounded in his view that the abstract reality of set theory was as real as the concrete reality of the symbolic expressions.

Model theory studies the relationship between theories and their models in general; the most fundamental developments in this field have emerged from the application of the model-theoretic viewpoint to the study of the models of axiomatic set theory. This process has a self-referral nature in that set theory is used to study the structure of set theory itself. In this context, one takes as the theory under consideration Zermelo-Fraenkel set theory. The models of interest are subsets or subclasses of the universe of sets $V$ that already satisfy internally all the Zermelo-Fraenkel axioms. These models are thus “mini-universes” contained within the true universe of sets $V$. One is especially interested in the existence of models having special additional properties, beyond the Zermelo-Fraenkel axioms; the existence of such models is intimately connected to the logical notions of consistency and independence, as we shall discuss below.

The first major development in the study of models of axiomatic set theory was Gödel’s (1938) description of the constructible universe $L$. Gödel took the notion of constructibility and applied it in a nonconstructive way to give rise to the abstract concept of a constructible set. Gödel showed that the class $L$ consisting of all possible constructible sets satisfied all the Zermelo-Fraenkel axioms; $L$ was a model of
Zermelo-Fraenkel set theory. Then he showed that the Continuum Hypothesis was valid in \( L \). This established the consistency of the Continuum Hypothesis: the Continuum Hypothesis could not be disproved from the Zermelo-Fraenkel axioms (otherwise it would have to be false in every model of Zermelo-Fraenkel set theory). But this did not establish that the continuum hypothesis could be proved from the Zermelo-Fraenkel axioms.

The great breakthrough occurred in 1963 when Paul Cohen showed that the Continuum Hypothesis indeed could not be proved from the Zermelo-Fraenkel axioms. Cohen’s discovery was based upon the invention of a very general and powerful technique for creating models of set theory, called the method of forcing. Applications of forcing have played a central role in almost all major developments in set theory since Cohen’s monumental breakthrough; the great power and universality of this technique derive from its deep insight into the mechanics of transformation through which an abstract intention can be sequentially transformed into a set-theoretic actuality. In the remainder of this section, we shall consider the main themes of the theory of forcing. We shall see how they give mathematical expression to those fundamental principles of Maharishi Vedic Science that identify and practically apply the level of natural law in which knowledge and organizing power are found in an undivided state, the level of the language of nature, the Veda.

We shall consider the formulation of the theory of forcing in terms of Boolean-valued models, following Jech (1978). The basic idea of forcing is to start with some model \( M \) of the Zermelo-Fraenkel axioms contained within \( V \); \( M \) could be, for example, the constructible universe \( L \). \( M \) is called the ground model. We would like to extend \( M \) to create a new model satisfying all the Zermelo-Fraenkel axioms but also satisfying some particular extraordinary property. This extension is achieved by adjoining to \( M \) a set \( G \) not in \( M \); the set \( G \) must have very special characteristics that establish the intended property in the new model.

Although the needed set \( G \) does not exist inside \( M \), one can create, inside \( M \), a forcing language that contains a natural name for \( G \). The forcing language in fact contains names for all the sets in the new model. By an abstract, set-theoretic transformation, the names are then transformed into the sets constituting the new model.
The names in the forcing language are generated by an iterative process completely parallel to the iterative mechanics of set formation, except that the power-set operation is replaced by another mathematical transformation based upon a specific structure \( B \) called a *Boolean algebra*. \( B \) is a specific set in \( M \), and \( G \) is a subset of \( B \) in \( V \), but \( G \) is not a member of the model \( M \).

The Boolean algebra \( B \) is generated from a set \( P \) of *forcing conditions*. The application of the technology of forcing involves judicious choice of the set of forcing conditions \( P \). It is the choice of \( P \) that expresses one’s intention; all else is then determined. In practice, \( P \) consists of partial specifications of a desired set \( S \) required to fulfill one’s intention. Even though the desired set \( S \) will not lie in \( M \), \( M \) will contain “approximations” to it that constitute partial specifications of \( S \). The subset \( G \) of \( P \) will then put together mutually compatible partial specifications to obtain a complete specification of \( S \). Specifying \( G \) will be equivalent to specifying \( S \), but \( G \) likewise will not be in \( M \).

The forcing language is a completely natural language. The names in the forcing language are abstract sets, generated in a natural way through a mechanics that directly mirrors the iterative mechanics of generation of the set theory universe. Because of this, the organizing power of the forcing language is fully lively in the structure of the names themselves, and can be concretely unfolded from the names.

The names in fact already provide an expression of the organizing power of the forcing language; the totality of names structures a *Boolean universe* \( V^B \), which satisfies all the Zermelo-Fraenkel axioms. Furthermore, the name of \( G \), as an element of this Boolean universe, has the desired property of \( G \). The Boolean universe \( V^B \) is similar to the ordinary universe of sets, except it is governed by a logic different from the ordinary two-valued classical logic, where there are only two truth values, “true” and “false.” For the Boolean universe \( V^B \), every element of the Boolean algebra \( B \) represents a possible truth value.

The name of \( G \) in the forcing language has a self-referral structure; \( G \) is represented by a function from \( B \) to \( B \) that takes each element of \( B \) into itself. This means that \( G \) represents the subset of \( B \) characterized by the following self-referral property: if \( G \) is any element of \( G \), then the formula “\( G \in G \)” has truth value \( G \).
The Boolean universe $V^B$ provides a Boolean-valued model having the desired properties. This abstract expression of the organizing power of the forcing language is presented by the structure of the forcing language itself. To obtain a classical model, it is necessary to collapse the infinite-valued Boolean logic $B$ to the two-valued classical logic. This transformation must be effected in a way that preserves the integrity of the structure of knowledge contained within the Boolean universe $V^B$.

The transformation is achieved by means of the construction of a generic ultrafilter on $B$ (if one starts for example with a countable model $\mathcal{M}$, this can always be done using the Axiom of Choice). The generic ultrafilter collapses the infinite Boolean algebra $B$ to the two-element Boolean algebra of classical logic; on this basis the Boolean universe $V^B$ can then be sequentially collapsed to a classical model. This gives rise to a localized, classical expression of the organizing power contained in the forcing language. In itself, all classical possibilities are simultaneously lively in the gap between true and false; the process of collapse gives expression to a unique, well-defined classical possibility.

This phenomenon is strikingly parallel to the collapse of the wave function in quantum mechanics. In the quantum-mechanical state, all classical possibilities are simultaneously lively; through the process of observation, the wave function collapses to a configuration representing a single classical possibility. The quantum mechanical level of reality is in fact often described in terms of quantum logic, which differs from classical two-valued logic and contains many intermediate truth values between “true” and “false,” a situation analogous to the multi-valued logic describing the Boolean universe $V^B$. (The mathematical structure of quantum logic, however, is different from the structure of logic governing a Boolean universe. For an elementary account of quantum logic and the theory of measurement, see Herbert, 1985.)

The theory of forcing has brought to light a natural, abstract level of mathematical language, the forcing language, in which there is a natural correspondence between name and meaning; this is the level of language in which knowledge and organizing power are in an undivided state. This has a direct parallel in Maharishi Vedic Science in the structure of the language of the Veda, in which knowledge and organizing power are in an undivided state, and there is a natural correspondence between sound and meaning.
In the Vedic literature, Maharishi identifies the Brahmanas as presenting the mechanics of transformation of pure knowledge into the diverse expressions of its organizing power. This provides the basis for the technology of the Yagyas, specific prescribed procedures, whereby the mantras of the Veda, the expressions of pure knowledge, are made to concretely unfold their organizing power in such a way as to bring fulfillment to a specific intention. The technology of the Yagyas, Maharishi continues, thereby enables one to use the language of nature, the language of the Veda, to fulfill one’s desires. The theory of forcing presents the mathematical expression of this theme, utilizing the forcing language to bring forth a specific desired expression of mathematical structure.

In the performance of the Yagya, the sankalpa (intention) determines the outcome. In the technology of forcing, the intention is expressed in the choice of the forcing conditions, from which the Boolean algebra $B$ is generated. This determines the structure of the logic governing the sequential generation of the names of the forcing language, and thereby the specific character of the organizing power embodied in the forcing language. The different values of intention correspond to different ways of structuring the gap between “true” and “false,” giving rise to different structures of Boolean-valued logic.

When Paul Cohen (1963) introduced the method of forcing, he showed how it could be applied to construct models of Zermelo-Fraenkel set theory in which the continuum had virtually any desired cardinality: $\aleph_1, \aleph_2, \aleph_3, \ldots, \aleph_{\omega + 1}$, and so on. This established the independence of the Continuum Hypothesis, that is, that the statement $|P(\mathbb{N})| = \aleph_1$ could be neither proved nor refuted from the axioms of set theory. (If it could be proved, then every model would have to satisfy $|P(\mathbb{N})| = \aleph_1$, and if it could be disproved, then no model could satisfy $|P(\mathbb{N})| = \aleph_1$.) The independence of the Continuum Hypothesis is a statement purely about the structure of the symbolic language: the formula “$|P(\mathbb{N})| = \aleph_1$” is not a theorem of Zermelo-Fraenkel set theory, and likewise its negation “$|P(\mathbb{N})| \neq \aleph_1$” is not a theorem. Yet this metamathematical result could only be discovered by taking recourse to the most abstract constructions of set theory!

In the years since Cohen’s pioneering work, the method of forcing has been used to construct models of set theory having all kinds of
extraordinary properties, and on this basis many important independence results have been derived. In this we find a profound expression of the role of abstract set theory in providing the proper foundation for the complete understanding even of the concrete structure of the finite, symbolic language.

9. Categories, Toposes, and Intuitionism

In this section we shall examine the category-theoretic approach to the foundations of mathematics, which complements the set-theoretic approach, and also the approach called intuitionism, which directly contradicts the set-theoretic approach. We shall see how the theory of large cardinals provides a natural set-theoretic foundation for category theory; we shall see further how the recent development of topos theory, based upon category theory, provides a natural set-theoretic foundation for intuitionistic mathematics.

We shall see particularly that this final step of integration of classical and intuitionistic mathematics is based upon a self-referral structure of knowledge called sheaf semantics, in which the values of the knower (stages of knowing) and the known (sets) are identical. This will provide a striking mathematical parallel to the self-referral structure of pure knowledge at the foundation of Maharishi Vedic Science, which synthesizes all contrasting values of intellectual understanding in its own holistic structure.

We begin by considering category theory. Category theory is an abstract theory of mathematical theories introduced by Eilenberg and Mac Lane in 1945. Category theory has its basis in certain simple observations about the structure of abstract theories.

Each abstract theory describes a certain category of objects; these objects are simply the models of the theory, the mathematical structures satisfying the axioms of the theory. The abstract theories are concerned not only with the isolated properties of models, but also with the relationships of the models to one another. In particular, each theory has as one of its fundamental concepts the concept of a transformation from one object to another that preserves the integrity of the fundamental structural relationships of the theory. Such a transformation is called a morphism of the theory.
Despite the striking differences in the formulation of the different theories, there are two common features shared by the morphisms of all theories:

- If \( f \) is a morphism from an object \( A \) to an object \( B \) and \( g \) is a morphism from \( B \) to \( C \), then \( f \) and \( g \) can always be combined sequentially to yield a composite morphism \( g \circ f \) from \( A \) to \( C \). The composition operation \( \circ \) satisfies the Associative Law: 
  \[(a \circ b) \circ c = a \circ (b \circ c).\]
- For every object \( A \) there is an identity morphism \( 1_A \) from \( A \) to \( A \) that takes each element of \( A \) into itself. The identity morphism satisfies the identity laws: 
  \[1_A \circ f = f \quad \text{and} \quad g \circ 1_A = g,\] whenever these compositions are defined.

The observation of these simple properties led to the formulation of the abstract concept of a category. A category consists of objects \( A, B, C, \ldots \) and arrows \( f, g, h, \ldots \) such that:

1. Every arrow has a source and a target, which are objects.
2. If \( f \) is an arrow with source \( A \) and target \( B \), and \( g \) is an arrow with source \( B \) and target \( C \), then \( f \) and \( g \) can be composed to yield a unique arrow \( g \circ f \) with source \( A \) and target \( C \). The composition operation \( \circ \) is required to satisfy the associative law, 
  \[f \circ (g \circ h) = (f \circ g) \circ h.\]
3. For every object \( A \) there is an identity arrow \( 1_A \) from \( A \) to itself such that 
  \[1_A \circ f = f \quad \text{and} \quad g \circ 1_A = g,\] whenever the compositions are defined.

Category theory is developed axiomatically starting from these simple assumptions. What is most striking about the language of category theory is that it does not talk about elements; the objects of the category are intended to represent mathematical structures, yet there is no way to talk about the elements of the structure. The only relationship one can talk about is the composition operation for arrows, the way two arrows combine in sequence to yield a third arrow.

The language of category theory thus provides a striking contrast to the language of set theory. In set theory, the membership relation
is everything; it is the primordial relationship from which all values of relationship are unfolded. In category theory, the membership relation cannot even be expressed directly in the language; all one can express are relationships between arrows.

Because of its abstract viewpoint, category theory is able to describe relationships between mathematical structures in a language simultaneously applicable to all abstract theories. Familiar mathematical constructions thereby become appreciated in a totally new light in terms of arrows rather than elements; they are seen to be special cases of universal constructions applicable to all possible theories.

Category theory not only provides a universal language for describing the constructions within a mathematical theory, but further provides a natural framework for describing transformations between theories. These are the arrows of category theory, transformations from one category to another that preserve the categorical structure; these transformations are called functors.

The study of functors has provided new insights into the relationship between the diverse abstract theories of modern mathematics and the “Samhita” of set theory, the universe of sets. When the universe of sets is seen from the viewpoint of category theory, it is seen as the category of sets. This is the category whose objects are all possible sets, and whose arrows are all possible functions, or transformations, from one set to another. The category-theoretic vision does not see, however, the internal membership relation; it sees only the composition operation for functions.

The category-theoretic vision most naturally describes the internal dynamics of the Samhita, the category of sets, in terms of functors from the category of sets to itself. A deep result in category theory called Beck’s theorem shows how the categories corresponding to a wide class of abstract theories can be actualized within the category of sets in terms of such functors. On this basis, one can locate the “blueprint” for the expressed level of mathematical structure in the internal dynamics of the Samhita, the category of sets.

Specifically Beck’s theorem shows how the category of models for an abstract algebraic theory can be actualized as the category of algebras for a triple in the category of sets (for example, Mac Lane, 1971, pp. 133–155). The extraordinary aspect of this mathematical construc-
tion is that all the relevant transformations in the category of sets are derived from the forgetful functor.

The forgetful functor is the functor from the category of models of an abstract theory to the category of sets that acts in the simplest possible way: applied to a mathematical structure (a model of the theory), the forgetful functor simply forgets the structure, that is, it forgets all the operations and relations, leaving just the underlying set of elements of the structure, an undifferentiated set of point values.

The forgetful functor thus represents the most fundamental and natural transformation from the expressed, differentiated level of mathematical structure to the undifferentiated field of the Samhita, the category of sets. The mathematical description of this simplest value of transformation that does nothing except “forget” could only be achieved, however, on the basis of the abstract vision of category theory.

From the forgetful functor one can construct a triple in the category of sets, a specific set of three transformations internal to the category of sets. Beck’s theorem shows how the complete structure of the algebraic category can be reconstructed from this triple. This reveals how the “memory” of the structure of the algebraic category can be “reenlivened” in the category of sets on the basis of the forgetful functor.

The mathematical study of the forgetful functor in category theory provides an interesting mathematical commentary on the theme of “forgetting” at the basis of the Transcendental Meditation and TM-Sidhi programs. In the TM-Sidhi program, the mechanics of forgetting is utilized to systematically “forget” the boundaries of a thought and thereby experience the unbounded wholeness of the Samhita, the undifferentiated field of pure consciousness. The effect is to enliven in awareness the internal, self-interacting dynamics of the Samhita, which contains the blueprint of creation.

In a parallel way, Beck’s theorem locates the blueprint for the expressed level of mathematical reality in the internal dynamics of the category of sets; this is achieved on the basis of the forgetful functor, the mathematical expression of the mechanics of “forgetting.” In this way category theory, from its own unique perspective, provides a fascinating new insight into the relationship of the expressed level of mathematical reality described by abstract theories and the Samhita of mathematics, the universe of sets (seen as the category of sets).
The category-theoretic viewpoint beautifully complements the set-theoretic approach to the development of mathematics. From our foundational viewpoint, however, we would like to establish the completeness of the set-theoretic foundation. In order to do this, it is necessary to provide a set-theoretic foundation for category theory.

The most direct approach is simply to treat category theory as any other abstract theory, regarding a category as a special type of set with structure. This is not so simple, however, because many fundamental category-theoretic constructions utilize categories that are too big to be sets. For example, a construction might involve the category of sets, which contains as objects all possible sets, and as arrows all possible functions. This category is as big as the universe of sets itself.

The solution is to recognize that one never really needs the category of sets (or other categories that are proper classes); one only needs categories that are big enough to reflect the requisite properties of the universe of sets. Thus we can replace the category of sets by the category of small sets, where small sets are taken to be the elements of a partial universe $V_\alpha$ for a sufficiently large cardinal $\alpha$. In this way the theory of large cardinals provides a natural set-theoretic foundation for category theory. (For a general discussion of set-theoretic approaches to the foundation of category theory, see Blass, 1984.)

Category theory, however, has its own style of thinking, very much independent of the set-theoretic approach. With the continued development and maturation of category theory, it was natural for the category theorists to try to provide an independent foundation for modern mathematics.

One such attempt involved a self-referral approach, in which category theory was applied to describe its own categorical structure. Category theory is itself an abstract theory; the models of this theory are all possible categories, and the arrows of this theory are all possible functors. Category theory, as an abstract theory, thus describes the category of all categories. When one describes the category of all categories in categorical terms, that is, in terms of functors and the composition operation for functors, one obtains the theory of the category of all categories. This theory applies the categorical viewpoint to describe the structure of category theory itself; thus it employs category theory in a self-referral way. This approach was systematically applied by
Lawvere (1966) in his attempt to provide a category-theoretic foundation for modern mathematics.

Lawvere’s idea in developing the theory of the category of all categories was to use special functors to probe inside categories to determine their internal structure. He could then use the categorical description of the category of sets to determine which category was the category of sets, and on this basis reconstruct the set-theoretic foundation.

The appropriate categorical description of the category of sets was worked out by Lawvere (1964). The category of sets, as indicated above, is the universe of sets seen from the categorical viewpoint, so one does not see the membership relation, one only sees the composition of functions. The problem is to characterize, by categorical properties, those special properties of the category of sets that distinguish it from other categories. Although Lawvere succeeded in this, his characterization of the category of sets involved a non-elementary property, that is, a property requiring set-theoretic language for its formulation. To provide a foundation truly independent of set theory, it was necessary to find a suitable characterization of the category of sets in terms of elementary properties. The “right” elementary concept was discovered by Lawvere and Tierney in 1969 (see Gray, 1979, pp. 60–64); this was the concept of an *elementary topos* (henceforth referred to simply as a *topos*).

A topos is a category having certain special properties that make it “resemble” the category of sets. The defining properties of a topos do not uniquely characterize the category of sets; in fact, a topos can be strikingly different from the category of sets in many fundamental ways. The defining properties of a topos make it resemble the category of sets to the extent that fundamental set-theoretic constructions can be carried out in any topos, and on that basis the topos can be interpreted as a generalized universe of sets. In this way topos theory provides a generalization of set theory to include many possible universes displaying extraordinarily diverse properties. This has had profound implications for the unification of modern mathematics, as we shall see below.

The most distinctive of the special properties of a topos is the existence of an object in the topos called the subobject classifier, designated by $\Omega$ (not to be confused with the absolute ordinal $\Omega$). The special feature of $\Omega$ is that the subobjects of any object $A$ correspond naturally to the different possible arrows from $A$ to $\Omega$. 
The motivation for this requirement comes from the category of sets, where the two-element set \( \Omega = \{ \text{true, false} \} \), plays the role of the subobject classifier. This can be seen as follows.

To form a subset \( B \) of a set \( A \), we can freely choose, for each element of \( A \), whether or not to include it in the subset \( B \). The pattern of choices corresponds to the function from \( A \) to \( \Omega \) that takes each element chosen into true and each element rejected into false.

The fact that there are precisely two possible choices for each element is intimately connected with the two-valued nature of classical logic; if \( s \) is any element of \( A \), then the statement “\( s \in B \)” will have truth value either true or false depending upon the two possible choices: includes or excludes. In the development of topos theory, it is shown how the subobject classifier \( \Omega \) plays the role of a “set of truth values” for the internal logic of a topos. For toposes other than the category of sets, however, the internal logic will typically be governed, not by the familiar laws of classical logic, but by the laws of intuitionistic logic. This will usually mean that there are additional truth values in the gap between true and false; these truth values can even have the structure of a continuum.

Intuitionistic logic is the logic governing the intuitionistic formulation of mathematics. The intuitionistic school of mathematics was founded by the Dutch mathematician L.E.J. Brouwer in the early decades of the 20th century; the seed is already contained in his famous paper of 1908, in which he challenges the a priori validity of classical Aristotelian logic.

The fundamental premise of Brouwer’s intuitionism is that a mathematical construction is meaningful only if it can, in principle, be effectively carried out by the localized, sequential functioning of the human intellect. This led Brouwer to reject fundamental principles of set theory, such as the Axiom of Choice, which required consideration of an infinity of simultaneous choices. This further led Brouwer to reject several fundamental laws of classical logic, particularly the law of the excluded middle (either \( P \) or “not \( P \)” and the double negation law (“not not \( P \)” is equivalent to \( P \)). Intuitionistic mathematics thus requires a different set of principles of logical inference for its development; these are the principles of intuitionistic logic.
The intuitionistic formulation of mathematics directly contradicts the standard set-theoretic formulation. Topos theory provides, for the first time, a way in which the complete development of intuitionistic mathematics can be integrated into the framework of classical mathematics, and hence grounded in set theory. This is achieved by showing how the different intuitionistic theories, which are formulated in terms of elements, correspond to the development of mathematics internally within specific toposes, whose internal logic is intuitionistic and not classical.

Specifically, every topos has a natural set-theoretic language, capable of expressing the membership relation, called its Mitchell-Bénabou language. Corresponding to each object in the topos, this language contains specific variables that talk about the “elements” of that object; corresponding to each arrow of the topos there is a specific function symbol in the language that allows one to talk about the way the arrow transforms the elements of its source into the elements of its target.

The Mitchell-Bénabou language is prima facie a purely formal device since a topos is presented as a category: the objects are not presented as sets, and there is therefore no obvious meaning to the concept of an “element” of an object. Nevertheless, one can construct an internal interpretation of this language, in which the interpretation of a formula is given by a specific arrow having target $\Omega$, the “set” of truth values. On this basis, one can then define a notion of internal validity of a formula of the Mitchell-Bénabou language.

When one does this, one finds that the rules of intuitionistic logic are internally valid: applied to internally valid formulas, they yield internally valid formulas. Furthermore, certain fundamental axioms of set theory (excluding the Axiom of Choice) are always internally valid. This means that set theory can be developed internally in any topos, provided one restricts oneself to intuitionistic logic.

What is the meaning of this internal set theory? The answer is provided by sheaf semantics, or Kripke-Joyal semantics, a development of the Kripke semantics for intuitionistic logic. Kripke (1965) presents a way to understand intuitionistic logic from a classical perspective. The fundamental idea is that a mathematical structure is viewed as containing elements that are not fixed but variable, where the elements are in different “stages of knowing.” A natural definition of validity that takes
into account the variability of stages of knowing gives rise to a logical structure governed by the rules of intuitionistic logic rather than the rules of classical logic.

The adaptation of the Kripke semantics to topos theory yields the sheaf semantics of a topos. In the most general formulation of this semantics (see for example Johnstone, 1977), the objects of the topos represent simultaneously (1) sets, the objects of knowledge, and (2) the stages of knowing, the values of the knower. The arrows from an object \( A \) to an object \( B \) represent simultaneously (1) the elements of the set \( B \) in the stage of knowing \( A \) (the knower-known relationship); (2) functions from the set \( A \) to the set \( B \) (transformations between the objects of knowledge); and (3) passages from the stage of knowing \( B \) to the stage of knowing \( A \) (transformations between the values of the knower).

The sheaf semantics of a topos provides a striking mathematical expression of the self-referral structure of pure knowledge. In this semantics, the objects of the topos, viewed as stages of knowing, present the value of Rishi. These same objects, viewed as sets, present the value of Chhandas. The arrows of the topos, viewed as elements, present the value of Devata. Because the same objects simultaneously express the values of the Rishi (knower) and Chhandas (known), the structure of knowledge presented by sheaf semantics is self-referral in its nature.

The Kripke semantics for intuitionistic logic provides what is perhaps the most explicit description of the knower-known relationship in mathematics. Brouwer’s formulation of intuitionism reflected a change of mathematical viewpoint that gave prominence to the knower, rather than the known; mathematical constructions were viewed as meaningful to the extent that they could be actualized in the awareness of the mathematician. The Kripke semantics formalizes this idea of the “creative subject” (see Van Dalen, 1983, p. 175) in terms of different stages of knowing and thereby provides a semantical description of the structure of knowledge appropriate to intuitionistic mathematics. The most natural and complete expression of the Kripke semantics is found in the sheaf semantics of a topos, in which the structure of knowledge is self-referral: the values of the knower and the known are identical.

The sheaf semantics of a topos provides a striking mathematical parallel to the three-in-one structure of pure knowledge as described by Maharishi Vedic Science. The Samhita value here corresponds to the...
underlying categorical structure of the topos. The sheaf semantics itself is then just a viewpoint that “sees” the topos as a field of sets on the basis of a particular intellectual formulation of a knower-known relationship within the topos. More fundamentally, the topos is just a category and there are no elements of the objects; there are no sets. Likewise, in his formulation of Maharishi Vedic Science, Maharishi has emphasized that the three-in-one structure of pure knowledge is ultimately just an intellectual conception: Rishi, Devata, and Chhandas are creations of the intellect.

In the three-in-one structure of the Samhita of Maharishi Vedic Science, one locates not only the source of all streams of knowledge, but also the synthesis and unification of all contrasting values of intellectual understanding. The central role of topos theory in modern mathematics has been in providing this value of synthesis, as we shall now discuss.

We have seen that the structure of a topos can be described in two different languages: the language of category theory, the language of arrows, and its internal, set-theoretic language, the Mitchell-Bénabou language. In the development of topos theory it is shown how these two languages can be translated into one another and provide equally valid descriptions of the topos. In this way topos theory provides a basis for the integration and unification of the viewpoints of set theory and category theory.

Topos theory provides at the same time a profound integration of intuitionistic and classical mathematics by showing how to construct specific toposes that are models for the fundamental theories of intuitionistic mathematics. In this way an intuitionistic theory is understood to be the development of mathematics in the internal language of a particular topos, whose logic is intuitionistic and not classical. The “meaning” of the principles of the intuitionistic theory is then brought to light by sheaf semantics.

The construction of the topos-theoretic models for intuitionistic theories is based upon a very powerful and general construction that is intimately connected to the set-theoretic technique of forcing. This is the construction of the classifying topos of a geometric theory. See Makkai & Reyes (1977).

A geometric theory is an axiomatic theory for which the symbolic structure of the axioms satisfies certain technical restrictions. Geomet-
ric theories form a very rich class of mathematical theories, and the study of the models of such theories is of great foundational interest. Every geometric theory has a classifying topos, which contains the generic model of the theory. The generic model is a specific model of the theory that embodies the infinite organizing power of the theory; the classifying topos is the universe in which the generic model lives.

The generic model has the following special property: all possible models of the theory, in all possible toposes, can be obtained from the generic model by applying different transformations (geometric morphisms) to the classifying topos. Thus the total organizing power of the theory is captured, in seed form, in the generic model of the theory. The generic model has the ability to speak for all possible models, and it alone has this ability.

The classifying topos can itself be constructed from the syntactical structure of the theory, that is, the symbolic structure of the expressions of knowledge. This fundamental construction of topos theory provides a new insight into the mechanics of transformation of the structure of pure knowledge (the symbolic structure of a theory) into the total expression of its organizing power (the generic model in the classifying topos of the theory).

This theme of transformation of pure knowledge into the expressions of its organizing power is the theme of the Brahmana aspect of the Vedic literature, as we considered in some detail in Section 8 in the context of the set-theoretic technique of forcing. The topos-theoretic technique of forcing, based upon the theory of the classifying topos, provides a further step of extension and perfection of this theme of set theory.

The topos-theoretic technique of forcing is applied to create toposes displaying desired properties. In this technique, one first formulates one’s intention in the form of axioms for a geometric theory and then constructs the classifying topos of the theory, using the standard topos-theoretic construction. See Scedrov (1984).

Here the intention is directly expressed in the symbolic language of a geometrical theory, the mathematical expression of “pure knowledge.” This structure of knowledge is then mathematically transformed into the expression of its organizing power, which is the fulfillment of the intention.
The set-theoretic applications of forcing can be understood as special cases of the more general topos-theoretic technique. The Boolean universes that arise in this context are special examples of toposes, in which the “set of truth values” corresponds to a Boolean algebra. In this special case the rules of inference of classical logic are valid, even though there are infinitely many truth values, not just true and false. In the set-theoretic applications, one wants to create universes in which classical logic is valid; one therefore includes in the axioms of the geometric theory special axioms that force the logic to be boolean.

The topos-theoretic approach to forcing can also be naturally applied to construct models of intuitionistic theories; it has in fact been used to construct models of virtually every intuitionistic theory of interest. Topos theory in this way has integrated intuitionistic mathematics into the framework of classical mathematics.

As topos theory is itself a development of category theory, the set-theoretic foundation for category theory provides a natural foundation for topos theory as well. The different toposes that provide the models for the different theories within intuitionistic mathematics are then actualized as specific sets within the universe of sets, the Samhita of set theory. Topos theory thus provides that supremely integrative viewpoint capable of locating the totality of intuitionistic mathematics, which directly contradicts the set-theoretic formulation, within the Samhita of set theory, the universe of sets.

Topos theory plays the role of Vedanta in the unified structure of modern mathematics. Vedanta is that area of the Vedic literature devoted to the development of Brahman Consciousness, the fully matured state of enlightenment. Maharishi has described how Brahman Consciousness is structured in awareness through a process of synthesis, in which the infinite diversity of creation is synthesized into the unmanifest, transcendental wholeness of the Self. Maharishi has further commented on the way in which different aspects of the Vedic literature present different intellectual approaches to the same underlying reality; Vedanta presents that holistic vision that synthesizes all these contrasting values of intellectual understanding. Topos theory very naturally plays the role of Vedanta in the unified structure of modern mathematics: it synthesizes all the contrasting foundational viewpoints in the holistic structure of the universe of sets, the math-
mathematical expression of the transcendental wholeness of the Samhita of Maharishi Vedic Science.

It has been suggested by some mathematicians that topos theory might provide a new foundation for mathematics. It seems to the author that the great contribution of topos theory to the foundations of mathematics has not been to usurp the set-theoretic foundation, but rather to establish its completeness by synthesizing intuitionistic mathematics within it. The most fundamental applications of topos theory have been in the construction of models for intuitionistic theories, including totally new theories, such as synthetic differential geometry. These models are provided by Grothendieck toposes, whose construction makes essential use of set-theoretic arguments. It seems therefore most natural to base topos theory on set theory, and see the role of topos theory as bringing to light the extraordinary self-referral structure of knowledge in certain sets called toposes, and thereby establishing one further value of synthesis in the universe of sets, the Samhita of set theory.

10. The Axiom of Determinacy

In the last section we saw how the viewpoint of intuitionistic mathematics, which directly contradicts the set-theoretic approach, can be integrated into the set-theoretic foundation; this step of synthesis was based upon topos theory. In this section we shall consider an exciting new development in set theory that presents one further step of synthesis in the holistic structure of the universe of sets. Here the synthesis will involve two contradictory formulations of set theory and will be achieved only on the basis of the insight into the nature of the infinite provided by “large” large cardinals. The main theme of this section was suggested to the author in January 1986 by Paul Corazza.

We considered in Section 6 the standard formulation of set theory based on the Zermelo-Fraenkel axioms, including the Axiom of Choice. An alternative formulation of set theory much studied in recent years involves replacing the Axiom of Choice by the Axiom of Determinacy (see Jech, 1978; Kanamori & Magidor, 1978).

The Axiom of Determinacy is formulated in terms of the following infinitary game. There are two players, $A$ and $B$. At the beginning of the game a set $S$ is specified, whose elements are infinite sequences of
natural numbers. The players then alternately choose natural numbers, starting with player \( A \). This generates an infinite sequence of natural numbers. If the sequence generated is in the set \( S \), then player \( A \) wins; otherwise player \( B \) wins. The Axiom of Determinacy asserts that for any set \( S \) of sequences of natural numbers, there will always be a winning strategy for either player \( A \) or player \( B \).

A winning strategy for \( A \) means that, whatever choices \( B \) makes, \( A \) can make each choice (based upon \( B \)’s previous choices) in such a way that he is assured to win. A winning strategy for \( B \) means that, whatever choices \( A \) makes, \( B \) can make each choice in such a way that he is assured to win. Obviously, for a given set \( S \), there can never be a winning strategy for both \( A \) and \( B \). However, it is not at all obvious that for every set \( S \) there must be some winning strategy for either \( A \) or \( B \).

It can be shown in fact, using the Axiom of Choice, that there must exist some set \( S \) for which neither \( A \) nor \( B \) has a winning strategy. This means that the Axiom of Choice is not compatible with the Axiom of Determinacy; if the Axiom of Choice is true, then the Axiom of Determinacy must be false.

When the Axiom of Determinacy was introduced by Mycielski and Steinhaus in 1962, it was known to be incompatible with the Axiom of Choice, but it seemed to be consistent with the other Zermelo-Fraenkel axioms. It was soon discovered that the development of mathematics based upon the Axiom of Determinacy, rather than the Axiom of Choice, gave rise to an extraordinarily interesting theory.

It was found in particular that in an Axiom-of-Determinacy world, sets of real numbers are much better behaved than in the standard Axiom-of-Choice world. For example, all sets of real numbers are Lebesgue measurable, so the Banach-Tarski paradox does not arise. The type of irregular, random structure displayed by nonmeasurable sets, whose existence can be inferred from the Axiom of Choice, is simply not found in an Axiom-of-Determinacy world.

Furthermore, it was found that the Axiom of Determinacy had an extraordinary ability to resolve combinatorial questions that were in principle unanswerable on the basis of the standard set-theory axioms. It seemed that the Axiom of Determinacy gave one the ability to compute almost anything one would like; these computations were just not possible in the usual Axiom of Choice world.
The elegance of the development of set theory based upon the Axiom of Determinacy suggested the fundamental importance of this axiom for the development of mathematics, yet its incompatibility with the Axiom of Choice made it unclear how it could be incorporated within the framework of the standard set-theoretic foundation. The answer, discovered by Martin and Steel in 1986, is that the Axiom of Determinacy describes a world called \( L(R) \), which is a subuniverse of the universe of sets, \( V \). The subuniverse \( L(R) \) consists of all sets constructible from the set \( R \) of real numbers. These sets are generated by a process of transfinite recursion, resembling the iterative mechanics of set formation, with the following differences:

1. One begins with (the transitive closure of) \( R \) rather than the null set.
2. The partial universe at level \( \alpha + 1 \) is defined to consist of all definable subsets of the partial universe at level \( \alpha \).

Here a definable subset is one for which the criterion for membership in the subset can be expressed by a formula in the symbolic language of Zermelo-Fraenkel set theory.

This definition of successive layers of the universe \( L(R) \) contrasts with the iterative mechanics that generates \( V \), in which the partial universe at level \( \alpha + 1 \) consists of all possible subsets of the partial universe at level \( \alpha \).

The significance of step (2) is that the dynamics of cosmic intelligence involved in creating arbitrary subsets of an infinite set is disallowed; one includes only subsets that can be characterized in a concrete way as “the set of all elements having such and such a property.”

It is not difficult to show that the subuniverse \( L(R) \) satisfies all the Zermelo-Fraenkel axioms, excluding the Axiom of Choice. Martin and Steel showed that if a supercompact cardinal exists, then \( L(R) \) also satisfies the Axiom of Determinacy. On this basis, \( L(R) \) has emerged as the natural example of an Axiom-of-Determinacy world, lying within the holistic structure of the universe \( V \) of sets.

These recent developments have brought to light the way in which the holistic knowledge of the infinite contained in the theory of large cardinals can provide a natural foundation for the computing ability.
available through the Axiom of Determinacy. This theme has a profound parallel in Maharishi Vedic Science in the field of Jyotish. Jyotish is that area of the Vedic literature that is most directly concerned with mathematical values. It contains procedures to precisely compute the way in which the diverse expressions of natural law sequentially unfold in time. Maharishi has explained how the computing ability of Jyotish derives from the knowledge of the precise mechanics of sequential progression of the Veda, which provides the blueprint for all sequential processes in nature.

The computing ability of Jyotish finds its ultimate expression in the level of awareness called Jyotish Mati Pragya. Jyotish Mati Pragya is described as an “all-knowing” level of awareness, capable of spontaneously computing the sequential values of transformation governing any process of change in creation. Maharishi has stated that this supreme computing ability dawns naturally in the state of enlightenment, in which the individual awareness is fully open to the self-interacting dynamics of the Samhita. It is the liveliness of the holistic value of natural law in awareness (the theme of Vedanta) that lies at the basis of the spontaneous computing ability of Jyotish Mati Pragya.

The recent developments in definability theory have given mathematical expression to this theme of relationship of Jyotish to Vedanta: the holistic knowledge of the infinite provided by supercompact cardinals lies at the basis of the extraordinary computing ability available through the Axiom of Determinacy.

This striking development in the theory of large cardinals has led to a new approach to the continuum problem, the problem of determining the precise infinite cardinality of the continuum, $|P(N)|$. This question, posed by Cantor over a hundred years ago, has been shown through the technique of forcing to be unanswerable on the basis of the Zermelo-Fraenkel axioms and the known large cardinal axioms. This situation has created some despair in the mathematical community of ever answering this question, and has even raised serious concern whether the question is meaningful in any ultimate sense.

The recent developments regarding the Axiom of Determinacy, however, suggest that this problem might be solved based upon the formulation of some new large cardinal axiom. The idea is that the subuniverse $L(P(R))$, consisting of all sets constructible from the power set $P(R)$,
should be characterized by some special, as-yet-unknown axiom, playing the role that the Axiom of Determinacy does for $L(R)$. This axiom should be powerful enough to allow one to compute the cardinality of the continuum in $L(P(R))$, which is then necessarily equal to the cardinality of the continuum in $V$. The fact that $L(P(R))$ satisfies this new axiom should itself be a consequence of some new large cardinal axiom, not yet formulated. The implementation of this program, if successful, will provide a deep mathematical expression of the theme of “computing the uncomputable” on the basis of the holistic knowledge of the infinite provided by the theory of large cardinals.

The parallel between the recent mathematical developments in definability theory and the theme of Jyotish suggests that the theory of large cardinals may have a significant role to play in the future development of modern science, and in particular the physics of the unified field. Jyotish, after all, is that area of Maharishi Vedic Science concerned with the mathematical computation of the sequential unfoldment of natural law in the diverse phenomena of creation; these computations are based upon the holistic knowledge of the self-interacting dynamics of the Samhita.

We analyzed in Section 7 how the mathematical description of the sequential unfoldment of natural law in modern science is directly grounded in the mathematics of the infinite, in particular in the theory of the continuum. Although this identifies the basis of the mathematics of modern science in a reasonably deep level of understanding of the infinite, this level of mathematical understanding does not yet fathom the more holistic values of infinity expressed by the large cardinals. It seems only natural that the complete understanding of the holistic value of natural law in modern physics should be grounded in a level of mathematics that most fully comprehends the holistic value of the Samhita, and this is the theory of large cardinals. Such a development would unify the deepest levels of knowledge of mathematics and physics, and would exalt modern science to a level at which it begins to do justice to the theme of complete knowledge in Maharishi Vedic Science exemplified in Jyotish. [See the article The Wholeness Axiom, this volume, for recent developments along these lines—Ed.]
11. The Fulfillment of Mathematics

A discussion of the fulfillment of mathematics, or of any academic discipline for that matter, is certain to raise the eyebrows of academicians. After all, is it not a cornerstone of the scientific method that knowledge is gained through an open-ended inquiry, in which new paradigms continue to emerge to displace the old. Any suggestion of a specific paradigm presenting the fulfillment of a discipline must be suspiciously viewed as an intrusion of dogma into the open-minded spirit of inquiry of science.

When one extends the theme of fulfillment to the daily life of the individual, presenting the concept of a life free from problems with satisfaction of desires on every level, then the response must surely be unequivocal disbelief. Have not centuries of human experience attested to the inevitability of problems and suffering in life? Furthermore, is it not the problems of life that kindle the creative spirit of mankind to surmount, through his own intelligence, these problems and on this basis bring about all progress in life?

The suggestion that fulfillment is possible not only for the individual but for the world as a whole, where the family of nations functions in perfect harmony, with each nation fully maintaining its own cultural integrity and bestowing each of its citizens with prosperity, certainly is a vision that must be met with incredulity. Have not the millennia of recorded history demonstrated the inevitability of conflict both between nations and within society?

Against this background of skepticism, Maharishi has boldly presented in his Maharishi Vedic Science an understanding of life that promises to bring fulfillment to every aspect of life of the individual, the society, and the whole world. In the same breath, Maharishi has upheld the integrity of the objective approach to knowledge of Western science and has insisted that any intellectual understanding, to be accepted as valid, must be substantiated through the systematic procedure of repeatable experimentation, which is the very essence of the scientific method. He has, in fact, continually encouraged scientists in all fields to subject the claims of Maharishi Vedic Science to the most rigorous scientific inquiry and experimentation.

To date, well over six hundred research studies have been published that have systematically tested the predictions of Maharishi Vedic Sci-
ence for the life of the individual, the society, and the world; these studies cover the full range of scientific inquiry into the nature of life, from physiology and psychology to sociology and ecology. The striking outcome of this comprehensive research program has been that essentially all of the predictions tested have been substantiated. For a comprehensive review of the literature, see Wallace (1986); Orme-Johnson and Dillbeck (1987) contains a review of the research documenting the sociological benefits resulting from the group practice of the Transcendental Meditation and TM-Sidhi programs.

Beyond these experimental studies, there is the subjective validation of the experience of higher states of consciousness by millions of individuals throughout the world who practice Transcendental Meditation and the TM-Sidhi programs. The power and completeness of Maharishi Vedic Science derives from the fact that it is grounded in experience and contains systematic and repeatable procedures for sequentially unfolding the higher stages of human development.

At the heart of Maharishi Vedic Science is the experience of the simplest state of awareness, the state of Transcendental Consciousness; this is the field of pure intelligence, in which the three-in-one structure of the Samhita can be directly perceived. In the experience of pure intelligence, consciousness knows itself. The reality of this experience cannot be challenged by any amount of intellectual disputation; it is the common experience of meditators throughout the world. Further, with the growth of clarity and permanence of the inner experience of Transcendental Consciousness these individuals have enjoyed the practical benefits in all areas of life predicted by Maharishi.

Maharishi presents a paradigm of life in fulfillment, a fully developed state of human life in which consciousness is fully awake within itself. In this state of enlightenment the individual enjoys not only inner peace and fulfillment, but also supreme enjoyment of every aspect of living. Life in this state is completely natural and spontaneous and has a powerful, beneficial influence on the environment. Maharishi describes the life of an enlightened man as a life of perfection, a life free from mistakes.

The experience of meditators throughout the world has documented the growth of life in the direction of enlightenment. There is furthermore abundant experience of the state of pure consciousness, with the
great fulfillment it brings to the heart and the mind, an experience that by its nature is the expression of perfection itself. It does not seem an unreasonable extrapolation to envision a stage of human development in which the inner experience of enlightenment is fully established, and the perfection of its nature is fully expressed in the diverse spheres of activity of the individual. Those of us who have been fortunate enough to work with Maharishi over the years have witnessed, through Maharishi’s own example, the supreme height of intelligence, creativity, dynamism, and compassion to which man can rise.

The state of enlightenment, as the fulfillment of life, very naturally fulfills every channel of intellectual inquiry, and therefore should be identifiable as the fulfillment of each academic discipline. To conclude our analysis of the relationship between Maharishi Vedic Science and modern mathematics, it is natural to see to what extent the structure of the state of enlightenment, available through Maharishi Vedic Science, can be identified as the goal of mathematical progress.

There are certainly many directions in which mathematical knowledge progresses. Most reduce to exploration of different consequences of the Zermelo-Fraenkel axioms. But one direction is of a fundamentally different character, the direction of expansion of the foundation of mathematical knowledge. This involves the systematic formulation, study, and adoption of new axioms to extend the axiomatic foundation of set theory towards completeness. The most important of these are the large cardinal axioms, asserting the existence of mathematical infinities displaying extraordinary new qualities of wholeness.

One of the striking features of the theory of large cardinals is that all the important large cardinal concepts thus far formulated are themselves linearly ordered in terms of their strength (or consistency strength), that is, they are organized in a sequence of progressively more and more powerful axioms, describing more and more holistic expressions of infinity. This sequence defines a natural direction of development in the theory of large cardinals.

A fundamental characteristic of this direction of development is the attainment of increasingly profound stages of synthesis or unification of mathematical knowledge. We examined in Section 9 how the existence of large cardinals makes possible the integration of category theory, and on that basis topos theory and intuitionistic mathematics, into the
framework of the set-theoretic foundation. Further, we saw in Section 10 that the existence of a supercompact cardinal makes possible the integration of Axiom-of-Determinacy set theory into this framework.

A second feature is the growth of knowledge in the direction of completeness. In a very obvious way, additional axioms lead to new theorems and therefore more complete knowledge. However, to really substantiate the growth of completeness, one would like to see large cardinal axioms resolve long-standing mathematical questions of fundamental significance. We saw in the preceding section that the most recent developments in the theory of large cardinals have identified a new direction of expansion of computing ability, which is likely to lead to a solution of the continuum problem as well as new applications of mathematics in the sciences. This growth of computing ability and its application is a natural expression of the increased organizing power available in the more holistic levels of mathematical knowledge.

Having identified a direction of evolution in the theory of large cardinals, it is natural to inquire whether it has a goal. It is here that Maharishi Vedic Science provides the requisite insight. Since the direction is that of expansion of intellectual comprehension to embrace more and more holistic expressions of infinity, the natural culmination and fulfillment of this process must be the awakening of the intellect to comprehend the ultimate value of infinity—the absolutely infinite value of the universe of sets, the mathematical expression of the Samhita of Maharishi Vedic Science. It is here that the goal of modern mathematics can be identified as the goal of Maharishi Vedic Science.

Maharishi Vedic Science, through its subjective technology of consciousness, the Transcendental Meditation and TM-Sidhi programs, provides the practical means to fully enliven in awareness the transcendental wholeness of the Samhita, bringing life to a state of fulfillment in which the absolutely infinite value of the Samhita is a living reality of day-to-day life. This state of enlightenment very naturally fulfills the quest in the theory of large cardinals for complete, holistic knowledge of the infinite.

What is most fundamental in creating the quality of fulfillment belonging to the state of enlightenment is the blissful nature of the experience of pure consciousness. This quality of bliss is appreciated by the most delicate level of one’s feeling. Thus to understand the relation
of mathematics to the state of enlightenment, we are led to consider the aesthetic element in mathematics.

Although it is perhaps not widely appreciated by non-mathematicians, the aesthetic element plays a prominent role in the activity of the mathematician. Most mathematicians have little concern for the applied value of their subject; they are drawn to the study of mathematics solely on the basis of the aesthetic beauty of the knowledge. This great charm in the study of mathematics has its roots in the way mathematics so directly describes the most fundamental aspects of the dynamics of intelligence.

In his famous essay, “Mathematical Creation,” Poincaré (1956) discussed the way the “aesthetic sensibility” of the mathematician guides the process of mathematical discovery. Poincaré observed that the combinations of ideas that were most beautiful were, at the same time, most useful, and therefore the aesthetic sensibility of the mathematician should spontaneously lead to the proper synthesis of ideas required for the solution of a particular mathematical problem.

On a more general level, one could say that the whole process of the development of mathematics is guided by this aesthetic value. Mathematics evolves spontaneously in the direction of knowledge that is more beautiful, where the element of beauty can be directly correlated with the conceptual elegance, universality, and abstractness of a particular theory, concept, or construction.

What is found afterwards is that the knowledge that is most beautiful proves to be most useful, both in other areas of mathematics and in the sciences. In modern physics we have witnessed the way in which more fundamental levels of understanding of natural law require progressively more abstract mathematical models for their description. The physicist today working in the most fundamental area of unified quantum field theories and superstring theories is required to be conversant with more areas of abstract mathematics than almost any pure mathematician. It is beginning to appear that every major area of abstract mathematics might very well have an important contribution to the final formulation of the physics of the unified field!

When one considers the way abstract mathematical knowledge is applied in the sciences, one sees the two sides of mathematics, pure and applied, as expressing the complementary values of pure knowledge.
and organizing power. The ability of abstract mathematical principles to organize our understanding of the physical world in terms of a few fundamental formulas is just an expression of the organizing power inherent in the structure of pure knowledge. The direction of development of pure and applied mathematics reveals that the knowledge that is most aesthetically fulfilling is at the same time the knowledge that has the greatest organizing power and is consequently the knowledge that is most powerful and practically useful to life.

The fulfillment of this theme in the development of mathematics is found in the state of enlightenment. The Samhita presents the self-referral structure of pure knowledge, in which knowledge and organizing power are in an undivided state. The awareness open to this self-referral structure of knowledge gains the quality of bliss: the aesthetic sensibility is completely fulfilled. At the same time, the infinite organizing power of natural law spontaneously organizes one’s life in a way that brings fulfillment to every aspect of living. This fulfills both the thirst for the great charm belonging to the abstract study of pure mathematics and the desire for the practical benefits of the knowledge of natural law conferred by applied mathematics.

We have observed the way the theory of large cardinals so clearly presents the theme of growth of holistic knowledge of the infinite; on this basis, we should expect the aesthetic element to have a special significance in this area of mathematical study. The author frankly does not know whether mathematicians working in this area experience greater “bliss” than mathematicians involved in other areas of mathematics, and suspects that perhaps this is not always the case. Nevertheless, if the knowledge of the large cardinals is truly lively in the awareness of the mathematician, so that his intellect really grasps these very great expressions of unboundedness, then the aesthetic sensibility must be stirred at a very profound level. This level of appreciation of the mathematics of the infinite should naturally grow in the awareness of the mathematician as he grows in enlightenment through the Transcendental Meditation and TM-Sidhi programs.

A further aspect of the relationship between Maharishi Vedic Science and the fulfillment of mathematics is the way in which Maharishi Vedic Science can lead to significant new mathematical developments. In a very obvious way, this is achieved by developing the creativity and
intelligence of the mathematician, which form the basis for all mathematical progress. The systematic growth of both intelligence and creativity through the Transcendental Meditation and TM-Sidhi programs has been substantiated by a large body of scientific research. See Wallace (1986).

In addition to the generic benefits of Maharishi’s technology of consciousness, there are several ways in which his Vedic Science can specifically contribute to the theory of large cardinals and thereby expand the axiomatic foundation of set theory, as we shall now discuss.

In Section 6 we indicated that the justification for new, more powerful large cardinal axioms must take direct recourse to the subjective faculty of mathematical intuition, that very fine level of intellect that can directly recognize the truth of the first principles of mathematics, the axioms of set theory. To validate more and more powerful large cardinal axioms, progressively more profound aspects of mathematical intuition must be called into play. This means that continued progress in this direction of mathematical development requires the continued refinement of the subjective faculty of mathematical intuition.

Gödel himself recognized the importance of this direction of development of the mind of the mathematician, but he commented that any procedure for systematically accelerating this development would require a substantial deepening of our understanding of the basic operations of the mind. See Wang (1974, pp. 324–6). Maharishi Vedic Science provides precisely the required understanding of the fundamental dynamics of intelligence; the Transcendental Meditation and TM-Sidhi programs provide the practical formula for applying this knowledge for the systematic development of mathematical intuition. Maharishi Vedic Science can in this way accelerate the development of mathematics in the direction of complete, holistic knowledge of the infinite.

In this context, the theoretical aspect of Maharishi Vedic Science can also play a fundamental role in providing motivation for new set-theoretic axioms. In this article we have seen the way the principles of Maharishi Vedic Science, which describe the intrinsic dynamics of intelligence, are directly mirrored in the deepest principles of set theory describing the holistic nature of the infinite. This was seen in the Reflection Principle, which provides the primary justification for the
“smaller” large cardinals, and was further developed in our consideration of elementary embeddings of the universe of sets, which provided the justification for the known “large” large cardinals. For the continued development of the theory of large cardinals in the direction of more and more holistic comprehension of the infinite, motivation must be found for introducing new axioms. The principles of Maharishi Vedic Science, which describes the ultimate value of infinity and its internal dynamics, can be expected to provide the necessary motivation. [See the article, *The Wholeness Axiom*, this volume, for new developments in this area.—Ed.]

In considering the relationship between mathematics and Maharishi Vedic Science, we have identified the role of Maharishi Vedic Science both in contributing to the development of mathematics and in locating the fulfillment of this process of development in the state of enlightenment. At the same time, modern mathematics can be seen to offer its own contribution to the growth of enlightenment through Maharishi Vedic Science.

Through the Transcendental Meditation and TM-Sidhi programs, the intellect of the student is made lively at progressively more refined levels; the study of mathematics then provides a channel through which the mind gains facility in functioning, in a perfectly orderly way, at increasingly abstract levels of thought. The principles of mathematics are themselves seen to present, in exact, quantitative terms, the structure and dynamics of the student’s own intelligence; this makes the intellectual understanding of the self-referral structure of intelligence more complete and brings satisfaction to the intellect of the student. The student grows in the awareness that all expressions of structure and relationship in creation are just the quantified values of his own pure intelligence.

This balanced growth of experience and understanding finds its culmination in the state of enlightenment, in which the complete enlivenment of the self-referral structure of the absolute infinite in one’s life brings fulfillment to both the heart and the mind. Maharishi Vedic Science in this way makes the study of mathematics significant as a component of the path to enlightenment.


12. New Directions

In this paper we have presented the first steps of exploration of the relationship of modern mathematics to Maharishi Vedic Science. We have described a number of parallels between principles of Maharishi Vedic Science and the most fundamental principles on which the whole structure of modern mathematics is based. This has allowed us to place the discipline of mathematics within the holistic framework of Maharishi Vedic Science. The fulfillment of mathematics can thereby be located in the state of enlightenment, available through Maharishi Vedic Science.

Foundational areas of mathematics were chosen as the focus of this paper because they seem to be a natural starting point for the study of the relationship between mathematics and Maharishi Vedic Science. We conclude this paper by briefly suggesting several further directions for research into this theme.

The primary concern of this paper has been to identify the three-in-one structure of the Samhita at the foundation of set theory. Maharishi has described in detail the sequential emergence of all the diverse aspects of the Vedic literature from the self-interacting dynamics of the Samhita. If mathematics indeed reflects the complete structure of knowledge that constitutes Maharishi Vedic Science, then it should be possible to identify the mathematical parallels to these diverse aspects of the Vedic literature and their complex network of interrelationships. Investigation in this area promises to uncover important new insights into the unified structure of the totality of mathematical knowledge.

A second direction is to consider other active foundational areas, particularly areas connected with computer science. The lambda calculus, for example, provides a striking mathematical expression of self-referral in the structure of language and should therefore provide significant mathematical parallels to the self-referral structure of pure knowledge identified in Maharishi Vedic Science.

An intimately related field is denotational semantics, the mathematical study of the intrinsic meaning of programming languages. This new field has already provided remarkable insight into the mechanics of transformation of syntax into semantics, the transformation of the symbolic structure of the language into its mathematical meaning. Denotational semantics should provide a fascinating commentary on the relationship between the structure of pure knowledge and the
expression of its organizing power, the theme of the Brahmana aspect of the Vedic literature. (For an introduction to the lambda calculus and denotational semantics, the reader is referred to Barendregt, 1981, and Stoy, 1977.)

Another relevant foundational development concerns the possible extension of ordinary set theory to a reflexive theory of classes. Maddy (1983) has shown how Zermelo-Fraenkel set theory can be imbedded in a reflexive class theory for which there is a class of all classes that contains itself as an element. The paradoxes associated with a “set of all sets” are avoided through a modification of logic, whereby the class theory is governed by an essentially three-valued logic rather than the classical two-valued logic. Such a theory makes possible the explicit self-referral expression of the membership relation and can thereby be expected to provide new insights into the relationship of Maharishi Vedic Science to the foundations of mathematics.

A final suggestion is to apply Maharishi Vedic Science to develop new mathematics. In almost every area of mathematics Maharishi Vedic Science should be able to provide insights that motivate new developments. It would be particularly gratifying to see significant progress in the theory of large cardinals motivated by the holistic knowledge of the infinite contained in Maharishi Vedic Science.

In reflecting on the richness of structure of both Maharishi Vedic Science and modern mathematics, it is obvious that the study of the relationship between these two great towers of knowledge has scarcely begun. Future developments promise to contribute much towards the enrichment and fulfillment of mathematics, and the author looks forward to the growing interest and involvement of the community of mathematicians in this area of study.

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References


THE SAMHITA OF SETS


Weinless, M. (2011). Categories and toposes: Dynamism at the foundation of mathematics. In P. Corazza and A. Dow (Eds.). *Con-


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Unified Field Chart

for

Mathematics

Faculty of the Mathematics Department
of Maharishi University of Management
MAHARISHI TECHNOLOGY
OF THE UNIFIED FIELD

The TRANSCENDENTAL
MEDITATION® Program

THE TRANSCENDENTAL
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ALLOWS THE CONSCIOUS
MIND TO IDENTIFY
ITSELF WITH THE
UNIFIED FIELD OF
ALL THE LAWS OF
NATURE, THE TOTAL
POTENTIAL OF
NATURAL LAW, IN
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TRANSCENDENTAL CONSCIOUSNESS
PURE INTELLIGENCE
UNIFIED FIELD
Maharishi Vedic Science and Technology: an Integrated Approach to Mathematics

The “Unified Field Chart for Mathematics” (previous pages) presents an overview of a new integrated approach to mathematics. The left side of the chart illustrates the interconnections of the major areas of mathematics, from its fundamental principles to its applications throughout society, and shows how the whole field of mathematics has its basis in the unified field of natural law, from where all the laws of nature govern the orderly evolution of the universe. This vision of the whole field of mathematics at a glance, on the left side of the chart, helps the student to “connect the parts with the whole,” to connect the specific areas of knowledge with the whole discipline, and with the source of all disciplines in the unified field of natural law.

The right side of the chart shows how the unified field of natural law becomes a living reality for the student through the Maharishi Technology of the unified field—the Transcendental Meditation and TM-Sidhi programs—which allows individual awareness to identify itself with the unified field of natural law, the field of pure consciousness, in the simplest state of one’s own awareness, Transcendental Consciousness. This technology enlivens the infinite organizing power of natural law inherent in the unified field in one’s own thoughts and actions, for a life increasingly lived in full accord with natural law.

Discovery of the Unified Field of Natural Law

The developments in modern science, in particular in quantum physics, have opened new perspectives for a unified understanding of nature. By probing nature’s functioning at finer distance scales, modern science has made profound advances towards a unified view of the structure and dynamics of matter, culminating in the discovery of a unified field of all force and matter fields, a single self-interacting field from which the physical universe, and all the laws of nature, emerge.

Historically, the analysis of the microscopic structure of matter began with the idea that all substances are composed of tiny particles, like atoms and their subatomic constituents. With the development of quantum theory, however, physicists soon had to conclude that the classical particle picture is inadequate for the description of these con-
stiuents of matter; they realized that the different elementary particles have to be conceived as specific resonant excitations of fundamental quantum fields.

Prior to the development of unified field theories, scientists had discovered a variety of separate quantum fields, such as the four force fields (electromagnetic, weak, strong, and gravitational interactions) and the various matter fields. In the last few decades it was realized that with the progression towards finer distance scales, laws of nature are increasingly unified, so that previously separate quantum fields are found to be merely different components of underlying unified quantum fields. This process of unification culminates in a complete unification at the level of the Planck scale (10^{-33}cm) where all the various force and matter fields are unified into a single unified field of natural law, the holistic unmanifest field underlying the manifest universe. The most recent and successful unified field theory, superstring theory, derives all the different force and matter fields from the different vibrational modes of a single underlying supersymmetric unified field—the superstring field. The dynamism at this fundamental level of the Planck scale is governed by natural law in its completely unified form. All diversity at more superficial levels emerges from this unified state of natural law by a process which is called sequential symmetry breaking.

The unified field can be understood as the fountainhead of natural law, since all the laws of nature expressed in the effective field theories governing physics at larger distance scales are already contained in their most compact form in the original supersymmetric Lagrangian of the unified field. As such, the unified field represents the most concentrated field of intelligence in nature.

The Properties of the Unified Field
Are the Properties of Pure Self-Referral Consciousness

The fundamental properties of the unified field include the properties of being a non-Abelian gauge field that incorporates quantum gravity. These include the property of self-referral, or self-interaction, which is reflected in the Lagrangian, the fundamental mathematical formula quantifying the laws of nature at the level of the unified field.

Maharishi Vedic Science, the science of consciousness, brings to light that pure consciousness alone is fully self-referral: that is, only
consciousness can know itself fully from within itself. Consciousness in its self-referral state—pure consciousness, or Transcendental Consciousness—is the silent source of all mental activity, a field of pure intelligence, which has inherent within it infinite creative dynamism. Since the fundamental properties of the unified field are identical to those of consciousness in its self-referral state, it is natural to conclude that the unified field of natural law, as described by modern physics, and the field of pure consciousness, as described by Maharishi Vedic Science, are equivalent. Through the subjective methodology provided by the Maharishi Technology of the Unified Field—the Transcendental Meditation and TM-Sidhi programs—it is possible to verify that these two fields are identical.

The Maharishi Technology of the Unified Field—
The Transcendental Meditation and TM-Sidhi Programs
Through the Transcendental Meditation program, individual consciousness expands to its unbounded value, pure consciousness, Transcendental Consciousness, experienced as the simplest form of awareness. At this level, consciousness is identified with the unified field of natural law. The more advanced TM-Sidhi program, including Yogic Flying, habituates the awareness to think and act from this level of pure consciousness, the unified field; this practice has been shown to greatly enhance coherent brain functioning and mind-body integration. As this field becomes established in individual awareness through these technologies of the unified field, the individual grows in greater achievements and success, and experiences increasing fulfillment. Moreover, individual life is supported by the laws that govern life, since individual aspirations no longer meet with resistance, and life is lived spontaneously in accord with natural law. This expansion of life as a result of regular experience of Transcendental Consciousness suggests that the field of pure consciousness is truly the field that underlies the functioning of all of natural law; that pure consciousness and the unified field discovered by modern physics are one and the same. See Hagelin, (1987) for a fuller discussion of these points.
Scientific Validation of the Technology of the Unified Field

More than 600 scientific research studies, conducted at over 200 universities and research institutes in 30 countries, validate the benefits of the Maharishi Transcendental Meditation and TM-Sidhi programs for mind, body, behavior, and society. One of the most important implications of research on these programs is that they have the effect of enlivening total functioning of the brain, which leads to the full expression of human intelligence and creativity. Research in developmental neurophysiology has found that specific types of experience are necessary for specific areas of the brain to develop. For total brain functioning to develop, the experience of self-referral consciousness—unbounded awareness—is required, which is easily gained through the Maharishi Transcendental Meditation program.

Maharishi Vedic Science—The Science of Consciousness

Complete knowledge of consciousness is contained in the most ancient record of knowledge, Rk Veda and the Vedic literature, which Maharishi has organized and brought to light for its full theoretical and practical values in his Vedic Science. Unlike other sciences based solely on objective investigation, Maharishi Vedic Science offers total knowledge of the subject, the knower; the object, the known; and the process of knowing, which connects them. These are the fundamental qualities within the unified wholeness of consciousness. These qualities spontaneously arise due to the self-referral property of the unified field—it knows itself. In the language of the Vedic literature, the unified field is called Samhitā. The knower is Rishi, the process of knowing is Devatā, the known is Chhandas. Thus Maharishi Vedic Science is the science of the unified field of knower, knowing, and known—the science of unity and diversity at the same time.

The Samhitā of Rishi, Devatā, and Chhandas is available in Rk Veda, which expresses the fundamental reverberations of natural law within the unified field of pure consciousness. These reverberations or self-interacting dynamics of consciousness expressed in Rk Veda are further elaborated in the whole Vedic literature and expressed in the infinite diversity of natural law governing the ever-expanding universe. Rk Veda is therefore the Constitution of the Universe, the fundamental structure of natural law whose perfect order is expressed in the order
and harmony displayed throughout the universe. Rk Veda—the Constitution of the Universe—is the unseen level of intelligence present within every grain of creation.

Professor Tony Nader, M.D., Ph.D., has made the profound scientific discovery that the detailed structures of the fundamental laws of nature expressed by Rk Veda and the Vedic literature correspond precisely to the fundamental structures and functions of human physiology. This historic discovery brings to light that the total intelligence of natural law governing the universe is the inner intelligence of every human being; every individual is really cosmic, and has the potential to live perfection in daily life.

The Unified Field as the Basis of Mathematics

Unified Structure of Modern Mathematics

The left side of the chart shows the unified structure of modern mathematics, grounded in the field of pure intelligence, the unified field of natural law. The whole range of mathematics is shown: its unmanifest source in the field of pure intelligence, the unified field of natural law (Level 1); the conceptual representation of the infinite wholeness of the unified field of pure intelligence provided by set theory, which is the unified foundational theory of modern mathematics (Level 2); the elements that collectively structure mathematical knowledge—the fundamental concepts, methodology, and fundamental structures (Level 3), which are themselves formally grounded in set theory; the diverse theories of modern mathematics (Level 4), which emerge from the interaction of these elements; and the applications of mathematics (Level 5), serving all areas of national life administered by government.

The Set-Theoretic Foundation of Modern Mathematics

Set theory plays the role of the “unified field theory” of modern mathematics; it provides a unified foundation for the diverse theories constituting modern mathematics (Level 4). The unified structure of set theory is expressed in terms of a single type of object, a set, and a single primitive value of relationship, the membership relation. From this
unified structure of knowledge, all the diverse expressions constituting the body of modern mathematical knowledge can be unfolded.

The basic elements that underlie the structure of mathematical knowledge are presented in Level 3 of the chart. These elements of mathematics are organized into three boxes that express the three values of knower, process of knowing, and known: fundamental concepts (knower), methodology (process of knowing), and fundamental structures (known).

In axiomatic set theory, these three values are represented by the three diagrams within the box at Level 2 of the chart. The basic structural concepts are the concepts of set, element, and membership relation. The formal methodology of axiomatic set theory is provided by the sequential unfoldment of the theorems from the axioms, using the principles of logical inference. The structures studied in axiomatic set theory are all possible sets, sequentially unfolding from the unmanifest point value of the null set and collectively structuring the transcendental wholeness of the universe of sets, the ultimate mathematical expression of infinity.

In the context of the set-theoretic foundation, the elements of modern mathematics depicted in Level 3 of the chart are given a formal foundation in the corresponding elements of axiomatic set theory in Level 2. In this way, the structure of knowledge of axiomatic set theory in Level 2 formally contains the totality of knowledge of modern mathematics depicted at Level 4, and provides the foundation for the practical applications of modern mathematics depicted at Level 5. The power of these diverse applications of mathematical knowledge reveals the extraordinary organizing power contained in the abstract knowledge of the infinite provided by set theory.

**Maharishi Vedic Mathematics**

The unified foundation for modern mathematics provided by set theory is based upon the conceptual representation of the infinite—a conceptual hierarchy of greater and greater infinite values that collectively structure the ultimate mathematical infinity, the transcendental wholeness of the universe of sets. Maharishi Vedic Mathematics fulfills this intellectual understanding of infinity through the direct experience of the infinite, unbounded field of pure intelligence and the self-interacting dynamics
through which this absolute infinity quantifies itself, giving rise to all quantitative values and their relationships. This is the experience of the Absolute Number, which is the source of all numbers and all values of mathematical relationship. Indeed, Maharishi’s Absolute Number initiates and maintains orderliness and precision throughout the entire universe. The Transcendental Meditation and TM-Sidhi programs provide the practical means for anyone to gain the experience of the Absolute Number in one’s own simplest state of awareness. On this basis one rises to the status of a Vedic Mathematician, when natural law spontaneously computes all the laws of nature required to fulfill every desire. The mathematician enjoys the fruit of all mathematical knowledge: life free from mistakes, life lived with mathematical precision in perfect accord with natural law.

**Life in Accordance with Natural Law**

The Transcendental Meditation and TM-Sidhi programs culture the ability to think and act from the level of the unified field of natural law, spontaneously in harmony with the evolutionary direction of natural law. Such thought and action gains the full support of natural law for the effortless achievement of desires. When the unified field of natural law is fully enlivened and stabilized in individual awareness, one is living the full potential of one’s creative intelligence in higher states of consciousness; one’s total brain functioning is fully enlivened. Gaining the full support of nature through development of the full creative potential of consciousness makes individuals masters of their lives. They spontaneously command situations and circumstances and control their environment; their behavior is always spontaneously nourishing to themselves and everyone around them. They become ideal citizens, able to fulfill their interests without jeopardizing the interests of others.

**National Law Upheld by Natural Law**

Natural law governs life on every level of creation, from the submicroscopic world of elementary particles to the large-scale structure of the universe. National law governs the life of the nation and has its ultimate basis in natural law. Natural law has its unified foundation in the unified field of all the laws of nature. The chart displays how all the diversified values of
natural law, as discovered by modern science, emerge from this unified level of natural law. Ultimately, the diversified structure of natural law displayed throughout creation is reflected in the diversity of human nature and in the innumerable tendencies expressed in different lands throughout the world, and even within the borders of individual nations. These diverse trends and tendencies displayed by the individual citizens of a nation give rise to the necessity for man-made laws; national law regulates and administers the trends and needs of the various segments of society. Ideally, national law is in harmony with natural law, promoting all good for everyone, and upholding the progress and evolution of all citizens, the nation, and the family of nations.

**Enlivening the Unified Field of Natural Law in the Whole Society**

Successful administration of the nation requires the government to satisfy all citizens and sectors of society, and organize for the mutual fulfillment of all areas of national life. The only practical and proven way to achieve this is to create an integrated national consciousness through a large group of Yogic Flyers—experts in the TM-Sidhi program. This group, by enlivening the holistic, evolutionary qualities of the unified field of natural law in the whole collective consciousness, creates an indomitable influence of harmony, or coherence, in the nation, neutralizing collective stress, and raising national life to be increasingly in accord with natural law. Scientific research has documented that the practice of the TM-Sidhi program in large groups results in increased positive trends in national and international life, including decreased crime rate, decreased social turbulence and violence, decreased international conflicts, decreased accidents, improved social health, and improved economic trends.

When the government maintains a coherence-creating group of Yogic Flyers—a “Group for a Government”—natural law fully supports national law: the government is able to satisfy all the different interests in society, and the nation enjoys increasing progress, prosperity, cultural integrity, and invincibility. A permanent Group for a Government practicing the Transcendental Meditation and TM-Sidhi programs can be easily established through education, the military, or
business and industry. Then government administration will become as perfect as the administration of nature through natural law, enjoying “automation in administration,” functioning as effortlessly and efficiently as the government of nature, which administers the entire universe without a problem. When the man-made constitution of every nation is in full alliance with the eternal Constitution of the Universe, perfection in individual life and society becomes a reality and the world will enjoy lasting harmony and peace.

References
Creating Heaven on Earth

go through Maharishi Vedic Science and Technology:

Mathematics

Faculty of the Mathematics Department
of Maharishi University of Management
Abstract

The present article is one chapter from the book Creating Heaven On Earth Through Maharishi Vedic Science and Technology, first published in 1989. It provides an accessible look into the way in which the dynamics of pure intelligence, as described in Maharishi Vedic Science, are given expression in the field of mathematics. As such, it provides a commentary on the Unified Field Chart (see previous pages), which depicts diagrammatically the unfoldment of the entire range of mathematics from its unified source in the foundations of mathematics, which in turn has its basis in the structuring dynamics of pure consciousness itself, within the transcendental field.

The article discusses the relationship between pure intelligence and the intelligence of the mathematician, and how this relationship leads to the unique methodology that is found in mathematics, and how it results in a precise language for describing the functioning of nature. The home of all mathematical knowledge is found to exist in the “unified field” of mathematics—Set Theory. The sequential unfoldment of the universe from the empty set through self-referral applications of set-theoretic operators and the unfoldment of all mathematics from the fundamental axioms about sets are seen as the first sprouting of pure intelligence within the field of mathematics.

The unified nature of pure consciousness itself is found represented within mathematics in the concept of the mathematical continuum. And mathematics itself is found to be lively within the field of pure knowledge, the Veda, in the form of Maharishi Vedic Mathematics, expressed in the sequential unfoldment of the various aspects of the Vedic literature.

This fundamental relationship between mathematics, the mathematician, and their source in pure consciousness is found to be the key to unlock the full potential of the awareness of the mathematician; to unlock the treasures of mathematics in the awareness of a student of mathematics; to bring mathematics itself to fulfillment by bringing fulfillment to its deepest aspirations as a field of knowledge; and to bring to all mankind the blessings of the mathematics of pure intelligence, through the Maharishi Technology of Consciousness, by which the mistake-free orderliness of nature’s functioning can be spontaneously lived in daily life.
Overview

The goal of this chapter is to instill a deep appreciation of the great beauty, charm, and power of mathematics through an understanding of the basic methodology and structure of modern mathematics in the light of Maharishi Vedic Science. We shall find that mathematics has its source in the process whereby the field of pure intelligence first quantifies itself into knower, known, and process of knowing, and is intrinsic to the entire sequential unfoldment of creation from this first step of quantification.

We shall see how this most holistic understanding of mathematics explains why mathematics is the language of science, why the study of mathematics contributes to one’s own inner development, why Maharishi Vedic Science and Technology enhances success at and enjoyment of mathematics, and how the state of enlightenment brings the discipline of mathematics to its supreme fulfillment, both on the personal level and on the level of society.

We shall glimpse the great power, universality, elegance, and unity given to mathematics by its methodology, its hierarchical organization, and its unified foundation in set theory. We shall also see how deep principles of Maharishi Vedic Science are elegantly expressed in some of the fundamental concepts of mathematics, such as the continuum of real numbers, which captures the qualities of the transcendental continuum of consciousness; the derivative of a function, which locates the dynamic structure of the laws of nature at a point; and the universe of sets, which is the mathematical expression of the transcendental wholesness of life, transcending the intellect.

By diving deeply into Maharishi Vedic Mathematics, and by our daily practice of the Transcendental Meditation and TM-Sidhi programs, we come to the ultimate appreciation of mathematics, namely, that mathematics is part of our own nature. It is simply an expression of the self-interacting dynamics of our own intelligence.

The complete fulfillment of mathematics is found in the supreme state of enlightenment for the individual and in heavenly life on earth for society, where the supreme computing ability of nature is fully enlivened in the individual and society, so that life is lived without mistakes, supported by all the laws of nature.
1. Mathematics: Fundamental Purpose and Approach

1.1 The Source and Goal of Mathematics

The field of mathematics is the exact science of order. The unbounded range of mathematical knowledge embraces, in principle, all possible patterns of orderly structure and transformation conceivable by the human mind. Deeply grounded in the exact orderly nature of intelligence, mathematics provides the language used by the sciences to quantify the exact orderly structure of the laws of nature governing every level of creation. In the words of nineteenth century mathematician James Joseph Sylvester, “The object of pure Physic is the unfolding of the laws of the intelligible world; the object of pure Mathematic that of unfolding the laws of human intelligence” [18].

In this section, we shall see that the whole discipline of mathematics emerges in a natural way from the self-referral nature of intelligence, and that this accounts for the great power and effectiveness of mathematics as the language of modern science. We shall also see that this makes mathematics fulfilling.

Mathematics is abstract by its very nature. The objects and structures studied by mathematics are purely conceptual; they include, for example, the infinite collection of counting numbers, \( \{1, 2, 3, \ldots\} \), and the geometric straight line, having infinite length but neither width nor depth. The creation of mathematical objects may be inspired by the mathematician’s observations and experiences of the physical world, but they are ultimately creations and expressions of the intelligence of the mathematician (see Figure 1). The infinite extent of the counting numbers and the line, for example, is derived from the ability of the intelligence of the mathematician to experience, comprehend, and quantify the infinite, rather than from any value of infinity perceived in the physical world. The study of mathematics is the study of the orderliness of the structure and dynamics of intelligence itself.

According to Maharishi Vedic Science, the most fundamental level of the mathematician’s intelligence is the transcendental field of pure intelligence, which is experienced in one’s own awareness during the practice of the Transcendental Meditation and TM-Sidhi programs.

Maharishi has elaborated how mathematics, as an aspect of Vedic Science, has its ultimate source in the field of pure intelligence [11].
Within the undivided unity of the Samhita, two values, existence and intelligence, can be located. It is the intelligence value that knows itself and thereby creates three values within itself—the knower (Rishi), the process of knowing (Devata), and the known (Chhandas). Thus 1 becomes 2, and 2 becomes 3. Maharishi Vedic Science thus locates the source of mathematics in the self-interacting dynamics of consciousness, whereby consciousness quantifies itself: the infinite, undivided value of pure intelligence sequentially expresses itself in quantified values of pure intelligence (refer to Figure 2). In the continuation of this sequential process, all mathematical structures and relationships unfold.

Figure 1. Careful study of a natural process such as planetary motion leads to the formulation of a mathematical model.

Maharishi describes how the self-interacting dynamics of this field of pure intelligence gives rise also to all of physical creation [9, pp. 24–35]. Just as mathematics is the intellectual expression of the self-interacting dynamics of consciousness, so the universe is the physical expression of the self-interacting dynamics of consciousness. Thus mathematics expresses intellectually what the universe expresses physically. It is no surprise, therefore, that mathematics is the language of science. Maharishi Vedic Science in this way provides a natural explanation of the “unreasonable effectiveness of mathematics in the natural sciences.” [16].

Maharishi Vedic Science identifies the goal of all human activity as life enriched by the fruit of all knowledge, life lived in its fullest values on all levels from most abstract to most concrete [2, pp. 63–68].
Mathematics contributes to the attainment of this goal by providing a foundation for science and technology, which serve society.

The Maharishi Transcendental Meditation and TM-Sidhi programs, however, make possible the attainment of this goal by allowing the individual to experience the self-interacting dynamics of pure intelligence at the most abstract level of life and to begin incorporating its full organizing power into daily life [6, pp. 399–410]. This experience, connecting the mathematician to the source of mathematics in his own awareness, makes the study of mathematics personally relevant and fulfilling.

The study of mathematics itself also contributes to the inner development of an individual. Mathematics cultures precise and orderly thinking and the ability to think and reason at more abstract levels of the mind. Mathematical research and problem solving develop intuition, creativity, and the ability to maintain broad awareness even while focusing sharply.

Mathematics inspires in the student a profound appreciation for all of creation. The patterns uncovered by mathematical inquiry have such perfect harmony and exquisite beauty that any mathematician must marvel at their source, human consciousness. That these patterns can also be found throughout nature leads the mathematician to realize that there must be a subtle yet all-pervasive connection between the physical world and the field of consciousness. This appreciation of the nature of life finds its fulfillment in the state of enlightenment, in which an
individual experiences all of creation in terms of the self-interacting dynamics of his own awareness [3].

1.2 The Methodology of Modern Mathematics

In this section, we shall examine two crucial ingredients in the methodology of modern mathematics: the systematic use of deductive logic as a criterion of right knowledge and the systematic use of abstraction to simplify the process of investigation and broaden the field of application.

The structures studied by mathematics are completely abstract, and mathematical knowledge is gained by subjective means. Nevertheless, mathematical knowledge is completely reliable because mathematicians have developed a precise, systematic, and non-variable criterion of right knowledge: deductive logic.

Logic is the science of inference; deductive logic identifies those patterns of inference that are universally valid and absolutely reliable. The use of deductive logic to validate mathematical knowledge was first employed systematically by the Greek mathematician Euclid in his presentation of geometry in the *Elements*. Euclid located a few fundamental principles, and then logically derived from them almost 500 theorems concerning plane and solid geometry.

Euclid’s *Elements* is an example of an axiomatic theory. From the first principles of the theory, called the axioms, one sequentially unfolds the theorems of the theory on the basis of logical inference. Each theorem has a proof, consisting of a sequence of steps demonstrating how the theorem can be logically derived from the axioms and the previously established theorems.

In his *Elements*, Euclid attempted to locate a unified foundation for all known mathematics in a small number of fundamental, self-evident principles of geometry. In the ensuing millennia, mathematics has evolved far beyond the conceptual framework of Euclidean geometry. Nevertheless, the logical theme of the *Elements* has continued to guide the destiny of mathematics. In modern mathematics, axiomatic set theory has assumed the role of the unified foundational theory of mathematics. This theory is developed axiomatically from the Zermelo-Fraenkel axioms, from which all of modern mathematics sequentially unfolds (refer to Part 3).
The principles of logic that are used to sequentially unfold the theorems of an axiomatic theory are universally valid laws of inference. Deeply grounded in the nature of intelligence, they constitute an aspect of the dynamics of intelligence responsible for the sequential emergence of elaborated expressions of knowledge from more concise expressions of knowledge. The sequential development of an axiomatic theory thereby provides a mathematical parallel to the theme of Maharishi’s *Apaurusheya Bhashya* of Rik Veda, whereby the Rik Veda presents the sequential elaboration of the totality of knowledge contained in its first syllable, AK, which is the most compact expression of the self-interacting dynamics of the Samhita [12, pp. 210–213].

A second pervasive aspect of the methodology of mathematics is the use of abstraction. At the most concrete level of abstraction, one may use a mathematical model to describe a complicated physical phenomenon. The model aims to capture the essential structure and processes of the phenomenon—the most relevant values of organizing power. The benefits are twofold: first, the model is much simpler to investigate than the phenomenon itself and, second, the model may also govern other physical phenomena, so that the results of investigating the model may have wider application.

At another level of abstraction, mathematical concepts themselves are generalized. For example, consider the integers \{\ldots, -2, -1, 0, 1, 2, \ldots\} with addition. An investigation into the essential organization of this system yields the more general, and hence more powerful, theory of groups. Once again, the benefits are twofold: the group is easier to investigate because one need only work with certain essential features of the integers, and the resulting theory of groups applies not only to the integers, but universally to any system satisfying the axioms of a group (refer to Figure 3). The usefulness of abstraction in mathematics profoundly illustrates the principle from Maharishi Vedic Science that knowledge of more abstract levels of life is more powerful [13, pp. 563–564].

At each level of abstraction, we can think of the more abstract level or theory as a structure of knowledge in which all possibilities are simultaneously lively, in the same way that, according to Maharishi Vedic Science, all possibilities are simultaneously lively in the abstract structure of pure knowledge [9, p. 97]; the infinite organizing power of
Group Theory

**Theorem.** In any group, $e$ and $a^{-1}$ also satisfy $a \ast e = e$ and $a \ast a^{-1} = e$.

**Proof**

**Axioms**

1. **Associativity** $a \ast (b \ast c) = (a \ast b) \ast c$
2. **Identity** $e \ast a = a$
3. **Inverses** $a^{-1} \ast a = e$

Figure 3. A group is any set with a binary operation $\ast$ which satisfies the axioms above. The theorems of group theory derived from these axioms are true for all possible groups.

This knowledge is then expressed in all the specific mathematical structures that satisfy the axioms and are therefore governed by the theory. Abstraction as a unifying principle in mathematics is discussed more fully in Section 2.

The abstract approach not only provides knowledge of greatest comprehensiveness and universality, but provides knowledge in a structure that is most charming to the intellect and most easily assimilated. In doing abstract mathematics, the mind of the mathematician functions in the field of all possibilities, freed from the boundaries of any particular structure. At this abstract level, the mind can work directly with the fundamental and universal organizational principles simultaneously governing all the structures, and perceive directly the relationships of these principles, without getting lost in the local details of any specific structure. The result is that the knowledge gained on the abstract level is not only most universal and powerful, but also most easily and delightfully gained. This is Maharishi’s theme of skill in action, “Do less and accomplish more,” which finds its fulfillment in the state of enlightenment. When the mind is spontaneously functioning from its most abstract and powerful level, that is, from the transcendental level of pure intelligence, anything can be accomplished by mere intention [9, p. 126].
2. Current Theories of Mathematics

2.1 The Diverse Structures of Modern Mathematics

This section will provide a glimpse of the great diversity of modern mathematics and its organization into three main branches of knowledge: algebra, topology/geometry, and analysis. Each area of mathematical knowledge explores its own characteristic type of abstract mathematical objects, called mathematical structures.

A mathematical structure is simply a set of elements together with a specified structure. For example, the most fundamental of all mathematical structures are the natural numbers \(\{1, 2, 3, \ldots\}\) with the structure provided by the arithmetic operations of addition and multiplication. The two aspects of a mathematical structure express the two fundamental qualities of consciousness identified in Maharishi Vedic Science, existence and intelligence [3]: the set of elements is the underlying existence aspect of the structure and the structure on the set, which describes the relationships between the elements, is the intelligence aspect of the structure. The intelligence aspect gives rise to patterns and behavior, the dynamic expressions of the structure.

The types of structure that can be given to a set can be broadly classified into two main categories: algebraic structure and topological (or geometric) structure.

Algebraic structure is defined in terms of algebraic operations and/or relations. Most common are binary operations and order relations. A binary operation on a set is a rule for combining two elements of a set to produce an element of the set. For example, addition combines 5 and 3 to produce 8. An ordering on a set describes how the elements of the set are related, often by size. For example, the relation “less than or equal to” is an ordering on the natural numbers, and the subset relation is an ordering on any collection of sets. Modern algebra studies a wide variety of structures with binary operations or orderings and is widely applied in all other branches of mathematics and in many branches of science, particularly in theoretical physics, chemistry, computer science, and communication theory.

Topological and geometric structure describe the way points are organized into wholes possessing shape or form. Geometry uses the concept of a metric, or distance, between two points to organize points
into a geometric figure. For example, a circle can be described as the set of all points at an equal distance (the radius of the circle) from a given point (the center of the circle). Geometry studies all properties of geometric figures that can be described in terms of a metric. Topology, on the other hand, studies those properties of shapes and forms that do not depend on length or measurement. Since these are properties that remain unchanged even when a figure is stretched, twisted, or shrunk, topology is sometimes called “rubber-sheet geometry.” From the point of view of topology, a donut is indistinguishable from a teacup! Topology uses just the simple concepts of subset, union, and intersection from set theory to organize a collection of points into a shape or topological space. Topology and geometry are applied in other branches of mathematics and in most branches of science, particularly general relativity and unified field theories of physics, art and architecture, computer-generated graphics, and computer-aided design.

The theories of modern mathematics are traditionally organized into the threefold division of algebra, topology/geometry, and analysis. Whereas the first two areas focus on the structure of sets of points, the third area, analysis, focuses on transformation. At the heart of analysis is a mathematical technique for quantifying the dynamic impulse of change of a function at a point, a process called differentiation. Through this ability, analysis has become one of the principal mathematical tools employed in the sciences to quantify the laws of nature.

Analysis has its foundation in the real number system. The real number system, in which algebraic, geometric, and topological properties are integrated into one coherent mathematical structure, can be viewed as a mathematical representation of the continuum of pure consciousness. This theme will be more fully discussed in Section 4.

The three main branches of mathematical knowledge—topology, analysis, and algebra—form the body of modern mathematics. They interact closely, each contributing to the development of the others, and together form the basis of applications in all the modern sciences. They have a unified foundation in set theory, which will be elaborated in later sections.

One mathematician who strove to unify mathematics was David Hilbert.
“Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts.” —David Hilbert

David Hilbert was born in 1862 in Königsberg. He received his doctorate from the University of Königsberg in 1885. In 1892, Hilbert became a professor at the University, working there until his appointment as professor at the University of Göttingen, a position he held until his death in 1943.

Hilbert’s method of work was to master one field at a time, and to leave it when his success had reached its peak. In this way, through repeated specialization, Hilbert became an expert in an extraordinary range of mathematical fields. The complete list of his lectures covers fifty-six different specializations!

Hilbert contributed to many branches of mathematics by working on key problems. In 1900, he posed his famous list of twenty-three problems summarizing fruitful directions for mathematics.

In Hilbert’s view, mathematics is an indivisible whole in which every question can be answered by the power of human reason. He was a leader in developing the modern axiomatic formulation of mathematics.

Kurt Gödel, however, showed that Hilbert’s vision of a complete and irrefutable structure for mathematics cannot be realized in an axiomatic formulation. According to Maharishi Vedic Science, knowledge is structured in consciousness, and so Hilbert’s goal of complete, irrefutable knowledge is not compatible with the ever changing nature of waking-state consciousness. Hilbert’s goal can be fulfilled by developing Unity Consciousness, a state of consciousness in which knowledge is complete and self-validating.
2.2 The Unity of Modern Mathematics

The great diversity within modern mathematics we have already described. Modern mathematics is very deeply grounded in the intellect, the faculty of discrimination, which according to Maharishi Vedic Science is that aspect of intelligence responsible for creating and recognizing differences [4, pp. 152, 304]. In the midst of great diversification, however, mathematics is by its nature highly unified, perhaps more so than any other field.

At the most holistic level, the set-theoretic foundation has placed all streams of mathematical knowledge in the context of a single unified axiomatic theory (refer to Section 3). Because of this, a development in any area of mathematics can be applied, in a uniform and consistent way, in any other area of mathematics.

Mathematics is also unified by the principle of abstraction. Mathematical knowledge becomes progressively more unified at more abstract levels. Consider the example in Figure 4. We saw in Section 2.1 that by extracting the salient quantities and processes of a physical phenomenon, an abstract mathematical model may be constructed. A very simple example is that of using numbers to talk about quantities of fruit. Numbers in themselves, without reference to the fruit, can be added, multiplied, and so on. The results of doing so can be applied to any physical phenomenon described by the numbers. This gives a unity to the collection of physical phenomena described by the numbers.

At the next level of abstraction, numbers can be replaced by variables. In this way one can describe, for example, a property of addition of numbers generally, \( x + y = y + x \), without reference to the numbers themselves. In this way, an axiomatic theory of natural numbers with addition can be constructed. The theorems derived from the axioms can be applied to all natural numbers.

At the next step of abstraction, operations can also be replaced by variables. This is the level of abstraction of the various theories of abstract algebra. For example, \( \ast \) can be used to denote any operation satisfying the axioms of a commutative group, that is, the group axioms (refer to Figure 3) together with the commutative law \( x \ast y = y \ast x \). The theory derived from the axioms of a commutative group can be applied to any system of objects satisfying these axioms.
Levels of Abstraction

1. Objects
   - Objects: $\bullet \bullet \bullet$ $\bullet \bullet \bullet \bullet \bullet \bullet$

2. Numbers
   - Numbers: $3 + 5 = 8 = 5 + 3$

3. Variables
   - Variables: $x + y = y + x$

4. Abstract Algebra
   - Abstract Algebra: $x * y = y * x$

5. Category Theory
   - Category Theory: $h = f \circ g$

Figure 4. Mathematics is structured in levels of abstraction. Knowledge at deeper levels of abstraction is more powerful because it governs all the more concrete levels.

In 1945, Eilenberg and Mac Lane introduced a further level of abstraction, category theory, based upon the concept of a morphism. A central aspect of each abstract theory is the description of relationships between structures; these relationships are described in terms of transformations, called morphisms, between structures. For example, in group theory, the relationships between groups are described in terms of transformations from one group to another called group homomorphisms. In category theory, one introduces variables representing morphisms, and introduces two simple algebraic axioms governing morphisms; these axioms are satisfied by the morphisms of every abstract theory. The theorems of category theory therefore apply to all abstract theories. This makes category theory a unified theory of abstract theories.

We see in these steps of abstraction that each step corresponds to a greater unification of mathematical knowledge. This theme of unification through abstraction finds its supreme expression in Maharishi Vedic Science in the field of pure intelligence, that most abstract and orderly field of life, in which all the laws of nature are completely unified [9, pp. 24–29].

Mathematics is also unified by its demand at every level for rigorous proof using the principles of deductive logic, as we discussed in Section 2.1. Mathematical logic itself has been unified recently by the develop-
ment of topos theory, which has successfully integrated two competing logical frameworks for the development of mathematics: intuitionistic logic and classical logic.

Throughout the history of mathematics, the greatest advances have emphasized the value of unification, of locating underlying unity and harmony even in the greatest opposites. The most highly regarded mathematical discoveries have been those that generalize specific mathematical principles, solve a wide variety of problems with one technique, locate deep and universal principles of structure and relationship, or integrate and unify different mathematical viewpoints. As the renowned German mathematician David Hilbert put it, “The organic unity of mathematics is inherent in the nature of the science, for mathematics is the foundation of all exact knowledge of natural phenomena.”

An example is provided by Descartes’ analytic geometry, which integrated the study of algebra and geometry by establishing a precise correspondence between algebraic equations and geometric curves, and which led to, among other things, the familiar concept of graphing an equation. Analytic geometry provided a framework for the development of the calculus by Newton and Leibniz, from which the whole field of mathematical analysis has sprouted.

During the past several decades, the theme of unification in mathematics has become more and more predominant. This trend is not only uniting diverse areas of mathematical study, but is even obliterating the traditional distinction between pure and applied mathematics. A striking example is provided by recent work in gauge field theories in mathematical physics, a development that has brought together deep aspects of differential geometry, algebraic geometry, algebraic topology, group theory, and functional analysis. It appears that the mathematical description of the most unified levels of natural law in modern physics requires the comprehensive knowledge of the deepest aspects of the dynamics of intelligence described by the abstract theories of modern mathematics. The final stage of unification in mathematics will be achieved when the mathematician locates the unified source of all values of mathematical knowledge in the self-referral structure of his own awareness. This will be the fulfillment of the discipline of mathematics, made practical through Maharishi Vedic Science and Technology.
3. The Unified Field as the Source of Mathematics

3.1 The Sequential Generation of the Universe of Sets

In this and the following section, we shall see how modern mathematics has developed, in set theory, a totally unified foundation for the diverse streams of mathematical knowledge we have glimpsed so far. This unified structure of mathematics is displayed in the Unified Field Chart for Mathematics. (See the article preceding this one.) We shall now see how set theory provides a precise description of the dynamics of intelligence that creates the abstract world of mathematical objects; this will provide a striking parallel to the description, in Maharishi Vedic Science, of the self-interacting dynamics of intelligence at the basis of creation.

Set theory plays the role of a “unified field theory” at the foundation of modern mathematics. Not only can all the diverse theories of modern mathematics be systematically derived from it, but set theory itself has a totally unified structure, in that set theory is formulated in terms of a single type of object, a set, and a single primordial value of relationship, the membership relation, the relation of a set to its elements.

The concept of a set is simply the idea of a collection of objects conceptually synthesized into a whole; the objects are the elements of the set. In the unified structure of modern set theory, every element of a set is itself a set. All sets are created through a sequential process, starting from the unmanifest point value of the null set, the set that contains no elements, designated 0.¹ The dynamics of this sequential process involves a set-theoretic transformation called the power-set operation, as well as certain steps of synthesis.

The power-set of a set \(X\), designated \(P(X)\), is defined to be the set consisting of all possible subsets of \(X\). This concept is illustrated in Figure 5. In this example, the set \(X\) contains 3 elements, and the power set \(P(X)\) contains 8 elements. The 8 elements of \(P(X)\) are all the different possible subsets of the set \(X\). At one extreme is the null subset, con-

¹ Maharishi makes the following observation about this starting point of the mathematical universe: “When the null set is made self-referral—when it becomes identified with the observer of sets, the mind of the mathematician—the null set is found to display the structure of pure intelligence. In the unmanifest dynamism of this first set, all sets are eventually found.”
taining no elements. At the other extreme is the set \( X \) itself, the “full” subset. Lying between these two extremes are three subsets containing one element and three subsets containing two elements.

The Power Set

![The power set](image)

Figure 5. The power set \( P(X) \) of a set \( X \) is the set of all possible subsets of \( X \). A famous result of Georg Cantor states that, for any set \( X \), \( P(X) \) has more elements than \( X \).

In modern set theory, the power-set operation is used to sequentially create greater and greater wholenesses, called partial universes (refer to Figure 6). We begin with the null set \( \emptyset \), which one calls the 0-th partial universe, designated \( V_0 \). We now apply the power-set operation to the null set to obtain the first partial universe, which we designate \( V_1 \). Applying the power-set operation again, now to \( V_1 \), we obtain the second partial universe, \( V_2 \). Applying the power-set operation over and over again, we generate an infinite sequence of partial universes \( V_0, V_1, V_2, \ldots \). In this sequence, each partial universe is simply the set of all possible subsets of the preceding partial universe.

These partial universes quickly get very large: \( V_0 \) contains 0 elements, \( V_1 \) contains 1 element, \( V_2 \) contains 2 elements, \( V_3 \) contains 4 elements, \( V_4 \) contains 16 elements, \( V_5 \) contains 65,536 elements, and so on. Nevertheless, all the partial universes in this sequence are still finite; each contains only a finite number of elements. The great power of set theory in modern mathematics derives from its ability to describe in a precise way infinite totalities, sets containing infinitely many different
elements. To generate such infinite sets starting from the null set, the sequential mechanics described above must somehow be transcended.

The Sequential Unfoldment of the Universe of Sets from the Null Set

![Diagram of the universe of sets](image)

Figure 6. The universe of sets is structured in expanding layers, sequentially unfolding from the point value of the null set through the application of the power-set and union operations.

This transcending process, which lies at the heart of the mathematics of the infinite, is achieved through a step of synthesis. We imagine the whole infinite sequential process of generating these partial universes $V_0$, $V_1$, $V_2$, ... as being conceptually completed, and then collect together all the elements of all these partial universes into one grand wholeness. In this way we obtain an infinite set, which represents the first infinite partial universe. We designate this infinite partial universe $V_\omega$. (In the language of set theory, $V_\omega$ is the infinite union: $V_\omega = V_0 \cup V_1 \cup V_2 \cup \ldots$.)

The partial universe $V_\omega$ is the starting point for a whole new sequence of expanding partial universes, greater and greater mathematical expressions of infinity! To generate these greater and greater infinite wholenesses, we utilize once again the power-set operation. Thus, applying the power-set operation to $V_\omega$ yields the next partial universe
designated $V_{\omega+1}$; applying the power-set operation to $V_{\omega+1}$ yields the next partial universe designated $V_{\omega+2}$; and so on. Georg Cantor, the founder of set theory, showed that applying the power-set operation to an infinite set always yields a set of greater infinite size, so this sequence of partial universes, $V_\omega, V_{\omega+1}, V_{\omega+2}, \ldots$, gives expression to ever greater mathematical values of infinity! \(^2\) Performing now a second step of synthesis, we obtain the partial universe $V_{\omega+\omega}$ defined to be the union of all these partial universes.

The partial universe $V_{\omega+\omega}$ is already large enough for the development of just about all “ordinary” theories of modern mathematics. Set theory, however, does not stop here. In modern set theory the process of generating sets is extended far into the infinite, to the very limits of the human intellect. In this way the totality of all possible sets, called the universe of sets, is found to provide a perfect mathematical expression of the ultimate value of infinity, the infinite value of the field of pure intelligence, lying at the unmanifest source of mathematics. This we shall elaborate in the next section.

### 3.2 The Dynamics of Intelligence in Set Theory

Upon examining the sequential mechanics of generating sets, we see that it is all the play of the field of intelligence, functioning within itself. Starting from nothing, that is, no mathematical objects, all of mathematical creation is sequentially unfolded. Where do all these sets exist? They exist in the field of intelligence itself. They are a purely conceptual reality, whose existence is inseparable from the field of intelligence that creates that reality.

Set theory provides a very precise description of the dynamics of intelligence that creates sets. This description provides deep insight into the nature of the field of intelligence at the basis of modern mathematics. One striking feature is its quality of infinite dynamism. There are a number of different ways this quality is expressed in the foundations of set theory. A particularly rich expression is found in the power-set operation, which, as we have seen, plays a central role in the sequential

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\(^2\) Commenting on Cantor’s remarkable theory of the infinite, philosopher/mathematician Bertrand Russell once remarked: “The solution of the difficulties which formerly surrounded the mathematical infinite is probably the greatest achievement of which our age has to boast.”
mechanics of generating sets. Let us analyze the dynamics of intelligence involved in this set theoretic operation.

Imagine starting with the infinite set of natural numbers: \( N = \{0, 1, 2, \ldots \} \). The power set \( P(N) \) consists of all possible subsets of \( N \). If we wish to think of examples of subsets of \( N \), we naturally think of such subsets as the set of even numbers \( \{0, 2, 4, 6, \ldots \} \), the set of perfect squares \( \{0, 1, 4, 9, 25, 36, \ldots \} \), the set of primes \( \{2, 3, 5, 7, 11, 13, 17, \ldots \} \), and so on. These subsets, however, are rather special; each is defined by a rule. Even though these subsets are infinite, the rules, in each case, are finite; that is, the rule can be communicated by a finite formula, or expressed in a finite number of words.

The concept of “all possible” subsets, however, is not restricted to such orderly subsets defined by finite rules. It includes also subsets that are determined combinatorially in an arbitrary way. This means simply that a subset can in principle be formed by freely choosing, for each natural number, whether or not to include it in the subset. For example, we might choose to include 0, exclude 1 and 2, include 3, and so on. Since the set \( N \) is infinite, an infinite number of choices will be required to completely determine a subset, and these choices can in principle be made independently of one another. There is no way, using finite expressions of language, to describe a specific subset formed in this way. There is likewise no way the human intellect can sequentially perform the infinity of choices necessary to create such a subset.

To create such a subset, what is required is a field of intelligence capable of making an infinity of simultaneous choices. The existence of such combinatorially determined subsets is a fundamental principle of set theory. The field of intelligence at the basis of set theory must therefore have an infinitely dynamic character, endowing it with the capability of making an infinity of simultaneous choices.

This infinitely dynamic process of creating a subset on the basis of an infinity of simultaneous choices is further expressed in a fundamental principle of set theory called the Axiom of Choice (refer to Figure 7). The Axiom of Choice is so named because it explicitly expresses the possibility of creating infinite sets simply by making infinitely many choices. This is one of the Zermelo-Fraenkel axioms; from these axioms, all the principles of set theory are sequentially unfolded by logical inference. The Axiom of Choice is a very powerful principle of set
theory. It can be used to prove the existence of specific sets having extraordinary properties that are never displayed by sets that can be concretely constructed by the human intellect [15, p. 164].

The Axiom of Choice

Figure 7. The Axiom of Choice describes the formation of a set on the basis of an infinite number of simultaneous choices of elements, one from each set of a given collection of sets.

At the basis of set theory then, we find the highly transcendental notion of a field of intelligence capable of an infinity of simultaneous choices. This provides a rather striking parallel to Maharishi’s description of the field of cosmic intelligence at the basis of creation, which is characterized as having the ability to know all things at one time, or to do all things at once [13, pp. 563–564]. The field of cosmic intelligence is the natural seat of that level of intellect capable of making infinitely many simultaneous choices; this field of intelligence alone could be capable of creating the abstract mathematical reality of set theory on the basis of its own infinitely dynamic self-interaction. The field of cosmic intelligence, as described in Maharishi Vedic Science, can be naturally identified with the field of intelligence at the source of set theory.

We have said that set theory provides a unified foundational theory for all the diverse theories of modern mathematics. In earlier sections we glimpsed the great diversity of structure described in modern mathematics: algebraic operations such as addition and multiplication, relations such as the order relation, functions describing values of mathematical transformation, values of geometrical relationship and
transformation, and so on. Set theory, we have seen, is formulated in terms of a single primordial relation: the membership relation. How can this provide a basis for all the diverse types of relationships that are required for the theories of modern mathematics?

The answer is simple and elegant: all structural concepts required for mathematics are sequentially unfolded from the membership relation through a sequence of definitions. One typically defines, in sequence, the concepts of ordered pair, Cartesian product, relation, function, operation, and so on. Ultimately all mathematical values of relationship are seen to sequentially emerge from the membership relation, the primordial relation of set theory. We shall see in Section 4 how the membership relation is a natural mathematical expression of the knower-known relationship in the field of consciousness, which Maharishi Vedic Science identifies as the primordial value of relationship at the basis of all values of relationship in creation [13, p. 497].

4. Mathematics from the Perspective of Maharishi Vedic Science

4.1 The Universe of Sets: The Ultimate Infinity

In Section 3 we saw how the sequential process of generating sets creates greater and greater mathematical expressions of infinity. If we collect all possible sets together into a single grand wholeness, we obtain the universe of sets, \( V \). The universe of sets, the transcendental wholeness of set theory, is not itself a set; it is too “large” to be consistently regarded as a set (refer to [15]). We shall examine in this section how \( V \) is a natural mathematical expression of the supreme infinite value of the Samhita, the transcendental wholeness of life as described by Maharishi Vedic Science.

The fundamental principle characterizing the nature of the universe of sets is a deep principle of set theory called the Reflection Principle. The Reflection Principle asserts that any conceivable structural property of \( V \) must be reflected in some set: If \( V \) has some property \( P \), then there must exist some set \( A \) such that \( A \) has property \( P \) (refer to Figure 8). This means that the holistic nature \( V \), which makes it greater than any possible set, cannot be captured by any intellectually conceivable property; that is, the holistic nature of \( V \) must transcend the intel-
lect. The Reflection Principle thus characterizes the universe of sets as the mathematical expression of an ultimate, holistic value of infinity that transcends the intellect.

**The Reflection Principle**

![Figure 8. The transcendental quality of the universe of sets $V$ is expressed in the Reflection Principle, which asserts that any structural property of $V$ must be reflected in some set.](image)

In Maharishi Vedic Science, the unbounded field of pure consciousness, experienced as the state of Transcendental Consciousness, is described as transcending the intellect [4, pp. 124–125]. This theme is expressed in the verse of the Bhagavad-Gita “That which is beyond even the intellect is he” (3.42) [4, pp. 242–243]. This is the supreme infinite value of the Samhita. Because the absolutely infinite value of the universe of sets likewise transcends the intellect, we shall view the universe of sets as a mathematical expression of the wholeness of the Samhita of Maharishi Vedic Science. This is in fact quite close in spirit to the vision of Georg Cantor, the founder of set theory, who viewed this ultimate mathematical infinity as the mathematical expression of the “single, completely individual unity in which everything is included, which includes the ‘Absolute,’ incomprehensible to the human understanding [i.e., transcending the intellect]” (cited in [17, p. 290]).

In set theory, the universe of sets is ordinarily conceived of as created through a process of synthesis, whereby all possible sets are synthesized into one grand wholeness. This corresponds to Maharishi’s description
of the way the supreme state of enlightenment, Brahman Consciousness, is structured through a process of synthesis in which all diversity is synthesized into the transcendental wholeness of the Self, the Samhita [12, pp. 216–217]. (This unbounded value of the Self is written with an uppercase “S” to distinguish it from the ordinary, localized self we typically experience.)

Maharishi has elaborated [8] how the range of diversity being synthesized extends to the farthest reaches of space and time, as expressed in the verse of Rik Veda “Far in the distance is seen the owner of the house, the Self, reverberating” (7.1.1). When one considers the sequential emergence of sets from the point value of the null set, one finds expanding values of infinity that express greater and greater values of synthesis. What is located “far in the distance” is the ultimate holistic value of infinity, arising through the synthesis of all possible sets to structure the transcendental wholeness of set theory, the universe of sets. This ultimate value of infinity we can view as the mathematical expression of the absolutely infinite value the Self, the “owner” of the edifice of set theory, that supreme value of cosmic intelligence that creates all sets through its own self-interacting dynamics.

The Samhita is the seat of the self-referral structure of pure knowledge [9, p. 27]. We shall examine now how the Reflection Principle gives mathematical expression to this self-referral theme in the structure of the universe of sets.

The primordial relation of set theory is the membership relation “∈” — the relation of the elements of a set to the set itself. (The formula “A ∈ B” designates that A is an element of the set B.) The membership relation in set theory can be viewed as a mathematical expression of the knower-known relationship in the field of consciousness [15, p. 147]. Now the universe of sets is not an element of itself (it is not a set), so it does not express the self-referral value of the membership relation in the form V ∈ V. Nevertheless, the Reflection Principle tells us that the universe of sets does contain elements reflecting each of its conceivable properties. This means that the impossible self-referral property V ∈ V can be “approximated” by S ∈ V, where the set S can be chosen to resemble V to any degree of intellectual discrimination. In this sense the universe of sets can be understood to express the self-referral value of the membership relation in a logically consistent way.
The Reflection Principle not only expresses the self-referral nature of $V$, but also can be understood to be a mathematical expression of the phenomenon of $\text{Akshara}$, the collapse of infinity to a point. Through the Reflection Principle, the structural qualities of $V$ are found reflected in specific sets, that is, specific point values within it. The sets involved may be in fact quite large infinite sets; nevertheless, in relation to the absolute infinite wholeness of $V$, they are merely insignificant point values.

Maharishi has described how the phenomenon of $\text{Akshara}$ gives expression to the self-interacting dynamics of consciousness at the source of the sequential unfoldment of the expressions of pure knowledge, the $\text{Richas}$ of Rik Veda [12, pp. 210–213]. This theme of sequential unfoldment of knowledge is expressed in modern set theory in the logical derivation of the theorems from the axioms of set theory. The Reflection Principle provides one of the main tools for motivating and justifying the axioms of set theory, the first principles of mathematics (refer to [15]). In this sense, the Reflection Principle too gives expression to an essential aspect of the dynamics of intelligence at the source of the sequential unfoldment of mathematical knowledge.

4.2 The Mathematical Continuum

In Section 4.1 we examined the universe of sets as a mathematical expression of the transcendental, holistic value of consciousness as described in Maharishi Vedic Science. In this section we shall see how the much more concrete and familiar mathematical reality of the real number continuum, which is central to the development of almost all areas of modern mathematics, also has its roots in the deepest values of consciousness, and indeed presents a mathematical model for the continuum of pure consciousness.

The concept of a continuous line is perhaps the most fundamental mathematical expression of the quality of continuity belonging to pure intelligence, the transcendental continuum of consciousness [9, p. 30]. In the real number system, the continuous line is conceived as a collection of distinct point values, each of which represents a number. This analysis of the continuous line in terms of discrete points is extremely subtle; it was not until the 19th century that an adequate theory of the real numbers was developed, based upon Cantor’s theory of infinite sets. At the heart of the theory of the real numbers is the completeness
principle, which makes precise the intuitive idea that the number system contains no holes or gaps. One way to express this is the principle of nested intervals, which describes the infinite, sequential collapse of the continuum to its own point values.

We shall illustrate this principle by an example (refer to Figure 9). Suppose we take an infinite decimal expression, for example 3.14159... (the number \( \pi \)). We can think of this expansion as describing a sequence of intervals of a line. The first digit, 3, tells us that we are considering the interval of points between 3 and 4; the second digit, 1, then tells us that we are considering the subinterval of points between 3.1 and 3.2; and so on. The infinite expression 3.14159... presents us with a sequence of intervals, each nested inside the previous one, such that the lengths of the intervals are shrinking to 0. The principle of nested intervals tells us that there is a unique limiting point to which this sequence of intervals converges.

The principle of nested intervals applies to any possible sequence of closed nested intervals whose lengths shrink to zero. This fundamental principle characterizing the continuum is perhaps the most natural and fundamental mathematical expression of the phenomenon of Akshara in Maharishi Vedic Science, in which the self-interacting dynamics of the Samhita is described in terms of the sequential collapse of the unbounded, infinite continuum of consciousness to its own point value [13, p. 497]. [Refer to the Richo Akshare Chart in this volume for a discussion of the phenomenon of Akshara in the theory of the continuum and in other major areas of modern mathematics. —Eds.]

It follows from the principle of nested intervals that any possible infinite decimal expansion must represent some real number. From this it follows that the set of all real numbers is a vast, “uncountably infinite” set, fundamentally greater in its infinite size than the infinite set of natural numbers \( N \). The set of real numbers is thus a transcendental mathematical wholeness and is a fitting mathematical expression for the continuum of pure consciousness, the transcendental wholeness of life.

The great power of the real number continuum in modern mathematics comes from its perfect synthesis and integration of the discrete algebraic structure of numbers with the continuous structure at the basis of geometry. This synthesis lies at the basis of the field of analysis discussed earlier, through which one can mathematically quantify con-
Constructing the Continuum of Real Numbers

Figure 9. The sequential collapse of the number line into ever smaller intervals associates an infinite decimal with every point on the line. The set of these decimals is the continuum of real numbers.

Figure 10. As the points Q move along the graph towards P, the secant lines PQ approach the tangent line PR and the slopes of the secants approach the slope of the tangent, which is the derivative of the function at P.
In physics, the derivative is used to quantify the unmanifest structure of natural law. Differential equations relate derivatives of functions, that is, they relate unmanifest impulses of activity at each point value in space and time. A single differential equation can capture, in a concise symbolic expression, a universal law of nature. The solutions of the differential equation then describe the diverse measurable consequences of the law as they evolve over time. The mathematician/physicist Henri Poincaré once remarked: “Without this language [mathematics] most of the intimate analogies of things would have remained forever unknown to us; and we would have been forever ignorant of the eternal harmony of the world, which is the only true objective reality.”

This theme of mathematical analysis has a parallel, in Maharishi Vedic Science, in the theme of Karma Mimamsa, or the analysis of action [12, pp. 238–239]. Maharishi has explained that Karma Mimamsa locates the dynamic structure of the laws of nature at the unmanifest point value, which is omnipresent in creation; the expression of the laws of nature sequentially unfolds from this unmanifest point value. In the TM-Sidhi program, the transcending process is used to directly enliven and experience this unmanifest dynamism of natural law, and thereby enjoy the full support of the laws of nature in daily life [6, pp. 406–410].

In Maharishi Vedic Science, the self referral structure of the continuum of consciousness gives rise to the natural laws governing all phenomena in nature [9, p. 27]. Similarly, the mathematical structure of the continuum of real numbers is seen to give rise, in mathematical analysis, to the quantification of the natural laws governing these phenomena.

5. Creating Heaven on Earth: Mathematics Rising to Fulfillment

5.1 Maharishi Vedic Mathematics: The Mathematics of the Veda

The comprehensive approach of Maharishi’s Master Plan to Create Heaven on Earth includes the full enlivenment of all aspects of Vedic knowledge. This includes in particular the mathematical aspect of Maharishi Vedic Science, called Maharishi Vedic Mathematics.
At its most fundamental level, Maharishi Vedic Mathematics is the mathematics of the Veda, the mathematics of the structure of pure knowledge. At this level, the source of all mathematical values and relationships is located in the self-interacting dynamics of the Samhita [11]. In particular, Maharishi Vedic Mathematics locates there both the discrete counting aspect of numbers, embodied in the natural numbers, 1, 2, 3, . . . , and, as we shall see below, the measuring aspect of numbers.

We introduced earlier Maharishi’s description of the sequential emergence of the counting numbers, 1, 2, and 3, from the internal dynamics of the field of pure intelligence. Maharishi has described the continuation of this sequential process and its elaboration in the specific numerical relationships among the syllables, Padās, Richas, and Suktas of Rik Veda based upon the theme of sequential unfoldment of knowledge, the Aparauroṣeya Bhashya [12, pp. 210–213].

Maharishi’s description of the mathematical structure of Rik Veda includes also the first expression of the measuring aspect of number in the quantities of wholeness expressed by the Suktas of the first mandala of Rik Veda [7]. He explains that the quantity of wholeness steadily decreases from its full value in the first Sukta, to its nil value in the 97th Sukta, the Avyakta or unmanifest Sukta, and then steadily increases to its full value again as one completes the circle of 192 Suktas. Maharishi has elaborated how, in this mathematical quantification of wholeness, opposite Suktas express complementary values in such a way that the sum of the two quantities of wholeness is always the same value, namely the wholeness of the first Sukta (see Figure 11 on next page).

We find here the primordial expression of the measurement aspect of numbers. The gradual collapse of wholeness to nil in the sequence of the first 97 Suktas is a quantitative elaboration of the phenomenon of Akṣhara expressed in the first syllable of Rik Veda, the collapse of infinity to its point value. Significantly, the theoretical development of the real number concept in modern mathematics is also deeply grounded in the phenomenon of Akṣhara, as discussed earlier.

Maharishi Vedic Mathematics thus provides a new insight into the genesis within the field of consciousness of the most fundamental mathematical structures and concepts. This insight is more than a new expression of intellectual understanding about the nature of mathematics; it is a reality that can be directly experienced through the practical
The Quantification of Wholeness in the First Mandala of Rik Veda

Figure 11. The 192 Suktras of the first Mandala of Rik Veda express gradually changing quantities of wholeness. This is the primordial expression of the measuring aspect of number.

...aspects of Maharishi Vedic Science and Technology, the Transcendental Meditation and TM-Sidhi programs, as illustrated on the right-hand side of the Unified Field Chart for Mathematics. (See preceding article.)

Vedic Mathematics, Maharishi [10, p. 385] explains, “is the mathematics of the Self; as Vedic Mathematics is the mathematics of one’s own consciousness, and as consciousness is a field of all possibilities, Vedic Mathematics is the mathematics of all possibilities—all possible computations, all possible derivations, all possible calculations wait at the door of Vedic Mathematics.”

Through the direct experience of pure intelligence, the abstract knowledge of mathematics becomes connected to one’s own most intimate experience of life. Mathematics is appreciated as the description of the orderly quantification of one’s own pure intelligence. This makes the study of mathematics a source of joy and fulfillment.

Maharishi Vedic Mathematics also includes the area of Vedic knowledge called Jyotish. Jyotish is the science of evolution. Maharishi has described [12, pp. 231–232] how Jyotish has its basis in the exact sequential unfoldment of the Richas of the Veda. This provides the blueprint for the sequential evolutionary development of any system in nature. On this basis, it is possible to compute mathematically the past, present, and future evolution of any system in nature. These math-
Mathematical computations of Jyotish are based upon the changing positions of the planets.

The practical application of the knowledge of Jyotish is found in its ability to predict influences an individual will experience from his environment during different periods of his life. On the basis of this knowledge, it is possible to take remedial measures to avert impending dangers before they arise, and thereby maintain the most rapid pace of evolution [9, p. 213].

The mathematical theme of Jyotish is similar to that of modern science, in that both use the language of mathematics to quantify the laws of nature and thereby mathematically predict the evolution of natural systems. Jyotish, however, by being based in the holistic knowledge of natural law contained in the Veda, is able to go far beyond the predictive range of modern science and describe the evolution of natural systems of arbitrary complexity, including people. Jyotish, therefore, is of great practical value.

The computational side of Jyotish is its objective aspect. According to Maharishi, the goal of Jyotish is to structure in human awareness the “all-knowing” level of consciousness called Jyotish Mati Pragya [12, pp. 231, 245]. This is the supremely developed state of consciousness in which the individual awareness has the ability to spontaneously compute and govern the past, present, and future of anything in creation. This supreme computing ability belongs naturally to the state of enlightenment, in which the individual awareness is fully established in the self-interacting dynamics of natural law at the basis of the entire sequential unfoldment of creation. Maharishi Vedic Science and Technology provides the practical means for anyone to rise to this supreme level of mathematical achievement.

5.2 The Full Development of the Mathematician

In this section we shall examine further how Maharishi Vedic Science and Technology can bring mathematicians to the supreme level of accomplishment and fulfillment.

As a starting point, we observed that the activity of the mathematician depends on very clear, precise, systematic, and orderly thinking. These analytical capabilities of the mind are directly developed through the Transcendental Meditation and TM-Sidhi programs, as
documented by a large body of scientific research (see [1, pp. 382–431] for a comprehensive bibliography). As one grows in enlightenment, the analytical abilities of the mind naturally mature to their full value. Also of great importance in mathematical creativity is the intuitive, synthetic functioning of the mind, and, according to Poincaré [14, pp. 2041–2050], the “aesthetic sensibility.” Poincaré observed that the combinations of ideas that were most beautiful were, at the same time, most useful. Therefore, the aesthetic sensibility of the mathematician should spontaneously lead to the proper synthesis of ideas required for the solution of a particular mathematical problem. According to Maharishi, the Transcendental Meditation and TM-Sidhi programs, in structuring higher states of consciousness, enliven the finest level of feeling responsible for intuition and aesthetic sensibility and thereby directly contribute to the growth of creativity [5, pp. 74–99].

In this regard, Maharishi Vedic Science and Technology can be expected to play a significant role in the development of set theory by helping to guide the adoption of new axioms. Because of Gödel’s famous Incompleteness Theorem, the axioms of set theory can continue to be expanded through the addition of new axioms that express ever deeper mathematical principles governing the nature of the infinite. These new axioms cannot be derived logically from the standard Zermelo-Fraenkel axioms; to motivate and validate these axioms, fundamental appeal must be made to the subjective faculty of mathematical intuition. The Transcendental Meditation and TM-Sidhi programs, by providing direct experience of the ultimate infinity, the infinite reality of pure intelligence, as well as the mechanics through which it quantifies itself, should directly develop those aspects of mathematical intuition most essential for a more complete understanding of the nature of the infinite in set theory. [For further discussion, see [15] and the article The Wholeness Axiom, in this volume.]

On a personal level, the fulfillment of the values of pure and applied mathematics is found in the state of enlightenment. The Samhita presents the self-referral structure of pure knowledge, the seat of the infinite organizing power of natural law. When human awareness is established in this supreme level of knowledge, it gains the quality of bliss: the aesthetic sensibility is completely fulfilled. At the same time, the infinite organizing power of natural law brings fulfillment to every aspect of
living. This fulfills both the thirst for the great charm of the abstract study of pure mathematics and the desire for the practical benefits of the knowledge of natural law conferred by applied mathematics. In the state of enlightenment, the life of the mathematician has risen to a state of mathematical perfection in which he makes no mistakes and enjoys the support of the totality of natural law [9, pp. 97–98].

The fulfillment of mathematics on the level of society has two aspects. First, mathematicians with clear, enlightened minds, who are using their full potential in their professional lives, will enable science and technology to contribute maximum to creating and sustaining heavenly life on earth for all. Being in tune with natural law at the deepest level [9, pp. 97–98], they will develop profound and fulfilling mathematical theories that unfold into applications of only a nourishing, evolutionary character.

Secondly, through direct experience of the source of mathematics in the dynamics of pure intelligence, anyone can rise to the state of mathematical perfection in which one makes no mistakes and enjoys the full organizing power of natural law. The full value of the orderly mathematical aspect of creation is available to everyone in the state of enlightenment.

In a very real and complete sense, the fulfillment of the discipline of mathematics as a whole lies in the establishment and maintenance of heavenly life on earth, in which both mathematicians and all others enjoy the fruit of all mathematical knowledge in a state of enlightenment.

Old and New Principles in the Field of Mathematics
In the past, each academic discipline has brought out the knowledge of specific laws of nature for improving life, and man acted according to each separate law. Now, through Maharishi Vedic Science and Technology man is learning to act from the ground state of all natural law, the pure field of self-referral consciousness. This is why the principles emerging now to create Heaven on Earth are so different from those which prevailed in the age of ignorance.
<table>
<thead>
<tr>
<th>Old Principles of Mathematics Predominant in the Age of Ignorance</th>
<th>New Principles of Mathematics to Create Heaven on Earth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical aptitude is rare.</td>
<td>Mathematical ability can be systematically developed by anyone through Maharishi Vedic Science and Technology.</td>
</tr>
<tr>
<td>Students suffer from “math anxiety.”</td>
<td>Through Maharishi Vedic Mathematics, students enjoy “math with smiles.”</td>
</tr>
<tr>
<td>Because of its abstractness, mathematics is remote from experience and disconnected from daily life.</td>
<td>Because of its abstractness, mathematics is directly connected to the most intimate experience of life, the experience of the abstract reality of pure intelligence.</td>
</tr>
<tr>
<td>Mathematics unfolds from concrete experience through steps of abstraction and generalization.</td>
<td>Maharishi Vedic Mathematics unfolds from Samhita through steps of quantification.</td>
</tr>
<tr>
<td>Infinity is merely a mathematical concept that can only be handled intellectually.</td>
<td>Infinity can be directly experienced as the nature of one’s own consciousness and spontaneously lived in daily life.</td>
</tr>
<tr>
<td>The computing ability of mathematicians is extended through the use of electronic computers.</td>
<td>The mathematician gains the supreme computing ability of nature by fully enlivening the “cosmic computer,” the human brain.</td>
</tr>
<tr>
<td>Even the best mathematicians make mistakes.</td>
<td>The enlightened mathematician lives a life of fulfillment, free from mistakes.</td>
</tr>
</tbody>
</table>
**Old Principles of Mathematics**  
Predominant in the Age of Ignorance  

| There is no reasonable explanation for the effectiveness of mathematics in the physical sciences. |

**New Principles of Mathematics**  
To Create Heaven on Earth  

| Mathematics is the natural language of science because it is the intellectual expression of the structure of pure knowledge, which is the home of all the laws of nature. |

| Gödel’s Incompleteness Theorem implies that no expression of mathematical knowledge can ever be complete. |

| The enlightened mathematician owns the home of all knowledge. |

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**Maharishi’s Absolute Theory of Heaven on Earth from the Perspective of Mathematics**

The field of mathematics is the intellectual expression of the quantification of consciousness. Its domain ranges from the abstract field of pure consciousness to the concrete, material value of the universe. The purpose of mathematics is to unfold complete knowledge of the self-interacting dynamics of consciousness by attuning the consciousness of the mathematician to the calculative value of natural law.

- It is the nature of Samhita, the undifferentiated, self-referral wholeness of consciousness, to know and thereby quantify itself.
- Within the unity of Samhita there are two values, the subjective value of wakefulness or intellect, and the objective value of existence. From the phenomenon of Samhita knowing itself, the intellect creates three conceptual components of Samhita: Rishi, the knower; Devata, the process of knowing; and Chhandas, the object of knowledge. Thus one becomes two, and two becomes three.
- The interactions of Rishi, Devata, and Chhandas are ultimately responsible for the further quantification of unity into the infinite diversity of creation.
• Maharishi Vedic Mathematics is the intellectual expression of the structure of pure knowledge, the sequential quantification of Samhita generated by the self-interacting dynamics of consciousness.

• The modern mathematician discovers the principles of mathematics in his own consciousness by sequentially refining experience through steps of abstraction and generalization.

• Just as the quantification of Samhita has its origin in Samhita and unfolds in a set sequence, just so mathematical knowledge has its origin in axioms and is sequentially unfolded from the axioms using formal logic. The universal use of axioms and formal logic in mathematics makes mathematics infallible and perfectly orderly, makes the advance of mathematics invincible, and makes the community of mathematicians absolutely coherent.

• Mathematical knowledge naturally finds application in all areas of life because it is the intellectual expression of the self-interacting dynamics of consciousness, which governs the emergence, growth, and dissolution of the universe.

• The Transcendental Meditation technique allows the conscious mind to identify itself with the self-referral state of consciousness, the source of mathematics. Regular practice of the Transcendental Meditation technique develops intuition, clarity in thinking, creativity, the ability to maintain sharp focus along with broad comprehension, and more profound aesthetic sensibility; all these qualities are essential for success in mathematics.

• When the mathematician is fully established in the self-referral state of consciousness, the mathematician lives the infinite reality that mathematics only portrays intellectually. In this fulfilled state of life the mathematician can spontaneously compute and govern the past, present, and future evolution of any system in nature.

**Conclusion**

We have glimpsed the great beauty, power, and universality of mathematics by examining the basic methodology and structure of mathematics in the light of Maharishi Vedic Science. We examined the purpose and approach of mathematics. We saw that the purpose of mathematics
is to understand all structures and processes of intelligence from the viewpoint of orderliness, from their most abstract source in one’s own pure intelligence, to their most expressed values in the phenomena of nature. The approach to gaining this knowledge involves systematic use of deductive logic, which provides an infallible criterion of right knowledge, and the systematic use of abstraction, which provides knowledge of greatest comprehensiveness and universality.

We examined the structure of modern mathematics, bringing out its great diversity and underlying unity. We then saw how set theory provides a unified foundational theory for modern mathematics, and describes the dynamics of intelligence that sequentially unfolds all the abstract structures of mathematics from their unmanifest source in the null set.

We considered next the correspondence between fundamental principles of Maharishi Vedic Science and the deepest concepts of mathematics. We saw that the universe of sets is a mathematical expression of the transcendental wholeness of pure consciousness, which transcends the intellect. We also saw that the continuum of real numbers quantifies the continuum of consciousness, providing a basis for the mathematical quantification of the laws of nature in modern science.

Finally, we saw that modern mathematics finds its fulfillment in Maharishi Vedic Mathematics. As the mathematics of the Veda, Maharishi Vedic Mathematics portrays the full range of quantification of intelligence from its source in the self-interacting dynamics of the Samhita to its most concrete expressions in the diverse phenomena of the physical world. Indeed, the practical aspect of Maharishi Vedic Mathematics includes the complete science of evolution in Jyotish.

By awakening the source of mathematics in everyone’s awareness through Maharishi Vedic Science and Technology, the infinite organizing power of the orderly mathematical structure of natural law will be available to bring every aspect of life to fulfillment and establish Heaven on Earth.
**References**


13. Maharishi Vedic University (1986). *His Holiness Maharishi Mahesh Yogi: Thirty years around the world—Dawn of the Age of Enlight-

This article, “Creating Heaven on Earth Through Maharishi Science and Technology: Mathematics,” here revised/updated, and reprinted with permission, was originally published as Chapter 3 in Heaven on Earth: Maharishi Vedic Science and Technology. Fairfield, IA: Maharishi University Press, 1989.
Richo Akshare Chart
for Mathematics

Faculty of the Mathematics Department
of Maharishi University of Management
<table>
<thead>
<tr>
<th>MATHEMATICS</th>
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<tr>
<td><strong>THEORY OF THE CONTINUUM</strong></td>
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<tr>
<td>The concepts and techniques of set theory underlie the construction of real numbers, permitting one to define and manipulate the infinite sequences of numbers, and to construct the resulting infinite decimal expansion. It is possible to model the set of integers and real numbers using an algebraic structure and a natural ordering.</td>
</tr>
</tbody>
</table>

| **ALGEBRA** |
| Algebraic theories (Richta) are derived from axioms governing the structure of operations and relations on sets. The axioms of each theory serve to distinguish all possible structures to those that satisfy the axioms (Zohari of A). |

| **ANALYSIS** |
| The fundamental concept in the theory of analysis (Richta) is the limit process, which is the limit process of a sequence of objects. The limit process involves an infinite totality of objects, such as points, sets, functions, or linear operators, in a unique limit (Zohari of A). |

| **TOPOLOGY** |
| The theory of topology (Richta) describes and quantifies the fundamental properties of shapes and forms. It is concerned with the characteristics of a system of open sets, which are organizing the topological space. The theory of topology studies the properties of continuous maps, or homeomorphisms, and how they transform the topological space. |

| **CATEGORY THEORY** |
| All mathematical objects can be described from a few fundamental constructions (Richta) described by category theory. Each of these constructions arises from the collapse of all possible constructions of a particular class. An object in a category is a point value called the universal element of the function (Zohari of A). |

| LOGIC |
| The fundamental laws of logic (Richta) emerge from the analysis of the semantics of logic; that is, from the analysis of the way in which a complex sentence can be systematically reduced to its truth value. This process is called the theory of truth (Zohari of A). |

| SET THEORY |
| The principles of set theory (Richta) describe the progressive achievement of greater and greater infinite totalities from the null set. The process of forming sets is based on the collapse of a finite set into a new set of a point value (Zohari of A), a single element of a new set. |

| ALGEBRAIC NOTATION |
| Algebraic notation uses symbolic expressions (Richta) to formulate the language of mathematics. In algebraic notation, a symbol represents a single possible mathematical object that satisfies a specific collection of properties. Algebraic notation collapses the multiplicity of all possibilities into a symbol (Zohari of A). |

| THEOREM OF THE CONTINUUM |
| The continuity is composed of an integrated diversity of structures (Zohari) that form the set theory. This fundamental aspect of the algebraic structure of the continuum provides the foundations for the set theory: Zohari of A. Each algebraic theory has its own transformations and constructions (Zohari). There are two processes enabling one to formulate all possible differential and integral equations, which quantify the mathematical laws governing continuous change. |

| MATHEMATICS |
| MATH |
The organizing power embodied in the axioms of set theory is capable of generating all known mathematical structures. All mathematical structures can be located in the set-theory universe, and all theorems of mathematics are in the reality theorems of set theory.

If one's awareness is not open to the infinite nature of consciousness, then the meaningful content of numbers would be limited to a certain finitary or constructive axioms, as expressed in the formalized or semi-constructive approaches to the foundations of mathematics.

When the symbolic expressions of set theory are examined from the purely finitary viewpoint of formalism, then there is no difficulty regarding the consistency of numerical knowledge. Further, without nonconstructive methods, one cannot formulate many of the statements which are true in such fundamental fields as analysis and topology.

The language and principles of formal logic, based on the mathematical analysis of universal deductive processes, because of its objective validity to the development of mathematical knowledge, provide a structure for the communication of knowledge, and link new knowledge to all mathematics.

In every branch of mathematics the same system of formal logic is used to structure the development of mathematical knowledge. This knowledge of logic provides a common and unified foundation for the development of formal communications, and application of all mathematics.

The implicit and explicit properties of algebraic transformations and transformations make algebra ideal for creating and characterizing the foundations of all mathematical sciences. For instance, mathematics sets the foundation of the continuum to construct function spaces, and topology uses algebraic structure to classify shapes.

Before set theory formalized the concept of the infinite, there was no way to understand the diversity of the continuum. For example, the set of all algebraic functions is a continuum, and its size is the same as the size of the continuum. Perhaps, the continuum is not a mathematical concept, but an implicit notion.

Without set theory one could not construct the uncountable set of transcendental numbers, nor properly investigate topological properties of the line such as compactness and connectedness, the concept of Lebesgue measure in the basis of the theory of integration, category theory in modern mathematics could not be developed.

The quantification of the continuum by set theory provides a foundation for dealing with all branches of mathematical theory, for it establishes functional analysis, differential topology, differential geometry, global analysis, and algebraic geometry.

The theory of the continuum unifies more than just the discrete possible algebraic theories. It unifies the space-time geometry, obtaining a single set-theoretic foundation for many different geometries and their inter-related structures. This is a very broad and fundamental perspective on the complexity of the universe of mathematical systems.

In mathematics, all shapes and forms can be analyzed on a common framework of mathematical properties. Furthermore, topological concepts are essential for defining and understanding continuous transformations between shapes.

Before the set-theoretic approach of topology was developed, the concept of form was limited to that of Euclidean and projective geometry. This limitation was due to the fact that there was no way to distinguish between intrinsic and extrinsic topological and metric properties of spaces.

Without the limit and the fully developed theory of the continuum, neither the world of infinite dimensional functions nor operator spaces nor the techniques for handling them were available to resolve definable properties of continuous functions, such as analytic analogs to the theory of singular integrals and the solution of partial differential equations.

After the limit has been correctly developed and the structure of the continuum fully analyzed, one could define differentiations and integration, and develop real, complex, and functional analysis as the proper setting for the study of both classical and modern problems related to natural phenomena.

In mathematics, all mathematical truths can be expressed in the category-theoretic language of commutative diagrams and morphisms. Moreover, all mathematical structures can be located within the metatheory, and all categories of all categories been located in the metatheory of all categories using the method of extracting universal elements from suitably chosen functors.

The unique viewpoint of category theory has been essential in new approaches to all branches of mathematics. This approach, which complements the general topology and other branches of mathematics, has, for example, provided important insights into the programming language and has established a framework for using algebraic techniques in topology.

The theorem of category theory has been instrumental in the development of algebraic geometry, and through the metatheory of all categories, category theory provides a foundation for all mathematics.
Modern Mathematics in Light of the 
Richo Akshare Verse of Ṛk Veda

Every academic discipline unfolds knowledge of one specific area of natural law. Maharishi Vedic Science unfolds knowledge of the Veda, the most basic element of creation, the lively blueprint of all the laws of nature, eternally existing in the unmanifest basis of the whole manifest universe.

Maharishi explains that Veda means pure knowledge, which exists in that transcendental state where consciousness is completely self-referral, when the awareness knows itself alone.

The Richo Akshare verse of Ṛk Veda describes how the totality of natural law, found in this self-referral state of nature’s functioning, gives rise to the diversity of creation. These universal dynamics find expression in the principles discovered by the modern disciplines.

The Richo Akshare Chart for Mathematics examines eight aspects of the mathematics discipline from the perspective of this verse from Ṛk Veda. Four of these are concerned with the primary areas of pure mathematics (analysis, algebra, topology) and four are concerned with foundational theories and topics. In each case, a core notion of the subject is discovered to arise in some way through the collapse of wholeness, and from that collapse there emerges a full expression of the field, subject, or in some cases, all of mathematics. These dynamics reflect the dynamics of pure consciousness itself, as expressed in the first four phrases of the verse, by which wholeness collapses to a point (the kshara of ‘A’) and expands to infinity to produce fully elaborated expressions of the Veda and ultimately the material creation.

Then, in each area is located a fundamental notion that represents, in a sense, a master key for this area: without this key, essential features of the field remain undeveloped or inaccessible; through the use of this key, on the other hand, the field is seen to thrive or unfold into its fullness. These dynamics mirror the principle, expressed in the remaining four phrases of the verse, that without knowing the transcendental field, the foundation of all knowledge, the verses of the Veda are of little value, whereas, having known this field, one is established in the fruit of all knowledge—in wholeness of life.

In general, the Richo Akshare Charts present the entire range of knowledge in terms of its source, the transcendental level of conscious-
ness of the knower. This opens the possibility for the knower to capture the totality of knowledge in his own awareness.

When individual human consciousness opens to the field of pure knowledge, it becomes a lively field of all possibilities. Then life is lived in perfection—spontaneously in accord with natural law. This complete knowledge of natural law, and a simple approach to it for everyone, is the basis of our aspirations to create Heaven on Earth.

Discovery of all the theories of all the disciplines of modern science within the structure of this Richo Akshare verse of Ṛk Veda explains why the human mind, identified with the transcendental field of consciousness, spontaneously receives the support of the infinite organizing power in which reside all the impulses of creative intelligence, the laws of nature responsible for the orderly administration of the whole manifest universe.

Now, through Maharishi Vedic Science and Technology of Consciousness, it is possible for everyone to align his awareness with the infinite organizing power of natural law and gain mastery over natural law.
Mathematical Foundations
in the Light of the Richo Akshare Verse

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Michael Weinless, Ph.D., received his B.S. from M.I.T. in 1964 and his Ph.D. in mathematics from M.I.T. in 1968. He went on to Harvard University where he held the positions of Benjamin Pierce Lecturer and Assistant Professor of Mathematics from 1968 to 1971. In 1972, Dr. Weinless became one of the founding faculty members of Maharishi International University (renamed Maharishi University of Management in 1995), where he pioneered the development of a unified-field-based mathematics curriculum, integrating principles of the Science of Creative Intelligence and Maharishi Vedic Science with the traditional mathematical content of the courses. Dr. Weinless was chairman of the Department of Mathematics from 1972 to 1990.
The Richo Akshare verse represents the Rik Veda’s own description of the principles and dynamics underlying the emergence of the Veda from within the field of unbounded silence. These fundamental dynamics, Maharishi has explained, lie at the root of the unfoldment of all knowledge. In this article, we examine the emergence of mathematical knowledge from the perspective of the Richo Akshare verse. We discuss how these fundamental principles and dynamics are seen in the context of the standard foundation for mathematics, ZFC Set Theory, and also discuss in the Appendices these same points in the context of alternative foundations formulated in the language of category theory.

§1

Richo Akshare

The verses of the Veda exist in the collapse of fullness (the kshara of “A”). . .

The principles of set theory (Richa) describe the progressive unfoldment of greater and greater infinite totalities starting from the null set. This process of generating sets is based on the collapse of a finite or infinite totality of sets to a point value (kshara of “A”), a single element of a new set.

Set theory was founded by the great German mathematician Georg Cantor in the latter part of the nineteenth century. Set theory today provides a unified foundation for all the diverse theories of modern mathematics. One aspect of the unified structure of set theory is the way it is formulated in terms of a single structural concept—the concept of a set. In Cantor’s monumental treatise of 1895 he defines a set in the following way (Wang, 1974, p. 188):

By a “set” we shall understand any collection into a whole $M$ of definite, distinct objects $m$ (which will be called the “elements” of $M$) of our intuition or thought.

In the concept of a set, the elements of a set are distinct point values that are synthesized into a conceptual wholeness. A set thus expresses the value of wholeness. At the same time, a set can be a single element of a new set; that is, one can form sets of sets, sets of sets of sets, and so
on. When we consider a set as an element of another set, then the set expresses the value of a point. In this way, each set has two sides to its nature: its value as a whole, and its value as a point. When one shifts from the viewpoint of a set as a whole to the viewpoint of a set as a point, we see the principle of fullness collapsing to its point value, the dynamic principle expressed in the first expression of the Richo Akshare verse.

In the study of infinite sets, which lies at the heart of set theory, the value of “fullness” of a set is literally infinite; in this context one finds infinity collapsing to its point value. This principle of fullness collapsing to its point value lies at the basis of the sequential process of creating sets in set theory. This we shall now elaborate.

We shall first review the basic notation of set theory. One uses curly brackets, \{\}, to symbolically represent sets. The elements of the set are written inside the brackets. For example, \{1, 3, 5\} is the set consisting of the three numbers 1, 3, and 5. The set \{3, \{1, 3, 5\}\} is a set containing two elements—one element is the number 3 and the second element is the set consisting of 1, 3, and 5. This illustrates the way one can form sets of sets.

As indicated above, the concept of a set is not limited to finite totalities. One of the main contributions of Cantor’s set theory is that it made it possible to precisely analyze the infinite through the study of infinite sets, sets having infinitely many different elements. An example of an infinite set is the set \(\mathbb{N}\) consisting of all the natural numbers: \(\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}\). (The three dots mean “and so on.”) Another example of an infinite set is the set of all even natural numbers \(\{0, 2, 4, 6, 8, \ldots\}\) or the set of all prime numbers \(\{2, 3, 5, 7, 11, 13, 17, \ldots\}\). If one liked, one could form a set containing these three infinite sets as its elements. In this process, we see an infinite totality becoming “collapsed” to a point value, a single element of a new set.

This process of forming sets of sets lies at the heart of modern set theory. It is applied in a sequential way to generate all possible sets starting from nothing. This is done in the following way.

One first creates the null set, the set that has no elements, \{\}. This set is designated by the symbol \(\emptyset\).

One then creates the set containing the null set as its single element: \{\(\emptyset\)\}. This new set contains one element.
One then creates a set containing each of these sets as its elements: \(\{\emptyset, \{\emptyset\}\}\). This set contains two distinct elements.

The continuation of this sequential process generates all possible sets. The totality of all possible sets is called the universe of sets. To describe more precisely the sequential generation of the universe of sets we must consider a basic set-theoretic operation called the power-set operation.

The power set of a set \(A\), designated \(P(A)\), is defined to be the set consisting of all possible subsets of \(A\). In this example, the set \(A\) contains three elements, and the power set \(P(A)\) contains eight elements. The eight elements of \(P(A)\) are all the different possible subsets of the set \(A\). At one extreme is the null subset, containing no elements. At the other extreme is the set \(A\) itself, the “full” subset. Lying between these two extremes are three subsets containing one element and three subsets containing two elements.

It can be shown that, in general, if a set \(A\) contains \(n\) elements, then its power set, \(P(A)\), contains \(2^n\) elements, which is always a bigger number than \(n\). Cantor extended this idea to infinite sets; he showed that if \(A\) contains infinitely many elements, then the power set \(P(A)\) contains a greater infinity of elements, that is, it has greater infinite size!

In modern set theory, the power set operation is used to sequentially create greater and greater wholenesses, called partial universes. One begins at stage 0 with the null set \(\emptyset\). We write \(V_0 = \emptyset\) to express that the partial universe at stage 0 is the null set. We then apply the power-set operation to obtain the partial universe \(V_1\) at stage 1, that is, \(V_1 = P(V_0)\). We now apply the power-set operation to \(V_1\) to obtain the partial universe \(V_2\) at stage 2, that is, \(V_2 = P(V_1)\). Continuing in this way, we generate an infinite sequence of partial universes \(V_0, V_1, V_2, \ldots\).

These partial universes quickly get very large: \(V_0\) contains 0 elements, \(V_1\) contains 1 element, \(V_2\) contains 2 elements, \(V_3\) contains 4 elements, \(V_4\) contains 16 elements, \(V_5\) contains \(2^{16} = 65,536\) elements, \(V_6\) contains \(2^{65,536}\) elements, and so on. Nevertheless, all the partial universes in this sequence are still finite; each contains only a finite number of elements. For the development of modern mathematics, it is essential that this sequential process be somehow extended into the infinite, in order to create infinite sets, sets containing infinitely many different elements. This is achieved through a step of synthesis, which we shall now analyze.
We have thus far considered a sequence of stages corresponding to the natural numbers 0, 1, 2, 3, . . . . For each stage there is a corresponding partial universe \( V_0, V_1, V_2, \ldots \). We now imagine a new stage, following the entire infinite sequence of natural numbers 0, 1, 2, . . . . We call this stage \( \omega \) (the Greek letter “omega”). At the stage \( \omega \) we create a partial universe \( V_\omega \). The mechanics here will be different, however. We will not use the power-set operation but rather a process of synthesis; that is, we will combine into a single wholeness all the partial universes \( V_0, V_1, V_2, \ldots \). In set-theoretic language, the partial universe \( V_\omega \) will be the infinite union: \( V_\omega = V_0 \cup V_1 \cup V_2 \cup \ldots \). This means simply the elements of all the sets \( V_0, V_1, V_2, \ldots \) are pooled together to form one big set \( V_\omega \). Therefore, \( V_\omega \) is an infinite set.

Even though \( V_\omega \) is itself an infinite set, each of its elements is still a finite set. To obtain partial universes containing infinite sets as elements, we must continue the process of generating sets beyond the level \( V_\omega \). This continuation utilizes once again the power set operation.

The next stage beyond \( \omega \) is called \( \omega + 1 \). The partial universe at this stage, \( V_{\omega + 1} \), is defined to be the power set of \( V_\omega \), that is, \( V_{\omega + 1} = P(V_\omega) \). The next partial universe \( V_{\omega + 2} \) is defined to be the power set of \( V_{\omega + 1} \), that is, \( V_{\omega + 2} = P(V_{\omega + 1}) \). Proceeding in this fashion we obtain the sequence of partial universes \( V_\omega, V_{\omega + 1}, V_{\omega + 2}, V_{\omega + 3}, \ldots \). At each stage in this process we see the mechanics of fullness collapsing to its point value, a shift in viewpoint from a set as a collection to a set as an element of another, larger set. These partial universes express greater and greater values of infinity, but each becomes just a single point, a single element, of the next partial universe. Performing now a second step of synthesis, we obtain the partial universe \( V_{\omega + \omega} \), defined to be the union of all these partial universes.

The partial universe \( V_{\omega + \omega} \) is already large enough for the development of just about all “ordinary” theories of modern mathematics. Set theory, however, does not stop here. In modern set theory the process of generating sets is extended far into the infinite, to the very limits of the human intellect. In this way the totality of all possible sets, called the universe of sets, is found to provide a perfect mathematical expression of the ultimate value of infinity, the infinite value of the field of pure intelligence, lying at the unmanifest source of the human intellect. This we shall elaborate in §2.
The infinite totalities and processes described by set theory transcend the boundaries of the finite localized expressions of nature in space and time and must be located in the infinite, unbounded nature of consciousness.

The concept of a set is totally abstract. This is clearly brought out in Cantor’s definition of a set as a collection into a whole of distinct objects of our intuition or thought. The wholeness of a set is a reality of the mind, a reality structured in the abstract field of intelligence.

When we examine the way all sets are sequentially generated starting from the null set, we see that the whole mechanics takes place within the field of intelligence in a self-sufficient way. It is not necessary to start with any objects from the outside. Starting from just the abstract concept of a set, the field of intelligence can start moving within itself and create sets literally out of nothing. Of course, the sets do not ultimately emerge from “nothing”; they emerge from the fullness of the unmanifest field of intelligence, which sequentially unfolds the sets from its own self-referral nature.

We saw in §1 that the process of generating sets extends into the infinite. This tells us that the field of intelligence at the basis of set theory must by its nature be infinite. Maharishi Vedic Science identifies the most fundamental value of intelligence to be the field of pure intelligence, the unbounded, infinite field of pure consciousness, the transcendental field of life. The absolutely unbounded, infinite nature of this abstract field of intelligence contrasts with the familiar experience of the relative aspects of life, which are localized within finite boundaries. The mathematical abstractions of infinity, which are the heart of set theory, are necessarily grounded in the infinite nature of the field of intelligence itself.

The different levels of infinity in set theory present “quantified values” of infinity, through which the infinite nature of pure intelligence expresses itself in quantitative terms that the intellect can analyze. The
hierarchy of greater and greater infinities in set theory expresses more and more fully the unbounded, infinite nature of the transcendental field of life.

We saw in §1 how the sequential process of generating sets creates greater and greater mathematical expressions of infinity. If we collect all possible sets together into a single grand wholeness, we obtain the universe of sets, $\mathcal{V}$. The universe of sets, the transcendental wholeness of set theory, is not itself a set; it is too “large” to be consistently regarded as a set (see Weinless, 1987). We shall examine in this section how $\mathcal{V}$ is a natural mathematical expression of the supreme infinite value of the field of pure consciousness, the transcendental wholeness of life as described by Maharishi Vedic Science.

The fundamental principle characterizing the nature of the universe of sets is a deep principle of set theory called the reflection principle. The reflection principle asserts that any conceivable structural property of $\mathcal{V}$ must be reflected in some set: If $\mathcal{V}$ has some property $P$, then there must exist some set $A$ such that $A$ has property $P$. This implies that the wholeness of $\mathcal{V}$ cannot be grasped in terms of any intellectually conceivable property. The intellect can “reflect” on $\mathcal{V}$ and discover that it has some property $P$, but then there must already exist some partial universe $\mathcal{V}_\alpha$ having property $P$. This means that the property $P$ captures some partial value of the wholeness of $\mathcal{V}$, but can never capture the holistic nature of $\mathcal{V}$ that makes it greater than any possible set. The reflection principle thus characterizes the universe of sets as the mathematical expression of an ultimate, holistic value of infinity, transcending the intellect.

In Maharishi Vedic Science, the unbounded field of pure consciousness, experienced as the state of Transcendental Consciousness, is described as transcending the intellect. This theme is expressed in the verse of the Bhagavad-Gita, “That which is beyond even the intellect is he” (Maharishi Mahesh Yogi 1969, 3.42, 242). Because the absolutely infinite value of the universe of sets likewise transcends the intellect, we can view the universe of sets as a mathematical expression of the absolute unboundedness of the field of pure consciousness.

In set theory, the universe of sets is ordinarily conceived of as created through a process of synthesis, whereby all possible sets are synthesized into one grand wholeness. This corresponds to Maharishi’s description
of the way the supreme state of enlightenment, Brahman Consciousness, is structured through a process of synthesis in which all diversity is synthesized into the transcendental wholeness of the Self.

Maharishi has elaborated how the range of diversity being synthesized extends to the farthest reaches of space and time, as expressed in the verse of Rik Veda, “Far in the distance is seen the owner of the house [the Self]” (7.1.1) (See also Weinless, 1987, p. 149.) When one considers the sequential emergence of sets from the point value of the null set, one finds expanding values of infinity that express greater and greater values of synthesis. What is located “far in the distance” is the ultimate holistic value of infinity, arising through the synthesis of all possible sets to structure the transcendental wholeness of set theory, the universe of sets. This ultimate value of infinity we can view as the mathematical expression of the absolutely infinite value of the Self, the “owner” of the edifice of set theory, that supreme value of intelligence that creates all sets through its own self-interacting dynamics.

When we examine in detail the dynamics of intelligence involved in generating the universe of sets, we find several striking expressions of the transcendental nature of the field of intelligence at the basis of set theory. One of these is found in the power-set operation, which we have seen is applied to generate a sequence of greater and greater infinite partial universes. Let us analyze the dynamics of intelligence involved in this set-theoretic operation.

What happens when the power set operation is applied to an infinite set? In one stroke, all possible subsets must be created and then synthesized into a new wholeness. The expression “all possible” is highly significant here. Imagine starting with the infinite set of natural numbers: \( \mathbb{N} = \{0, 1, 2, \ldots\} \). What is the meaning of “all possible” subsets of \( \mathbb{N} \)? If we wish to think of examples of subsets of \( \mathbb{N} \), we naturally think of such subsets as the set of even numbers \( \{0, 2, 4, 6, \ldots\} \), the set of perfect squares \( \{0, 1, 4, 9, 25, 36, \ldots\} \), the set of primes \( \{2, 3, 5, 7, 11, 13, \ldots\} \), and so on. These subsets, however, are rather special; each is defined by a rule. Even though these subsets are infinite, the rules, in each case, are finite; that is, the rule can be communicated by a finite formula, or expressed in a finite number of words.

The concept of “all possible” subsets, however, is not restricted to such orderly subsets defined by finite rules. It includes also subsets that
are determined combinatorially in an arbitrary way. This means simply that a subset can in principle be formed by freely choosing, for each natural number, whether or not to include it in the subset. For example, we might choose to include 0, exclude 1 and 2, include 3, and so on. Since the set $\mathbb{N}$ is infinite, an infinite number of choices will be required to completely determine a subset, and these choices can in principle be made independently of one another. There is no way, using finite expressions of language, to describe all the possible subsets formed in this way. There is likewise no way the human intellect can sequentially perform the infinity of choices necessary to create such a subset. To create subsets in this way, what is required is a field of intelligence capable of making an infinity of simultaneous choices. The existence of such combinatorially determined subsets is a fundamental principle of set theory. The field of intelligence at the basis of set theory must therefore have an infinitely dynamic character, endowing it with the capability of making an infinity of simultaneous choices.

At the basis of set theory, then, we find the highly transcendental notion of a field of intelligence capable of an infinity of simultaneous choices. This provides a rather striking parallel to Maharishi’s description of the field of cosmic intelligence at the basis of creation, which is characterized as having the ability to know all things at one time, or to do all things at once. The field of cosmic intelligence would be the natural seat of that level of intellect capable of making infinitely many simultaneous choices; this field of intelligence alone could be capable of creating the abstract mathematical reality of set theory on the basis of its own infinitely dynamic self-interaction. The transcendental field of cosmic intelligence, as described in Maharishi Vedic Science, can be naturally identified with the field of intelligence at the source of set theory.

In summary then, we have found two ways in which the transcendental, infinite nature of the field of pure intelligence can be located in the foundations of set theory:

1. Basic set-theoretic principles describe the formation of sets on the basis of the dynamics of a field of intelligence capable of making infinitely many simultaneous choices, which we can equate
with the field of cosmic intelligence as described in Maharishi Vedic Science.

2. The Reflection Principle tells us that the ultimate mathematical infinity, the infinite wholeness of the universe of sets, transcends the intellect and can therefore be viewed as a mathematical expression of the absolute infinite reality of the transcendental field of pure intelligence.

§ 3

... yasmin Deva

... in which reside all the Devas, the impulses of Creative Intelligence, the Laws of Nature

Set theory has at its basis axioms (Devas), which encapsulate the structuring dynamics of these infinite totalities and processes. These axioms embody both the organizing power that creates the mathematical universe from the null set, and the organizing power that structures mathematical knowledge on the basis of logical analysis of collections of objects.

Modern set theory is developed as an axiomatic theory. All the principles of set theory are sequentially unfolded, using logical inference, starting from a small number of first principles, called the axioms of set theory.

The axioms of set theory provide a very precise description of the dynamics of intelligence that creates sets. The usual formulation of set theory starts from the Zermelo-Fraenkel axioms; these axioms contain, in seed form, the total knowledge of modern mathematics. Following are the Zermelo-Fraenkel axioms:

**Axiom 1. Axiom of Extension.** Two sets that contain precisely the same elements must be equal.

**Axiom 2. Axiom of the Null Set.** There exists a set containing no elements.

**Axiom 3. Axiom of Pairs.** For any sets, \( A \) and \( B \), one can form the set \( \{ A, B \} \) containing precisely \( A \) and \( B \) as its elements.

**Axiom 4. Axiom of Unions.** For any set \( A \), one can form the set \( \cup A \) consisting of all elements of elements of \( A \).
**Axiom 5. Power-Set Axiom.** For any set \( A \), there exists a power set \( P(A) \) consisting of all possible subsets of \( A \).

**Axiom 6. Axiom of Infinity.** There exists an infinite set containing all the natural numbers: 0, 1, 2, 3, 4, . . .

**Axiom 7. Axiom of Subsets.** For any set \( A \) and any property \( P \), one can form the subset of \( A \) consisting precisely of those elements having property \( P \).

**Axiom 8. Axiom of Replacement.** If we have a set \( S \), and we replace each element of \( S \) by a new element according to some rule, the totality of the new elements will themselves form a set.

**Axiom 9. Axiom of Foundation (Axiom of Regularity).** There can be no infinite sequence of sets \( A_1, A_2, A_3, \ldots \) such that each is an element of the preceding set:

\[
\ldots \in A_4 \in A_3 \in A_2 \in A_1
\]

**Axiom 10. Axiom of Choice.** Let \( S \) be a set whose elements are non-empty and pairwise disjoint. Then there exists a set \( T \) containing exactly one element of each element of \( S \).

In the precise formulation of Zermelo-Fraenkel set theory, Axioms 7 and 8 are each expressed not by a single axiom but by an axiom scheme consisting of an infinite number of axioms, one for each property \( P \) in the case of Axiom 7, and one for each possible rule for replacing elements in the case of Axiom 8.

Axiom 1 formalizes the idea that a set is completely determined by its elements: two sets that contain precisely the same elements must be equal.

Axiom 9 says that if we start with any set, take an element, then an element of that element, and so on, this process must end after a finite number of steps. That means, after a finite number of steps we must arrive at the null set, the set that has no elements. This axiom therefore tells us that all sets are ultimately reducible to the null set; this implies that all sets can be sequentially unfolded from the null set, as informally discussed in §1.

The remaining axioms describe the dynamics of intelligence involved in forming sets. Axiom 2 tells us that we can begin the process by forming the set containing no elements, the null set. This gives us the first partial universe, \( V_0 \). Axiom 5 now tells us that we can form power sets;
thus we can sequentially create $V_1 = P(V_0)$, $V_2 = P(V_1)$, and so on. But these partial universes are all finite. In order to create the first infinite partial universe, $V_\omega$, we must use additional axioms. Let us consider how this works.

Let us first consider how one represents the natural numbers in set theory. In the set theoretic foundation of modern mathematics, every mathematical object is represented by a set. The most fundamental mathematical objects are the natural numbers: $0, 1, 2, 3, \ldots$. There is a natural way to represent these numbers as sets, originally developed by von Neumann. The idea is simply that each natural number should be the set of all the smaller natural numbers.

Thus $0$, the smallest natural number, must be represented by the empty set:

$$0 = \emptyset$$

The next natural number, $1$, is then represented by the set containing $0$ as its only element:

$$1 = \{0\} = \{\emptyset\}$$

The next natural number, $2$, is then represented by the set containing $0$ and $1$ as its only elements:

$$2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$$

And so on. In this way, each natural number becomes represented by a set.

When we represent natural numbers by sets in this way, $0$ is an element of $V_1$ (and all later partial universes), $1$ is an element of $V_2$ (and all later partial universes), $2$ is an element of $V_3$, and so on. Axiom 6 tells us that all the natural numbers can be combined into a single infinite set $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.

We can now create the partial universe $V_\omega$ as follows. We start with the set $\mathbb{N}$ of natural numbers, and use the Axiom of Replacement to replace each natural number $n$ by the partial universe $V_n$. (One can show that the rule that associates $n$ with $V_n$ can be expressed in the symbolic language of Zermelo-Fraenkel set theory, so the Axiom of Replacement can be applied.) In this way one obtains the replacement set $R = \{V_0, V_1, V_2, \ldots\}$. One can now apply Axiom 4 to form the union set of the set $R$, that is, the set consisting of all elements of elements of $R$. The set one obtains is then $V_\omega$. 

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Once we have constructed $V_\omega$, the Power-Set Axiom can be applied again to form $V_{\omega+1}$, $V_{\omega+2}$, and so on. We can now use again the Replacement Axiom for $\mathbb{N}$, replacing each natural number $n$ by the partial universe $V_{\omega+n}$ to obtain the set $\{V_\omega, V_{\omega+1}, V_{\omega+2}, \ldots\}$. Forming the union set of this set we obtain $V_{\omega+\omega}$. Continuing in this fashion, we see how the Zermelo-Fraenkel axioms formalize the fundamental aspects of the dynamics of intelligence involved in the sequential generation of sets.

In addition to this, the axioms express the basic principles that are used when one reasons about sets in mathematics. For example, when we infer that two sets must be equal because we show that they contain precisely the same elements, then we are applying Axiom 1. When we start with some set, and then consider the subset consisting of those elements having some given property, then we are applying Axiom 7. In reasoning about sets, one often combines two sets by using the operations of union or intersection. Let us see how these operations can be justified by the Zermelo-Fraenkel axioms.

The union of two sets, $A$ and $B$, designated $A \cup B$ is the set containing all sets that are either elements of $A$ or elements of $B$ (or both). That such a set exists follows from Axiom 3 and Axiom 4: Using Axiom 3, one first constructs the pair set $C = \{A, B\}$, and then using Axiom 4, one constructs the union set $D = \bigcup C$ so that $D = A \cup B$.

The intersection of two sets, $A$ and $B$, designated $A \cap B$, is the set containing all sets that are both elements of $A$ and elements of $B$. The existence of this set follows from Axiom 7: we simply form the subset of $A$ consisting of all those elements $x$ that have the property $x \in B$.

In general, all the different steps of inference that are utilized in mathematics when one is reasoning about sets can be formally derived from the Zermelo-Fraenkel axioms. This includes, in particular, non-constructive methods of reasoning, which are directly grounded in the infinitely dynamic nature of the underlying field of intelligence. The most dramatic of these involve applications of Axiom 10, the Axiom of Choice.

The Axiom of Choice is applied in the following pattern of reasoning. One first constructs some collection of sets, that is, a set of sets, and then forms a new set by arbitrarily “choosing” one element from each set in the collection. If the collection of sets is infinite, then an infinite number of free choices are involved in forming the new set. The
mathematician does not actually make the choices, but just invokes the axiom to assert that the set must exist. The actual dynamics of intelligence involved in creating the set necessarily transcends the localized, sequential functioning of the human intellect, which can make only one choice at a time; we are dealing rather with a direct expression of the dynamics of cosmic intelligence, which alone is capable of making infinitely many independent choices.

The Axiom of Choice is a very powerful principle of set theory. It can be used to prove the existence of specific sets having extraordinary properties that are never displayed by sets that can be concretely constructed by the human intellect.

The Zermelo-Fraenkel axioms thus express the basic impulses of intelligence that firstly create the abstract world of sets, and secondly are utilized by the mathematician in reasoning about sets. All the knowledge of modern set theory is sequentially unfolded from these axioms.

§ 4

... adhivishvē nishedubh

... responsible for the whole manifest universe.

The organizing power embodied in the axioms of set theory is capable of generating all known mathematics. All mathematical structures can be located in the set theory universe, and all theorems of mathematics are in reality theorems of set theory.

The theories of modern mathematics describe a great diversity of values of mathematical structure: different types of number systems, geometrical figures, functions, and so on. One feature of the set-theoretic foundation of modern mathematics is that every object of mathematical study is identified with a particular set, lying within the universe of sets. In this section we shall have a glimpse of how this is achieved. To begin this development, we shall need to find a set-theoretic representation of the concept of an ordered pair.

The concept of an ordered pair is simply two elements in a specified order. The notation for an ordered pair is \((a, b)\). Because order
is significant here, the ordered pair (1, 2) is different from the ordered pair (2, 1). This means that the ordered pair \((a, b)\) is a different concept than the concept of the two element set \(\{a, b\}\) where order is not considered: \(\{a, b\} = \{b, a\}\), so \(\{1, 2\} = \{2, 1\}\).

The idea of the set-theoretic foundation is that every mathematical object must be identified with some set lying in the universe of sets. This means that the ordered pair \((a, b)\) must somehow be identified with a set. What must be done is to construct from any two sets, \(a\) and \(b\), a set containing the knowledge of these two elements as well as the knowledge of which comes first. The solution that is utilized in modern set theory is the following: the ordered pair \((a, b)\) is defined to be the set \(\{\{a\}, \{a, b\}\}\). For example, \((1, 2) = \{\{1\}, \{1, 2\}\}\), and \((2, 1) = \{\{2\}, \{2, 1\}\}\).

These two sets are different. In general, the ordered pair \((a, b)\) is a two-element set. One of the two elements is a singleton (a one-element set) and the other element is a doubleton (a two-element set). The singleton tells us what the first element of the ordered pair is, and the doubleton tells us what the remaining element is. This thereby provides a systematic scheme for encoding the required information for the concept of ordered pair.

Using the concept of an ordered pair, we can construct sets to represent all the familiar objects of mathematics: fractions, negative numbers, real numbers, geometrical figures, functions, and so on. In this way every mathematical object becomes identified with a specific set lying within the universe of sets, \(V\). We shall consider now how this works.

We have already described the way each of the natural numbers 0, 1, 2, \ldots can be represented by a set. When we do this, the infinite set \(\mathbb{N}\) of natural numbers becomes a subset of the partial universe \(V_\omega\) and hence an element of the partial universe \(V_{\omega+1}\). We would like next to represent the negative numbers \(-1, -2, -3, \ldots\) also by sets. This can be done in the following way, using the concept of ordered pair.

What we shall do is start from the set of natural numbers \(\mathbb{N}\), and construct a new set \(\mathbb{Z}\) to represent all the integers, that is, the positive and negative whole numbers together. The idea we shall use is that every integer \(b\) can be expressed as a difference of two natural numbers, \(b = n - m\), so that we can use the ordered pair \((n, m)\) to represent the integer \(b\). For example, the negative number \(-3\) can be represented by the ordered pair \((2, 5)\), since \(-3 = 2 - 5\). When we do this, however,
there are many different ordered pairs that represent the same integer; for example, (2, 5) and (3, 6) both represent −3. The most elegant way of dealing with this ambiguity is to use all the ordered pairs, and represent each integer by a set of ordered pairs of natural numbers. Thus, for example the integer −3 is represented by the set:

\[-3 = \{(0, 3), (1, 4), (2, 5), (3, 6), \ldots \}.
\]

When one does this, each integer becomes an element of the partial universe \(V_{\omega+1}\) and the set \(\mathbb{Z}\) of integers becomes an element of \(V_{\omega+2}\).

Once we have represented the integers by sets, the next step is to represent all the rational numbers, all possible fractions, by sets. For this we use the same approach. Each rational number \(r\) can be expressed as a ratio \(i/j\) of one integer \(i\) divided by another integer \(j\). We can therefore use the ordered pair \((i, j)\) to represent the rational number \(r\). Again, we deal with the resulting ambiguity by considering all possible ordered pairs. In this way, for example, the rational number 1/3 becomes represented by the set:

\[\frac{1}{3} = \{(1, 3), (-1, -3), (-2, -6), (3, 9), (-3, -9), \ldots \}.
\]

When one does this, each rational number becomes an element of the partial universe \(V_{\omega+4}\) and the set \(\mathbb{Q}\) of rational numbers becomes an element of the partial universe \(V_{\omega+5}\).

The next step is to represent the real numbers by sets. The real numbers are all points on a continuous number line. These include all the rational numbers, but also numbers that are not rational, such as \(\sqrt{2}\). The technique here is to represent each real number \(r\) by the set of all rational numbers \(q\) with the property that \(q < r\), that is, \(q\) lies to the left of \(r\) on the number line. For example, \(\sqrt{2}\) is represented by the set consisting of all the negative rationals, together with all positive rationals whose square is less than 2. In this way, each real number becomes represented by a subset of \(\mathbb{Q}\), the set of rationals. Thus each real number becomes an element of the partial universe \(V_{\omega+5}\) and the set \(\mathbb{R}\) of real numbers becomes an element of the partial universe \(V_{\omega+6}\).

We can use the real numbers to build up geometrical objects. The real numbers represent all points on a continuous line. Using the familiar
approach of a rectangular coordinate system, one can represent points in a two-dimensional plane by ordered pairs \((x, y)\) of real numbers. The set of all such ordered pairs then represents the totality of points in a two-dimensional plane; this set, designated \(\mathbb{R}^2\), is an element of the partial universe \(\mathcal{V}_{\omega+8}\).

We can now represent the familiar objects of plane geometry: circles, triangles, rectangles, and so on. Each geometrical figure can be represented as a set of points in the plane, and hence a subset of \(\mathbb{R}^2\).

We can also represent functions. If \(f\) is a function from real numbers to real numbers, we can consider the graph of \(f\), which will be a subset of the plane. This set can then be used to represent the function \(f\).

In general, all the objects studied in all the branches of mathematics can be represented by sets lying within the universe of sets, \(\mathcal{V}\). When one does this, all the diverse values of mathematical structure become “coded” in terms of the membership relation, \(\epsilon\), which is the only primitive relation of set theory. We saw already how this is achieved for the concept of “ordered pair.” The definition of “ordered pair” is just the first in a sequence of definitions whereby all the structural concepts of modern mathematics are sequentially unfolded from the membership relation. One typically defines, in sequence, the concepts of ordered pair, cartesian product, relation, function, operation, and so on. We shall examine some of the details in a later section. Because these are all definitions, from the point of view of set theory they do not create anything new. This sequence of definitions rather gives expression to a sequence of viewpoints. These viewpoints locate all possible values of mathematical structure and relationship within the universe of sets, the abstract reality of set theory. Ultimately all mathematical values of relationship are seen in this way to sequentially emerge from the membership relation, the primordial relation of set theory. The membership relation is itself a natural mathematical expression of the knower-known relationship in the field of consciousness, which Maharishi Vedic Science identifies as the primordial value of relationship at the basis of all values of relationship in creation (see Weinless, 1987).

Through the sequential definition of mathematical concepts in terms of the membership relation, all mathematical principles become ultimately just statements about sets, expressible in the symbolic language of axiomatic set theory. These principles are validated, ultimately, on
the basis of the axioms of set theory, from which all the theorems of set theory follow through logical inference. In this way, the axioms of set theory embody the organizing power that not only generates all the diverse structures of modern mathematics, but also generates all the knowledge about these structures that constitute the different mathematical theories.

§ 5

. . . yastanna Veda

He whose awareness is not open to this field, . . .

If one’s awareness is not open to the infinite nature of consciousness, then the meaningful content of mathematics is limited to its finitary or constructive aspects, as expressed in the formalist or intuitionistic approaches to the foundations of mathematics.

This phrase of the *Richo Akshare* verse begins the explanation of the practical value of the knowledge contained in the first half of the verse. If an individual’s awareness is not open to the transcendental field in which the *Richas* reside, the next phrase says that they (the *Richas*) are of no practical value to him. They become simply some superficial statements about nature that may appear to be fanciful or mythical. On this level the hymns are relatively meaningless and there is no way to properly interpret them or apply them. Moreover, it is important to note that intellectual understanding of this field is not enough. The verse does not refer to a person who is “not aware of this field” but rather whose “awareness is not open to this field.” This implies that one can have the direct experience of the transcendental field, which is borne out by the experience of those practicing the Transcendental Meditation technique.

In terms of set theory, we can see the relevance of this phrase in several different ways. As was seen in §2, set theory was developed in order to provide a precise formalization of infinite totalities and processes. Without the formalization of infinite sets, many important concepts at the heart of mathematics, such as the analytic description of the continuum
of real numbers, or the limit process that underlies calculus and analysis, were used in an imprecise way that led to errors and confusion.

Although many of these problems were clarified with the formalization of the set concept, set theory itself went through several stages of development. In the early stages of development the naive use of the set concept to describe all infinite totalities gave rise to logical paradoxes, such as Russell’s paradox. Although these paradoxes could be eliminated by more carefully defining the concept of a set, many mathematicians felt that the use of any infinite totality in mathematical reasoning was not justified. They felt that since a completed infinite totality was beyond the range of human thought and experience, that one could not consider it a legitimate part of mathematics.

From the point of view of the Richo Akshare verse we see that if the awareness is not open to the transcendental field that underlies set theory, which has been located as the infinite, unbounded field of consciousness in the second part of this verse, one would not consider infinite totalities to be in the range of human thought or experience, and hence outside the scope of mathematics. This view led some prominent mathematicians around the turn of the century to reject the use of infinite totalities or infinite processes in the formal development of mathematics. These mathematicians generally fall into two of the several schools of the foundations of mathematics or of the philosophy of mathematics, intuitionism and formalism. The goal of both of these schools of thought was to verify beyond any doubt the absolute validity of mathematics. The intuitionists sought to do this by allowing only those concepts and constructions that are absolutely clear and self-evident on the level of the intellect or intuition, whereas the formalists sought to ensure the validity of mathematics by verifying the consistency of its logical development. We look first at the approach of the intuitionists and take up the formalists’ approach in §6.

Intuitionism regards mathematics solely as a production of the human mind. Arend Heyting, a prominent intuitionist says, “We do not attribute an existence independent of our thought, i.e., a transcendental existence, to the integers or to any other mathematical objects. . . . Mathematical objects are by their very nature dependent on human thought. Their existence is guaranteed only insofar as they can be determined by thought. . . . Faith in transcendental existence,
unsupported by concepts, must be rejected as a means of mathematical proof” (Heyting, 1983, 53).

For the intuitionist, then, only that portion of mathematics that can actually be constructed by iterating finite steps of human thought or reasoning is deemed to be valid. Although they do admit the existence of some infinite sets, such as the counting numbers, they do not allow one to use these infinite sets in an arbitrary way to create new infinite sets, since the construction would depend on the completion of an infinite process. This then would not be a construction that could be carried out in a clear and self-evident way on the level of the mind. The intuitionists’ way of doing mathematics has also become known as constructivism.

Taking this constructive approach to mathematics has, however, led to the development of new and interesting results. An important consequence of this approach is that it requires a new type of logic. Since the intuitionists require all mathematical objects to be constructed in a finitary way, they do not allow proof by contradiction. That is, even if the assumption of the nonexistence of some mathematical object can be shown to lead to a logical contradiction, the intuitionists do not regard that as a proof of its existence. From the point of view of logic this amounts to rejecting the Law of the Excluded Middle. This widely used law of logic states that for any proposition \( P \), either \( P \) or its negation \( \neg P \) is true. For an intuitionist, the fact that \( \neg P \) is false does not mean that \( P \) is true unless one has a way to constructively show that \( P \) is true. Thus the truth value of \( P \) is left open. This system of logic where the Law of the Excluded Middle is not assumed has become known as intuitionistic logic.

Although there have been many significant mathematical results derived from intuitionistic logic and the constructivist viewpoint, as a general philosophy of mathematics it appears to be incomplete. If one adheres strictly to the intuitionistic approach, one would not be able to develop a large portion of modern mathematics. For example, the construction of the ordinary set of real numbers can not be carried out according to the intuitionist criteria. It is also clear from Heyting’s comments that the fundamental reason for this incompleteness is the rejection of the transcendental basis of set theory and mathematics in
general. We will see in §6 the consequences of not being open to this field in terms of the formalist view of the foundations of mathematics.

§ 6

... kim richa karishyati

... what can the verses [of the Veda] accomplish for him?

When the symbolic expressions of set theory are examined from a purely finitary viewpoint of formalism, there is no intuitive basis for the consistency of mathematical knowledge. Furthermore, without nonconstructive methods, one cannot formulate many of the most basic concepts in such fundamental fields as analysis and topology.

As was mentioned in §5, this phrase emphasizes the lack of practical application of the verses of the Veda if one’s awareness is not open to the transcendental field in which they are structured. In this section we shall consider the foundational viewpoint of formalism, which examines the expressions of knowledge of set theory, the “verses” of mathematics, from a finitary perspective that rejects the transcendental reality of infinite sets. We shall see how Gödel’s incompleteness theorems demonstrate the inadequacy of this finitary approach to the foundations of mathematics.

The formalist school of mathematics was founded by the great German mathematician David Hilbert. Hilbert acknowledged the greatness of Cantor’s achievement in developing the theory of infinite sets: “This theory is, I think, the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity” (see Benacerraf & Putnam., 1964, p. 139). At the same time, Hilbert argued that the infinite transcended the range of human experience and that ultimately, infinite sets were a kind of illusion, having no real existence, so that statements about them were actually meaningless. Infinite sets had to be viewed as “ideal” elements. The use of these ideal elements made the development of mathematics extremely elegant and rich; from a foundational point of view, however, their use needed to be justified. Hilbert argued that the justification must be that the theory of infinite sets is consistent with the concrete areas of finitary
mathematics that are grounded directly in experience, such as ordinary arithmetic. This means that any arithmetical statement one proves in set theory must be a true statement about the natural numbers, so that, for example, one cannot give a proof that $1 = 0$. This consistency of set theory is something that needs to be demonstrated, however.

In this regard, Hilbert noted how the development of mathematical logic had made it possible to formalize the development of mathematical theories. In this process, the whole symbolic structure of set theory becomes described in finitary terms. This symbolic structure of set theory is something concrete and real, as opposed to the abstract, ideal realm described by the theory. Hilbert inaugurated a foundational program ("Hilbert's program") to provide an irrefutable foundation for mathematics by demonstrating that this symbolic structure of axiomatic set theory is formally consistent, that is, one cannot construct a proof, in the symbolic language, of the formula "$0 = 1.""

An essential aspect of Hilbert’s program was that the proof of consistency should use only finitary methods of reasoning. That is to say that one could not use infinite totalities or other infinitary objects in this chain of reasoning, since the actual existence of such objects was rejected by the formalists. The finitary structure of the symbolic language was the "real" part of mathematics, and it could only be legitimately studied using finitary reasoning.

Hilbert proposed his formalist program in 1925; just six years later Hilbert’s program was dealt its deathblow with Kurt Gödel’s discovery of his famous *Second Incompleteness Theorem*. The Second Incompleteness Theorem showed that any formal system strong enough to establish the fundamental theorems of arithmetic could never prove its own consistency. Gödel’s theorem applies in particular to axiomatic set theory. This means that, within the symbolic development of axiomatic set theory, one cannot derive a proof of the consistency of the theory itself. Since any finitary proof of the consistency of axiomatic set theory of the type Hilbert was proposing could be formulated within axiomatic set theory, this meant that if axiomatic set theory is indeed consistent, then no such finitary proof could be found.

The remarkable aspect of Gödel’s proofs was their essential use of the meaning of the symbolic statements of arithmetic. In these proofs certain statements were constructed whose meaning referred to prop-
erties of the formal system in which the statements were constructed. These kinds of self-referral arguments could never have been discovered by the formalists because of their commitment to ignoring the meaning of the symbols when studying the structure of the symbolic language. Gödel, on the other hand, attributed his discoveries to his unique “epistemological attitude” (Wang, 1974, pp. 7–13), in which he recognized as the primary reality of mathematics the abstract infinite reality of the concepts, rather than the more concrete finite reality of the symbolic language.

Gödel was not alone in his view of the importance of the meaning of mathematical statements. Gottlieb Frege, one of the founders of symbolic logic, was an early critic of formalism. Frege felt that a meaningless string of symbols could not be of any use in discovering the laws of nature underlying the physical universe. Without this ability for practical application, mathematics would be reduced to a mere game. In Frege’s words, “It is applicability alone which elevates arithmetic from a game to the rank of a science. So applicability necessarily belongs to it. Is it good, then to exclude from arithmetic what it needs in order to be a science?” (Frege, 1964)

We see from this then that the underlying meaning of mathematical statements is intrinsically tied to both their proofs and their applications.

In fact, the importance of infinitary (nonconstructive) methods in mathematics and the relationship between statements and their meaning was foreshadowed in an earlier result of Gödel’s, namely his Completeness Theorem. The Completeness Theorem asserts that if the axioms of a formal system are consistent, then the system must have a model; if the axioms of the system do not lead to logical contradictions, then there must exist some mathematical structure satisfying the axioms of the system. Hilbert and his students were attempting to prove this important result by using finitary methods. Gödel’s proof of the Completeness Theorem, however, showed how a model can be created from the symbolic language in which the axioms are expressed. This transformation of the symbolic formulas into the model involves a nonconstructive process, directly grounded in the infinitary methods of set theory. Again we see not only the importance of the infinite as a method of proof, but that it serves as a connecting link between the
symbolic representation of mathematical ideas and their meaning as interpreted in the models.

In conclusion, we see from the preceding examples from both intuitionism and formalism, that without the awareness open to the transcendental field of set theory—the unbounded, infinite field of consciousness—mathematicians would not be able to probe deep into the foundations of mathematical systems, nor would they be able to use the essentially infinitary methods of set theory to study the relationship between the symbolic expressions of mathematics and their meanings. The result would be an incomplete and fragmentary approach to mathematics that would limit its practical value.

§ 7

... ya ittadvidus

Those who know this level of reality. . .

The infinitary perspective at the heart of set theory has given rise to powerful technologies with which mathematicians have solved problems of importance to all of mathematics. Using these techniques, set theorists have transcended the limitations of their own field by establishing the independence of a broad class of fundamental propositions.

The theory of infinite sets was developed in the 19th century by Georg Cantor; one of Cantor’s main motivations was to provide a rigorous foundation for the field of analysis, which required a precise understanding of the continuum of real numbers. Set theory has since blossomed into a universal mathematical language, which not only provides powerful tools for reasoning about the infinite in all areas of mathematics, but provides in fact a unified foundation for all the diverse streams of mathematical knowledge. Mathematicians working in different mathematical areas certainly have their own specialized knowledge of their disciplines, but all share the knowledge of the basic principles of set theory, which are essential to the development of all areas of mathematics.
The three main streams of mathematical knowledge are analysis, algebra, and topology. We shall briefly consider the role of set theory in each of these areas.

Analysis provides the mathematical quantification of continuous change; it is used in the sciences to quantify the laws of nature by mathematical equations. At the foundation of analysis is the theory of the continuum of real numbers. The theory of infinite sets is necessary to characterize the continuum of real numbers in a precise way. One of the basic principles characterizing the continuum is the Completeness Principle, which describes its continuous nature, its property of having no holes or gaps. To formulate the Completeness Principle in a precise way, one must use the language of infinite sets. For example, one way of formulating the Completeness Principle is the Least Upper Bound Principle: Every nonempty set of real numbers that is bounded above has a least upper bound. This means that if \( S \) is a nonempty set of real numbers, and there is some real number \( r \) that is greater than or equal to each element of \( S \), then there is a least such number. We see how the formulation of this principle requires being able to talk about arbitrary infinite sets of real numbers, that is, it requires the abstract language and viewpoint of the theory of infinite sets.

The whole theoretical development of analysis is, in general, heavily dependent upon set-theoretic reasoning. The basic concept at the heart of analysis, the concept of a limit, expresses the idea of completion of an infinite sequential process of approximation to obtain an exact value. The theory of infinite sets provides the tools that can deal in a precise way with infinitary notions such as a limit, where an infinity of sequential steps can be conceptually completed.

Topology is the most abstract aspect of geometry, studying the notion of continuity in the most general setting. The basic structural concept of topology is the concept of a topological space, and this concept is directly defined in the language of set theory: a topological space is a set \( X \), with a specified subset \( \mathcal{O} \) of the power set \( P(X) \) (the “open” subsets) having the properties:

1. \( \emptyset \in \mathcal{O} \) and \( X \in \mathcal{O} \)
2. whenever \( U \in \mathcal{O} \) and \( V \in \mathcal{O} \) then \( U \cap V \in \mathcal{O} \)
3. if \( \mathcal{C} \subseteq \mathcal{O} \) then \( \cup \mathcal{C} \in \mathcal{O} \).
We have given the technical definition just to illustrate the point that it is entirely formulated in the language of set theory. Like analysis, the whole development of topology makes fundamental use of abstract set-theoretic reasoning.

In the field of algebra, one studies mathematical structures having binary operations satisfying algebraic axioms. Fragments of algebra can be developed without taking recourse to abstract set-theoretic reasoning. For example, one can study the logical consequences of the group theory axioms using just first order logic. The type of knowledge one can derive in this way is severely limited, however. In practice, the rich structure of modern algebra comes from higher-order reasoning about algebraic structures, and, in this, set theory plays the central role. This type of reasoning is involved for example in the study of substructures, quotient structures, and homomorphisms (structure preserving transformations). In describing finite structures, that is, structures having a finite number of elements, set theory plays a fundamental role even in counting the number of elements. In describing infinite structures, the abstract reasoning about infinite sets is brought into play. Even the Axiom of Choice has important applications in the theory of infinite algebraic structures (for example, in the proof that every vector space has a basis).

One fundamental role of abstract set-theoretic methods in all areas of mathematics is found in the mechanics of creating the fundamental examples of mathematical structure. We considered the beginnings of this process in §3, where we examined the creation of the natural numbers, integers, rationals, reals, the plane, and so on. In general, all the fundamental examples of mathematical structures can be created through the continuation of this sequential process, using a variety of set-theoretic constructions: unions, subsets, cartesian products, quotients, and so forth.

In the second half of the 20th century, the nature and scope of set theory’s contribution to the rest of mathematics took a dramatic leap forward in a completely unexpected direction, providing revolutionary tools and techniques for demonstrating that a wide variety of problems in many areas of mathematics were in principle unsolvable. The historical impetus for this new style of mathematics came from two major threads of investigation in foundational studies. The first of these was the work
of Gödel. We have already seen how his Second Incompleteness Theorem established that no sufficiently rich consistent theory, such as set theory, can demonstrate its own consistency. His other incompleteness result—the First Incompleteness Theorem—established another surprising fact: no sufficiently rich theory, like set theory, is capable of deciding the truth of all propositions that may be formalized in the theory. To prove this result, Gödel devised, for any such theory $T$, a sentence which asserted formally “I am unprovable from $T$”; he then demonstrated that this sentence could neither be proved nor disproved from $T$. Gödel’s theorem showed that no foundational theory would ever be able to provide enough tools to decide every question in mathematics; there would always be propositions that are undecidable by the theory. Such propositions are said to be independent propositions.

After Gödel’s discovery, many hoped that, even if certain peculiar logical anomalies, like Gödel’s sentence, could not be decided by set theory, at least all statements of genuine mathematical interest should be eventually decidable. Many years later, the work of Gödel and P. Cohen dashed this hope as well. Interestingly, the problem considered by Hilbert and many others to be the most significant open problem of the 20th century would turn out to be one of these undecidable propositions. We now briefly survey the historical developments that led to this discovery.

The work of Cantor showed that the continuum of real numbers was not only infinite, but had in fact an uncountably infinite size; in particular, he showed that there were “more” real numbers than there are natural numbers. He also showed that the “infinite dynamism” underlying this surprising phenomenon is to be found in the power-set operator $P$: applying $P$ to the set of natural numbers produces the set $P(\mathbb{N})$, the set of all subsets of the natural numbers, which can be shown to have the same size as the continuum of real numbers; the dynamism underlying the construction of the continuum from the set of rational numbers is due to the power-set operator.

Once it was established that the continuum $\mathbb{R}$ of reals represents an infinity that is bigger than $\mathbb{N}$, the natural question, which Cantor himself was unable to answer, is, How much bigger? Could there be an infinite size strictly between that of $\mathbb{N}$ and $\mathbb{R}$? For instance, the set of integers and the set of rational numbers both properly contain $\mathbb{N}$; one
might expect one or both of these to be of larger size than \( \mathbb{N} \). However, Cantor showed that both of these sets have the same size as \( \mathbb{N} \). Likewise, one can show that the interval of real numbers between 0 and 1 is, like \( \mathbb{R} \), uncountably infinite, and is also properly contained in \( \mathbb{R} \); one might expect that it represents a slightly smaller infinity than that of \( \mathbb{R} \) itself. However, again, Cantor showed that such intervals always have the same size as \( \mathbb{R} \). These results and others like them led Cantor to the strong belief that there could be no infinite size between these two—that \( \mathbb{R} \) represented the “next larger size” after \( \mathbb{N} \)—and his conjecture became known as the Continuum Hypothesis or CH.

Eventually, CH was shown to be independent of the axioms of set theory. In (1938), Gödel showed how to build a model of the axioms of set theory—that is, a new universe of sets—in which CH holds true. This universe was called Gödel’s constructible universe \( L \). \( L \) was built in stages, like the universe \( V \), but at each stage, only the minimum number of subsets of the previous stage—only those provably required by the axioms of set theory—were added in forming the next stage, and for this reason, the size of the continuum, which is the same as the number of subsets of \( \mathbb{N} \), was in this model as small as it could possibly be; from this it was shown that, at least in this model, there are no infinite sizes strictly between those of \( \mathbb{N} \) and \( \mathbb{R} \). Gödel’s proof thereby showed that at least it was possible for CH to hold true in a universe of mathematics in which the ZFC axioms hold; in other words, Gödel’s work established that CH is consistent with the axioms of set theory.

Later, Stanford University professor Paul Cohen (1963) showed how to build a different kind of universe in which the size of the continuum could be much bigger. For instance, he showed that in one of his universes, there are 50 different sizes of infinity strictly between the size of \( \mathbb{N} \) and the size of \( \mathbb{R} \). In particular, Cohen’s work showed that it was consistent for CH to be false—in other words, that the negation of CH is also consistent with the axioms of set theory.

Together, the work of Gödel and Cohen established that CH is independent of the axioms of set theory, being neither provable nor disprovable from the theory. Their work set the stage for an explosion of independence results impacting every major mathematical field, including analysis, topology, and algebra. Gödel’s early work expanded into an area now known as inner model theory; techniques have been
developed to build a wide variety of inner models, like \( L \), with specialized properties, which permit one to establish the consistency of various propositions. Cohen’s work, known today as forcing, has also developed considerably over the years to establish independence results; in contemporary research, consistency of a proposition and of its negation are typically both established using forcing, though sometimes still the technique of forcing is used in conjunction with inner model techniques to establish independence, as was originally done in the case of CH. (See Weinless, 1987, for a more detailed discussion of these points.)

We have therefore seen that set theory plays a central role in all areas of mathematical study, first in the conventional sense that the language of set theory is used to express the basic principles of the different mathematical theories and that set-theoretic methods are utilized to systematically generate the mathematical structures themselves. We have also seen how set theory has impacted virtually every area of mainstream mathematics in a highly unconventional way through the introduction of sophisticated techniques, such as creation of inner models and applications of forcing, for establishing the independence of certain mathematical propositions. With these advanced tools, set theory has found a way to transcend its own activity through the development of procedures for rigorously establishing the independence and undecidability of certain mathematical propositions.

§ 8

... time samasate

... are established in evenness, wholeness of life.

Set theory establishes a common language and structural foundation for all of mathematics. This, together with the powerful methods of independence proofs, provides a grand unification of mathematics, in which even contradictory propositions coexist in a single, coherent whole.

Modern mathematics enjoys a level of coherence and unification that is unmatched by any other discipline of modern science. This completely
integrated, unified structure of modern mathematics is provided by the set-theoretic foundation.

Set theory itself has a completely unified structure. It is formulated in terms of a single primitive concept, the concept of a set, and a single primitive relation, the membership relation, the relationship of one set being an element of another set. In the set-theoretic foundation of modern mathematics, all mathematical concepts are defined in terms of the membership relation, and all mathematical objects are identified with specific sets lying within the universe of sets.

We indicated in §4 the way in which mathematical concepts are defined in terms of the membership relation. This involves a sequence of definitions, whereby all the diverse values of relationship considered in modern mathematics are sequentially unfolded from the membership relations. In this way, all values of mathematical relationship are ultimately reducible to the membership relation, which is a mathematical expression of the knower-known relationship in the field of consciousness (see Weinless, 1987).

It is instructive to examine several of the steps in this sequential process. One first defines the concept of an ordered pair; as discussed earlier:

\[(a, b) = \{[a], \{a, b\}\}.
\]

One next defines the concept of the cartesian product \(S \times T\) of two sets \(S\) and \(T\). The set \(S \times T\) is the set of all ordered pairs \((x, y)\) such that \(x \in S\) and \(y \in T\). For example, if \(\mathbb{R}\) is the set of real numbers, then the cartesian product \(\mathbb{R} \times \mathbb{R}\) is the set \(\mathbb{R}^2\) used to represent all points in a two-dimensional plane.

One next defines the concept of a function from a set \(S\) to a set \(T\). The intuitive idea of a function from \(S\) to \(T\) is some rule that assigns, to each element \(x\) of \(S\), a well-defined element \(y\) of \(T\). In the set-theoretic foundation, one formally defines the concept of a function in the following way: a function from \(S\) to \(T\) is a subset of \(S \times T\) with the property that each element \(x\) of \(S\) occurs exactly once as the first element of an ordered pair \((x, y)\). The idea is that the second element \(y\) is the value of the function applied to the element \(x\). For example, the function from the natural numbers to the natural numbers that takes each number \(n\) into its square, \(n^2\), corresponds to the following subset of \(\mathbb{N} \times \mathbb{N}\):
{\((0, 0), (1, 1), (2, 4), (3, 9), (4, 16), (5, 25), \ldots\)}.{\]

This set encodes complete knowledge of the function—what the value of the function is for every element in N. This is similar to the way the set representing an ordered pair encodes complete knowledge of the pair—what the two elements are, and which comes first. In general, the approach in the set-theoretic foundation is to represent every mathematical concept by a set that efficiently encodes all the relevant knowledge.

Using the concepts of cartesian product and function, one can now define the concept of a binary operation on a set S. The intuitive idea of a binary operation on S is a rule that assigns to any two elements of S a well-defined element of S. Examples of binary operations are the operations of addition and multiplication on the set of natural numbers, or integers, or rationals, or reals. In the set-theoretic foundation, a binary operation on S is simply defined to be a function from S × S to S. For example, the operation of addition on the natural numbers is identified with the function that takes the ordered pair \((x, y)\) to \(x + y\), that is, the function that takes \((2, 3)\) to 5, \((4, 2)\) to 6, and so on. If we represent this function by a set, then the operation of addition on \(\mathbb{N}\) becomes represented by the following set:

\[
\{((0, 0), 0), ((0, 1), 1), ((1, 0), 1), ((1, 1), 2), ((0, 2), 2), \ldots\}.
\]

One can now define the concept of a type of algebraic structure, such as a group. The concept of a group is a set \(S\) on which there is specified a binary operation satisfying certain axioms (the group axioms). In the context of the set-theoretic foundation, we can define a group \(G\) to be an ordered pair \(G = (S, b)\) where \(S\) is a set and \(b\) is a binary operation on \(G\) such that the group axioms are satisfied.

Continuing in this way, all mathematical concepts can be formulated in the language of set theory, and all mathematical objects can be identified with specific sets. In this way, all of modern mathematics becomes formally reducible to set theory.

We mentioned above that set theory itself has a totally unified structure, being formulated in terms of a single structural concept and a single primitive relation. A further feature of modern set theory essential to the coherence of the set-theoretic foundation is that set theory is formulated in an absolutely precise way.
The precise formulation of modern set theory involves the formalization of mathematics using the symbolic language of mathematical logic. In this process, one introduces special symbols to express the basic logical constituents of language. The symbols used are the following: \( \land \) (and), \( \lor \) (or), \( \neg \) (not), \( \rightarrow \) (implies), \( \forall \) (for all), \( \exists \) (there exists). In addition, one uses variables \( x, y, z, \ldots \), parentheses ( ) for grouping, and the equals sign, =. For developing set theory one additional symbol is required to express the membership relation, \( \in \).

The Zermelo-Fraenkel axioms of set theory can all be expressed in this symbolic language of mathematical logic. Mathematical logic further provides an exact description of the rules of logical inference that are applied to sequentially unfold the theorems of set theory from these axioms. In this way, the entire structure of knowledge of axiomatic set theory can be described in an exact way.

The exact description of the rules of logical inference in mathematical logic gives modern mathematics an objective criterion of right knowledge. All mathematical knowledge is validated on the basis of logical proof, starting ultimately from the axioms of set theory. This is the way the knowledge is validated in the mathematical journals. Mathematical results are stated as theorems; each theorem has a proof, showing how the new result follows from previously established results on the basis of logical inference. The presentations of the proofs are not formalized in the sense that the proofs are not given in the symbolic language of mathematical logic. Nevertheless the underlying principle is that if the proof is valid it should be possible to fill in all the tiny steps of logic and completely formalize the presentation. In practice, the objectivity of the underlying formal structure has made the criteria of right knowledge in modern mathematics uniform and universal. The world mathematical community agrees on what constitutes a valid proof. It is true that there are many errors in published mathematical “proofs,” but when these errors are discovered and pointed out, everyone recognizes them, including the author.

The set-theoretic foundation places all mathematical knowledge in a common framework. Because of this, the diverse branches of mathematical knowledge naturally support and enrich one another. A mathematician working in any localized area of mathematical knowledge can apply results from any other area of mathematics. This is possible
because all areas of mathematics share a common language, the language of set theory, and common criteria of right knowledge, logical proof from the set theory axioms.

The coherent interaction of the diverse areas of mathematical study not only enriches each of the areas, but also continually gives rise to new areas of mathematical study integrating these different viewpoints. For example, the interaction of topology and algebra has given rise to the field of algebraic topology, in which abstract algebraic structures are associated with topological spaces. The result of the interaction of the diverse areas of mathematics in the unified context of the set-theoretic foundation is an extraordinary body of knowledge that, on the one hand, embraces greater diversity than is found in any other scientific discipline, and yet, on the other hand, is completely integrated and coherent.

One striking expression of the great power of unification and synthesis available on the ground of set theory is seen in the development of sophisticated techniques for establishing independence results, as discussed in Section 7. These techniques proceed by developing, for a given proposition $P$ (such as CH), one model of set theory in which $P$ holds true, and another in which $P$ is false; yet both models are found to coexist within the larger umbrella universe $V$. This phenomenon demonstrates the tremendous power of set theory to integrate and harmonize extreme opposite values—even contradictory propositions—within its own structure.

A similar force of unification is found in the recent development of topos theory. Previously, we described the intuitionistic approach to mathematics, which is based on a foundational viewpoint that rejects the infinitary methods of set theory. The viewpoint of intuitionistic mathematics is thus the antithesis of the set-theoretic viewpoint. Topos theory has shown how, within the universe of sets, one can construct subuniverses, called toposes, which are internally governed by intuitionistic logic and which provide models for all the main theories of intuitionistic mathematics. (See Weinless, 1987, for more details, and Appendix 2 below for connections between topos theory and the Richo Akshare verse.) In this way the whole body of intuitionistic mathematics can be integrated into the set-theoretic foundation of modern mathematics.
These examples highlight the supreme value of wholeness of knowledge contained in the unified structure of set theory, which is capable of synthesizing within its structure even those extreme aspects of mathematics that are grounded in the antithetical viewpoint. This wholeness of mathematical knowledge is symbolic of that ultimate wholeness of knowledge that is the content of the state of enlightenment, Brahman Consciousness, in which all diversity is found synthesized in the transcendental wholeness of the Self, the infinite field of life. With the enlightenment of this supreme value of knowledge through Maharishi Vedic Science and Technology, the level of coherence and harmony already displayed in the activity of mathematicians around the world will be displayed in all areas of life on earth, both individual and collective; this will be the reality of Heaven on Earth.

References


**Appendix 1:**

**Category Theory in the Light of the *Richo Akshare* Verse**

*Richo akshare*

The verses of the Veda exist in the collapse of fullness

*(the kshara of “A”)* . . .

All mathematical activity can be derived from a few fundamental constructions (Richa) described by category theory. Each of these constructions arises from the collapse of all possible constructions of a particular kind, encoded in a functor, to a point value called the universal element of the functor (kshara of “A”).

. . . parame vyoman . . .

. . . in the transcendental field, the Self, . . .

The action of all functors takes place within the metacategory of all categories, whose objects are categories and whose morphisms are these functors. As it includes all categories, this metacategory is so large that it is not itself a category; at the same time it gives rise to all categories through its internal dynamics.

. . . yasmin Deva . . .

. . . in which reside all the Devas, the impulses of Creative Intelligence, the Laws of Nature . . .

In any category or metacategory, the objects are completely characterized by their dynamic relationship with all other objects of the category.
This relationship is described using commutative diagrams comprised of transformations called the morphisms (Devas) of the category or metacategory.

\[\text{... adhivishve nisheduh}\]

\[\text{... responsible for the whole manifest universe.}\]

All mathematical truths can be expressed in the category-theoretic language of commutative diagrams and morphisms. Moreover, all mathematical structures can be located within the metacategory of all categories using the method of extracting universal elements from suitably chosen functors.

\[\text{yastanna Veda...}\]

\[\text{He whose awareness is not open to this field...}\]

Before the discovery of the foundational concepts of universal element and natural transformation between functors within the metacategory, important problems in mathematics, particularly in algebraic topology and homological algebra, remained unsolved and difficult to approach.

\[\text{... kim richa karishyati}\]

\[\text{... what can the verses [of the Veda] accomplish for him?}\]

Without the unified foundation given by the metacategory of all categories, category theory could be misunderstood as a mere generalization of mathematical theories, lacking sufficient mathematical content to be a mathematical theory itself.

\[\text{ya ittadvidus...}\]

\[\text{Those who know this level of reality...}\]

The unique viewpoint of category theory has resulted in new approaches to old problems and has unified apparently unrelated mathematical dis-
Category theory has, for example, provided important models in the theory of programming languages and has established a framework for using algebraic techniques in topology.

\[ \ldots \text{ta ime samasate} \]

\[ \ldots \text{are established in evenness, wholeness of life.} \]

Through the metacategory of all categories, category theory provides a foundation for all mathematics which complements that provided by set theory. With the concept of transformation as a unifying theme, this new foundation integrates the rich diversity of mathematical theories into a single coherent wholeness.

**Appendix 2:**

**Topos Theory in the Light of the Richo Akshare Verse**

*Richo Akshare . . .*

*The verses of the Veda exist in the collapse of fullness (the kshara of “A”) . . .*

Topos theory emerges when the universe of sets (“A”), the ultimate wholeness of mathematics, is examined from the point of view of category theory, and its essential structural features are abstracted (*kshara* of “A”). The precise expression of these structural features in the symbolic language of category theory constitute the defining properties of a topos.

\[ \ldots \text{parame vyoman} . \ldots \]

\[ \ldots \text{in the transcendental field, the Self, . . .} \]

At the heart of topos theory is the bidirectional transformation between the complementary viewpoints of set theory and category theory. The seat of this transformation is the transcendental field of the awareness of the mathematician—the lively field of all possibilities which simultaneously comprehends both of these complementary viewpoints.
From the defining properties of a topos, one constructs the internal logic of the topos, which can turn out to be either classical or intuitionistic; on this basis one constructs the internal interpretation of the symbolic language of set theory. Through this interpretation, all the set-theoretic concepts and principles necessary for the development of mathematics can be internally unfolded within the topos.

Sheaf semantics provides a self-referral viewpoint whereby a topos becomes interpreted as a generalized universe of sets in which sets have variable elements: the objects of the topos are the sets, and the arrows of the topos are the elements. In this way one locates within the structure of the topos the concrete expression of the organizing power contained in its internal set theory.

Before the development of topos theory, there was no way to provide a coherent interpretation of intuitionistic mathematics within the conceptual framework of classical mathematics. As a result, the classical and intuitionistic developments of mathematics were completely disjoint and unconnected, with each approach rejecting the other.
Prior to the development of topos theory, the body of intuitionistic mathematics simply made no sense to the classical mathematician, and consequently the expressions of knowledge of intuitionistic mathematics had no practical application to the body of classical mathematics, grounded in classical logic.

*ya ittadvidus . . .

Those who know this level of reality . . .

Topos theory today plays a fundamental role in many diverse areas of pure and applied mathematics ranging from algebraic geometry to logic and the theory of programming languages in computer science. In foundational research, topos theory has been applied to construct models for a rich variety of intuitionistic theories and has thereby made intuitionistic mathematics intelligible and useful to the classical mathematician.

*. . . ta ime samasate

. . . are established in evenness, wholeness of life.

Topos theory has unified intuitionistic and classical mathematics by showing how the two developments are just the internal development of mathematics in different toposes; it has also unified category theory and set theory by showing how a topos can be equivalently described in these two languages. Topos theory provides the holistic viewpoint that synthesizes the diverse foundational viewpoints into a coherent wholeness of mathematical knowledge.
The Wholeness Axiom

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ABOUT THE AUTHOR

Paul Corazza, Ph.D., received his Bachelor of Arts degree in Western Philosophy from Maharishi International University in 1978 and his M.S. and Ph.D. degrees in mathematics from Auburn University in 1981 and 1988, respectively. He was awarded a Van Vleck Assistant Professorship at University of Wisconsin for the years 1987–1990. He worked in the Mathematics Department at Maharishi International University in the years 1990–95. Following a career as a software engineer, he rejoined faculty at Maharishi University of Management in 2004 and currently serves a joint appointment in the Departments of Mathematics and Computer Science. Dr. Corazza has published more than a dozen papers in set theory, focused primarily on the quest for an axiomatic foundation for large cardinals based on a paradigm derived from Maharishi Vedic Science.
**Abstract**

The study of the Infinite in mathematics began with Cantor’s demonstration that unless truly infinite sets were admitted in the realm of mainstream mathematical investigation, it would not be possible to formulate a rigorous foundation for calculus. Moreover, once one infinite size is allowed into mathematics, Cantor showed there must also exist a vast infinite hierarchy of ever bigger infinite sizes. No sooner had the mathematical world accepted Cantor’s hierarchy of infinities than a new and deeper mystery about mathematical infinity appeared—the Problem of Large Cardinals.

Large cardinals represent infinite sizes so vast that they cannot be proven to exist using the standard axioms (ZFC) of set theory. Yet, no one in 100 years has proven that they don’t exist. A basic problem in foundations has been to give a natural, well-motivated set of axioms, to be added to the standard ZFC axioms, from which large cardinals can be derived.

This paper suggests one such axiom, the Wholeness Axiom, which is formulated on the basis of an intuitive conception of the Infinite that comes from Maharishi Vedic Science. We view the mathematical universe $V$ as an analogue to Vedic wholeness. We formulate intuitive principles regarding the nature of $V$ based on principles and dynamics of wholeness as described in Maharishi Vedic Science.

Based on these principles, we obtain the axiom schema that we call the Wholeness Axiom, and demonstrate that virtually all large cardinals can be derived from it. We pursue the analogy between wholeness and $V$, elaborating on the mathematical consequences that follow as we seek analogues to these Vedic principles in the realm of our new expanded set theory.

**1. Introduction**

For anyone who finds inspiration in the powerful conceptual unification provided by ZFC (or any of a number of other set theories), the need to give an account of large cardinal axioms\(^1\) cannot be ignored. Efforts to prove that even the strongest large cardinal axioms are inconsistent have failed; large cardinals have turned up with increasing frequency as the central element in the solution of a variety of mathematical problems; and yet, even the weakest large cardinals cannot be proven to exist using ZFC alone (see [EM], [Je], [KM],

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\(^1\) Please see [We] and [Co3] for an introduction to large cardinals.
It seems clear at this time in the history of research in the area that at least some large cardinal axioms ought to be admitted as legitimate additions to the axioms of ZFC, but which should be allowed?

Though large cardinals have been around at least since 1908 (see [Hf] in which the notion of “weakly inaccessible” first appears), they were not studied intensively until the 1960s. At that time, many who believed in the truth of certain large cardinal axioms attempted to justify them by drawing upon some form of intuition about the structure of the mathematical universe. Much of this intuition had its roots in Cantor’s vision of the Absolute Infinite. Indeed, one of the main tools for justifying large cardinals in this period was the Reflection Principle, a principle that takes its inspiration directly from the “incomprehensibility” of Cantor’s Absolute Infinite. It was quickly discovered, however, that only a handful of large cardinals could be justified using the Reflection Principle, and other intuitive principles emerged—principles which relied less and less on Cantor’s original conceptual framework. By now, there are dozens of large cardinal axioms and equally many intuitive principles that have been used to justify their existence.

In her article *Believing the Axioms I* [Ma1], Penelope Maddy, referring to this proliferation, remarks,

> . . . the axiomatization of set theory has led to the consideration of axiom candidates that no one finds obvious, not even their staunchest supporters. (p. 481)

Maddy’s remarks make evident the need for a more unified conceptual framework in which the role of large cardinals in the universe could be more clearly grasped. Rather than approaching the problem of justifying large cardinals by inventing *ad hoc* principles as the need arises, we believe that what is needed is a comprehensive vision, like Cantor’s vision of the Absolute Infinite, that would strongly suggest which large cardinals are “natural” and, for those deemed “unnatural,” even suggest a direction for demonstrating their inconsistency. We, perhaps naively, believe that, had it turned out that Cantor’s Absolute Infinite was rich enough to suggest the existence of even the largest large cardinals, these strong axioms of infinity would by now have found acceptance as true axiomatic assertions about sets, on an equal footing with the axioms of ZFC. The aim of this paper is to provide a
natural enrichment of Cantor’s notion of the Absolute Infinite in order to “fill in the details,” as it were, in the hope that a vision of the mathematical universe will emerge in which all large cardinals are seen as an expected feature rather than an unexplainable mystery.

Our source of “new details” concerning the Absolute Infinite comes from the most ancient vision of the infinite on record—the detailed vision of the seers of the Vedic tradition of knowledge. In our study of this Vedic perspective on the infinite, we were surprised to discover that this tradition sets forth a number of basic principles concerning the nature of the Absolute Infinite that, on the one hand, seemed fully compatible with Cantor’s own vision, and yet seemed at the same time to suggest the truth of the strongest large cardinal assertions, especially those assertions that are expressed in terms of elementary embeddings of the universe. Indeed, it has appeared to the author that these principles not only suggest the existence of the strongest possible type of elementary embedding of the universe (namely, a nontrivial elementary embedding from $V$ to itself), but even hint at a way of “seeing through” a belief commonly held in the community of set theorists, that such an embedding is inconsistent with ZFC!2

Our plan for this paper is to begin in Section 2 with some of Cantor’s own assertions concerning the Absolute Infinite, especially those that have had a lasting impact on the collective intuition among set theorists concerning the structure of the mathematical universe. We then proceed to outline the highlights of the Vedic perspective, noting similarities with Cantor’s perspective, but emphasizing new principles that will suggest a broader intuition concerning the structure of $V$ and possible existence of large cardinals. In Section 3, on the basis of these Vedic principles, we introduce a new large cardinal axiom, which we call the Wholeness Axiom (WA), which asserts the existence of an undefinable elementary embedding from $V$ to itself (and also specifies an additional technical requirement on $V$). In Section 4 we prove a number of math-

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2 In a well-known and celebrated proof, K. Kunen [K2] did show that in the usual class theories, such as Bernays–Gödel Set Theory or Kelley–Morse Set Theory, such embeddings are inconsistent. The proof, however, makes essential use of the additional properties that are added to such an embedding when it is treated as a class in such a theory; therefore, his proof does not imply (as we discuss further in the present paper) inconsistency with ZFC. See [Co1] and [Co2] for detailed discussions of this point. The unfounded belief that Kunen’s proof demonstrates that such an embedding is inconsistent with ZFC has become less widespread since the time this article was first authored (1994). —Ed.
ematical consequences of the Wholeness Axiom. We prove that WA implies the existence of a cardinal that is super-$n$-huge for every natural number $n$ and that WA is consistent relative to the large cardinal axiom $I$. We show how to formalize WA in an extension of the language of set theory obtained by adding a unary function symbol. And we prove a technical definability result that arises naturally in light of some of the structural consequences of WA. Finally, in Section 5, carrying the parallels with the Vedic paradigm even further, we are led to an analysis of the structure of Laver sequences, which seem to arise naturally when applying the Vedic paradigm.

We readily acknowledge that our use of the Vedic paradigm is somewhat unusual, and we do not claim to have the last word on the problem of justifying large cardinals. However, the parallels with this Vedic perspective have appeared quite striking and worthy of serious exploration. We believe our approach successfully provides a natural, unified framework for motivating virtually all large cardinals.

Let me conclude this introduction by expressing my gratitude to Maharishi Mahesh Yogi for his Vedic Science—the main inspiration for the present work.

2. The Cantorian Absolute and Vedic Wholeness

In this section, we review some of the main points concerning Cantor’s view of the Absolute Infinite that have especially contributed to the modern-day intuition of the structure of the universe $V$. We suggest, relying on M. Hallett’s perspective on Cantor’s contribution to modern set theory, a simple picture of how certain features of Cantor’s Absolute Infinite have naturally taken shape as intuitive principles and how these in turn have suggested the truth of certain assertions about the structure of $V$. We then give a brief overview of the Vedic perspective and extract a handful of central features of Vedic wholeness that seem to have set-theoretic relevance. Using our discussion of Cantor’s work as a template, we formulate five intuitive principles based on these features, and then discuss the implications of these principles for the structure of $V$. These considerations will come together in the form of a rather pleasing axiomatic principle, the Wholeness Axiom, whose mathematical consequences will be discussed in the next section.
Cantor introduced his theory of the Absolute Infinite as part of a general conceptual framework to give intuitive meaning to the “actual infinites” that he had introduced into mathematics. Prior to his work, the only notion of the infinite that was deemed appropriate for mathematical formulation was the “potential infinite” since this sort of infinite is readily exemplified in nature (for example, seasons—indeed all cyclic processes—recur in endless succession and are thus potentially infinite). Hence, for instance, natural numbers were to be understood as continuing indefinitely but not to be treated as a completed totality.

Cantor (and independently, Dedekind) found it necessary to make use of actual infinites in order to give a rigorous definition of the real numbers. And, once one such an infinite is permitted in the arena of mathematics, Cantor showed that there is an endless hierarchy of them of ever increasing magnitude; his transfinite ordinals and cardinals map out the terrain of the actual infinite.

But if actual infinites do not exist in nature, in what sense do they exist at all? For Cantor, the transfinite magnitudes represent thoughts in the Divine Intellect; certainly, even if we cannot conceive of the natural numbers as a completed totality, God can. Moreover, the Divine Intellect itself was to be understood as an Absolute Infinite, incapable of increase or diminution or any mathematical determination whatsoever (see [Ha, p. 13]). For Cantor, this Absolute Infinite represented the totality of all possible mathematical constructions, and, according to Hallett’s account, provided much of the intuition for the modern-day universe of sets (see [Ha, pp. xii, 38, 43–44, 48, and Chapter 4]). Indeed, many of the intuitive principles at the heart of the construction of $V$ can be traced directly to properties of the Absolute Infinite suggested by Cantor. We briefly discuss three such principles here: Limitation of Size, the Principle of Maximum Possibility, and the Reflection Principle. (See [Ha, pp. 20–23, 210–211, Chapter 4] for details concerning the first two of these, and [Ha, pp. 116–118] or [Re] for a discussion of the third.)

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3 Cantor argued that they must indeed exist in nature, but this is precisely because of their existence as thoughts of the Divine Intellect; see [Ha, pp. 23–24].
**Limitation of Size.** For Cantor, as far as mathematical practice is concerned, transfinite numbers resemble finite numbers in many important ways: both should be considered completed totalities on which we may perform operations, and both can be increased indefinitely and be given precise mathematical determination [Ha, Chapter 1.4]. By contrast, the Absolute Infinite is to be understood as an actual infinite of a very different kind; in Cantor’s words, the Absolute is “... a ‘true infinite’ whose magnitude is capable of no increase or diminution, and is therefore to be looked upon quantitatively as an absolute maximum.” (Quoted in [Ha, p. 44].) Indeed, Cantor makes the distinction between these two realms very clear when he says,

... we must make a **fundamental** distinction here between

IIa. Increasable actual-infinite or *transfinite*

IIb. Unincreasable actual-infinite or *Absolute*.

(Quoted in [Ha, p. 41].)

Clearly, Cantor has in mind what we would call nowadays a “type distinction”: The objects that admit mathematical investigation are to be understood as different in kind, and different in size, from the ultimate infinite which is quite simply beyond mathematical determination, and an “absolute maximum.” These notions contain the seeds of the Limitation of Size doctrine, according to which collections that are “too big” are not to be allowed as members of the mathematical universe; that is, they are not to be taken as *sets*. (See [Ha, 1.4 and Chapter 4].)

*Principle of Maximum Possibility.* For Cantor, “possibility implies existence;” if a notion is not blatantly inconsistent with the known theorems of mathematics, then this notion can be said to exist for the following reason: If the notion is not inconsistent then it is at least in principle possible for the object in question to exist; this means that it is creatable by the Divine Intellect.

As such, the notion has existence as a thought in the Divine Intellect. Cantor used reasoning of this kind to conclude that his transfinite ordinals and cardinals exist. (See [Ha, pp. 21–22].) The principle here is that “as much as possible exists” and is clearly one of the precursors
to the maximal iterative principle underlying the cumulative hierarchy, according to which “as many subsets as possible” are added to the universe with each application of the power-set operation (see [Ha], [W]).

**Reflection Principle.** According to Cantor, the Absolute Infinite cannot be grasped or comprehended by the rational mind and is certainly beyond any kind of mathematical description; in Cantor’s words, “In a certain sense it [the Absolute Infinite] transcends the human power of comprehension, and in particular is beyond mathematical determination.” (Quoted in [Ha, p. 13].)

This basic intuition was largely responsible for the formulation and wide acceptance of the Reflection Principle, which asserts that any first-order property that is true of the universe $V$ of sets (or the class $ON$ of all ordinals) must already be true of some set (or ordinal). The rationale behind the Reflection Principle is simply that if it were false, then some first-order property would be sufficient to characterize $V$, in violation of the precept that $V$, like Cantor’s Absolute, is too vast and complex to be determined by such a property.

The Reflection Principle has been used to motivate the inclusion of a number of large cardinal axioms among the basic axioms of set theory; see [Re]. For instance, because it is true that there are no cofinal sequences in $ON$ and that the size of the power-set of an ordinal is always another ordinal—that is, because $ON$ exhibits all the properties of an inaccessible ordinal—it follows from the Reflection Principle that some ordinal number should be inaccessible. (See [We] or [Co3] for more arguments of this kind.)

Having reviewed some of the more obvious ways in which Cantor’s vision of the Absolute Infinite has impacted the structure of the universe of sets and motivated the acceptance of certain large cardinals, we turn now to the Vedic paradigm of the Absolute or wholeness. As was the case for Cantor’s Absolute, Vedic wholeness is also unincreaseable and of greater magnitude than any other existent thing—and indeed
of an altogether different type, beyond intellectual determination; and the source of a manifest universe of greatest possible richness and cre-

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4 The Upanishads (representing a part of the Vedic literature) declare wholeness to be “bigger than the biggest and smaller than the smallest”: 
\[ \text{aŋorāṇīyān mahatamāhīyān} \]  
—Kaṭha Upanishad 1.2.20

A similar idea is expressed in the following, indicating that wholeness is an inexhaustibly rich field of existence: 
\[ \text{pārṇam adah pārṇam idam, pārṇat pārṇam udachyate} \]  
\[ \text{pārṇasya pārṇam adāya pārṇam evāvāsīyate} \]  
That is full; this is full. The full comes out of the full. Taking the full from the full, the full itself remains.  
—Bṛhad-āraṇyaka Upanishad 5.2.1

(Translations both here and in the sequel are due to Maharishi Mahesh Yogi unless referenced otherwise.)

5 As the following passage indicates, the Veda holds the thinking mind, like everything else in the created universe, to be a derivative of wholeness and as such, an inadequate tool for fully apprehending its source: 
\[ \text{yan manasā na manute yenāhur mano matam} \]  
\[ \text{tad eva brahma tvaṃ viddhi nedam yad idam upāsate} \]  
That which is not thought by the mind but by which, they say, the mind is thought; that verily, know thou, is Brahman [Wholeness], and not what people here adore.  
—Kena Upanishad 1.6 [Ra]

\[ \text{agṛhyah na bi gṛhyate} \]  
He is incomprehensible for he cannot be comprehended.  
—Bṛhad-āraṇyaka Upanishad 4.5.15 [Ra]

6 Maharishi explains that the manifest universe is the expression of the self-interacting dynamics of wholeness. He says, “It is the Creator, the process of creation, and the object of creation all together in itself” [M1, p. 213]. The Brahma Sutras (another aspect of the Vedic literature) declare wholeness or Brahman to be the source and ultimate reality of everything: 
\[ \text{janmādyasya yatāḥ} \]  
From which comes the birth, etc., of this (universe).  
—Brahma Sutra 1.1.2

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ative potential.\textsuperscript{7} However, in the Vedic view, wholeness is not static but, to the contrary, extremely dynamic by nature. Fundamentally, wholeness is pure wakefulness, and as such is awake to itself; consequently, it is perpetually in a state of self-knowing, self-transformation, and is always creating within itself, yet remaining fundamentally unaltered by these processes (much in the way that soft clay may be molded in various ways, as if bringing forth the potential of the clay, and yet the clay remains essentially unchanged by such transformations).

In the Vedic conception, the first step in the move of wholeness within itself involves a gathering of all its unbounded potential into a single focal point; this point of focus is said to be a point of “infinite dynamism,” teeming with the tremendous power and possibilities of the Absolute. This point then begins to expand, step by step, into the fundamental frequencies that constitute the manifest universe. Collectively, these frequencies are known as the \textit{Veda} and form a \textit{blueprint} for the manifest universe. The term “Veda” is translated as “knowledge”; these fundamental frequencies represent the self-knowledge that emerges sequentially within the pure wakefulness of wholeness.

It must be emphasized that this entire process of self-unfoldment of wholeness is \textit{unmanifest}, hidden from view, representing the hidden inner dynamics of the manifest universe. These dynamics are present deep within every grain of the universe, but are not obvious on the surface.

There are many passages in the Veda that elaborate upon these points at great length. We cite here one passage directly from Rk Veda, which expresses some of the main points just discussed, and then proceed to other quotations from Maharishi Mahesh Yogi’s extensive commentary on the Veda; for further quotes and references, see [Co3] and [M1].

\textsuperscript{7} Consider the following passage from Maharishi’s commentaries [M2, p. 30]: [Wholeness,] while remaining uninvolved with the creative process in nature, is an infinitely dynamic, inexhaustible source of energy and creativity. On that basis the whole creation goes on perpetually in its infinite variety, multiplying itself all the time. Change is perpetual, and change has come on the ground of non-change because of that non-changing value of infinite dynamism at the unmanifest basis of all creation.
The verses of the Veda exist in the collapse of fullness in the transcendental field, in which reside the fundamental frequencies responsible for the whole manifest universe.

—Rk Veda 1.164.39

The expression “verses” in the above passage is also sometimes translated “frequencies.” The quotation indicates how, within wholeness, the “transcendental field,” the basic frequencies of the universe emerge in the collapse of the unbounded fullness of the Absolute. The fact that this collapse is a gathering of potential to a point is important and is brought out by a basic feature in the structure of Vedic Sanskrit: the structure of the Vedic literature and the Vedic language is such that the first word in a verse, hymn, or longer section in the Veda contains in seed form the total content of this verse, hymn or section. Moreover, the sound of this first word—indeed, even the first syllable—is said to contain the fundamental impulses or frequencies that structure the content of the entire verse, hymn, or section.8

An example of this phenomenon of central importance here is the first syllable of the Veda itself: “AK.” The first letter “A” is a sound that is made with the throat fully open and embodies “fullness.” The second letter “K” is a sound made with the throat closed and is called a “stop”; it stands for the focal point within wholeness, the “point of all possibilities,” in which is located the “infinite creative potential of nature that in one stroke can give expression to the infinite diversity of creation” [M3, p. 278].

Commenting on the significance of the syllable “AK,” in relation to the passage from Rk Veda quoted above, Maharishi remarks,

The first syllable of Rk Veda, AK, expresses the dynamics of akshara—the “ksara of A” or collapse of infinity to its point value, which is the source of all the mechanics of self-interaction [M4, p. 1].

8 This observation about the structure of the Vedic literature is due to Maharishi, and is known as Maharishi’s Aparusheya Bhashya, the “uncreated commentary” of the Veda. See Section 5, and also [Ch], for a fuller discussion.
We now extract from this brief description of the Vedic dynamics of wholeness certain principles that we will use in the next section to motivate the Wholeness Axiom.

The Principle of Self-Transformation. This principle asserts that wholeness by nature moves within itself, knows itself, and creates within itself, and yet remains unchanged by these transformations.

The Focal Point Principle. When wholeness moves, the move becomes focused at a central point within wholeness in which all the knowledge and power of the Absolute becomes concentrated, in preparation for infinite expansion into the universe.

The Blueprint Principle. When the focal point of the move of wholeness begins to expand, it first takes shape as a blueprint of the domain that is about to be created. (In the Vedic paradigm, the Veda itself is this blueprint, and it is a blueprint for the manifest universe; but the principle we are giving here simply asserts that whenever wholeness moves, the focal point of the move unfolds into a basic blueprint of the thing being created.)

The Principle of Global Undefinability. This principle asserts that the self-interacting dynamics of wholeness take place not on the surface of existence but, rather, are totally unmanifest, logically prior to the manifest universe.

The Principle of Local Existence. The dynamics of wholeness are present at every point in its manifestation. (The point here is that, although the dynamics of wholeness are unmanifest and transcendental to the manifest universe, they are not divorced from it; rather, these dynamics are present in every grain of the manifest universe.)

This list of five Vedic principles provides us with the main “new details” that we wish to add to Cantor’s paradigm of the Absolute Infinite and apply to the structure of the mathematical universe. The new feature that the Vedic paradigm offers is a sequence of dynamic stages of unfoldment within the Absolute that we find nowhere in Cantor’s
treatment. We find these new principles rather suggestive of a view in which an elementary embedding of the universe (to itself) would be an expected feature and in which the critical point of the embedding would play a crucial role. In the next section, we show how these new principles can be used to motivate the Wholeness Axiom.

3. The Wholeness Axiom

Let us recall that our aim in introducing the Vedic version of the Absolute has been to provide a more detailed conceptual framework on which to base intuitions concerning the structure of the universe. Our hypothesis is that, given such a sufficiently rich, natural extension of Cantor’s Absolute Infinite as an intuitive model of the universe, it will be “obvious” that even the strongest large cardinal axioms should hold true.

The reader familiar with large cardinals will recall that the strongest large cardinal axioms assert the existence of nontrivial elementary embeddings of the universe into transitive proper class models of ZFC (in this paper, we reserve the letter $M$ to denote such models); the first ordinal moved by such an embedding, called the critical point of the embedding, is the large cardinal defined by the embedding. Generally speaking, the stronger the large cardinal axiom becomes, the more completely the image model $M$ is required to resemble $V$. From this perspective, the strongest large cardinal axiom of all would be the assertion of the existence of a nontrivial elementary embedding from $V$ to itself. Kunen [K2], however, showed (in the context of class theories) that such an embedding is too strong to be consistent, and so this type of embedding has not been studied extensively.

Because an embedding of this kind represents a natural upper limit to the strongest large cardinal axioms, our strategy has been to determine whether such an embedding is suggested to us by our Vedic principles; that is, does the picture of the universe $V$ that is painted using our Vedic principles suggest to us that an elementary embedding from $V$ to itself ought to exist? It was surprising to find that it does and that it even points to certain assumptions that are buried in Kunen’s proof, which are essential for the proof to go through, but which are not guaranteed to hold true in the context of ZFC (but which do hold true in the usual class theories).
To begin our analysis, let us recall from the last section that self-transformation suggests that the universe, the mathematical analogue of wholeness, “moves within itself, is transformed within itself, yet remains unchanged by the transformation.” This feature suggests that there is some sort of mapping naturally associated with the universe, with domain and codomain being the universe itself, and with the property that the “universe remains unchanged” by the mapping—a phrase we could interpret as meaning “all relationships are preserved.” These requirements on a mapping seem most naturally satisfied by an elementary embedding. Note that since the requirement is that there be some kind of “move” effected by the map, we can eliminate the identity mapping from consideration, and we seem to be left with a well-motivated nontrivial elementary embedding from $V$ to itself. 9

If we turn to the Focal Point Principle, we encounter additional requirements on the embedding: The embedding should give rise to a focal point within $V$ in which “all knowledge and power” of $V$ are found in seed form. A natural candidate for this “point” in the universe is the critical point $\kappa$ of the embedding. Certainly, of all sets in the universe, the critical point $\kappa$ is endowed by the embedding with the most powerful properties. As we will show, $\kappa$ necessarily has virtually all large cardinal properties; it also acts as a “seed” from which all sets in the universe can be located, as we will discuss more fully in the context of the Blueprint Principle, below.

These observations make a case for the “power” part of the focal point principle. As for the “knowledge” part, we draw upon one more mathematical result that we will demonstrate below: namely, that once we have such an embedding with critical point $\kappa$, it follows that the stage $V_\kappa$ is an elementary submodel of $V$; in symbols, $V_\kappa \prec V$. (This means that, for any possible relationship among sets $a_1, \ldots, a_n$ belonging to $V_\kappa$, this relationship holds in $V_\kappa$ if and only if it holds in $V$; more precisely, the inclusion map $\text{incl}: V_\kappa \to V: x \mapsto x$ is an elementary embedding. In particular, $V_\kappa$ and $V$ satisfy the same first-order sentences.) This relation provides a believable realization of the requirement that

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9 Roughly speaking, an elementary embedding $j$ from $V$ to a transitive model $M$ of ZFC is a function that “preserves all relationships” among sets; that is, for any first-order formula $\phi(x_1, \ldots, x_n)$ and any sets $a_1, \ldots, a_n$, $\phi(a_1, \ldots, a_n)$ holds true in $V$ if and only if $\phi(j(a_1), \ldots, j(a_n))$ holds true in $M$. Also, $j$ is nontrivial if for some ordinal $\alpha$, $j(\alpha) \neq \alpha$. See [We], [Co3], or [Je].
“all knowledge of $V$ can be found in $\kappa$ in seed form”: every first order statement true in $V$, and no false statements, must also hold at the $\kappa$th stage of the universe.

The Blueprint Principle suggests to us that the point value that we locate should in some way give rise to a blueprint that encodes the essential information about the structure of the “manifest universe.” We take “sets” to correspond to the manifest universe here. To fill the role of a “blueprint,” we have found it natural to consider a Laver sequence. A Laver sequence $[La]$ (see also [Co3]) at $\kappa$ is a function $f: \kappa \to V_\kappa$ such that for every set $A$ there is a supercompact embedding $i: V \to M$ such that $A = i(f)(\kappa)$, where $M$ is a transitive class model of ZFC. In other words, a Laver sequence “codes up” the information about the location of each set in the universe: Each set occurs as the $\kappa$th term of the image of the Laver sequence under a suitable supercompact embedding. As we will show below, given $j: V \to V$, the necessary supercompact embeddings arise naturally from $j$.

We have made use of our first three Vedic principles to motivate the existence of a $j: V \to V$ and have found that the mathematical implications of such an embedding correspond nicely with the Vedic requirements. At this point, we can no longer postpone our obligation to address the first major obstacle to our program so far:

**Obstacle #1: Kunen’s Theorem.** In [K2], K. Kunen showed, in the context of class theories, that there is no nontrivial elementary embedding from the universe to itself. A viewpoint held by many set theorists, especially when the discovery was first made, was that Kunen’s theorem proved that such embeddings are inconsistent with ZFC (which is not a class theory). However, since the existence of such an embedding cannot be formalized in the language of ZFC, care is needed in determining the impact of such an embedding on ZFC. As it turns out, what Kunen’s result forbids, in the context of ZFC, is elementary embeddings $j: V \to V$ that are weakly definable in $V$. We can rephrase Kunen’s theorem as follows:

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6 Suppose $\langle M, e \rangle$ is a model of ZF and $X \subseteq M$. Then $X$ is weakly definable in $M$ if the expanded model $\langle M, e, X \rangle$ satisfies all instances of Replacement for formulas of the expanded language. It is straightforward to show that whenever $X$ is definable in $M$, it must also be weakly definable in $M$. 

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3.1 Theorem. (Kunen) Every nontrivial elementary embedding of $V$ to itself is not weakly definable in $V$.

Kunen’s proof cannot be used to prove the inconsistency of a $j: V \to V$ with critical point $\kappa$ if $j$ is not weakly definable since, in the proof, we must be able to take the supremum of the sequence $\kappa$, $j(\kappa)$, $j^2(\kappa)$, $\ldots$, and this step requires an application of Replacement for formulas in the expanded language (formulas that have an occurrence of $j$). Since $j$ is not weakly definable, we are not allowed to freely apply Replacement to $j$-formulas, and so the supremum may not exist. Thus, the proof is not applicable to such an embedding. We propose, then, to introduce an axiom that asserts the existence of such a $j$ that is not weakly definable in $V$.

Mathematically speaking, then, obstacle #1 has been overcome. Still, we must ask whether or not it follows from our Vedic vision that an elementary embedding from $V$ to itself—if such a thing exists at all—really ought to be undefinable (or more precisely, not weakly definable). The applicable principle here is the Principle of Global Undefinability. This principle says that the self-transformation of wholeness, which in our context is embodied in a $j: V \to V$, is fundamentally unmanifest and ungraspable from the perspective of the manifest universe. This view suggests that if there is going to be a $j: V \to V$ at all, we should expect that it would be highly undefinable in $V$.

Based on this analysis, we would like to assert as a basic axiom that there is a nontrivial elementary embedding $j: V \to V$ that is not weakly definable in $V$. However, one other obstacle seems to darken our hopes:

**Obstacle #2:** Embeddings $j: V \to V$ that are not weakly definable can be weak. The fact is that elementary embeddings from a model of ZFC to itself that fail to be weakly definable are quite common: For instance, under certain mild large cardinal assumptions (for instance, the existence of a measurable cardinal), there is such an embedding from $L$ to $L$, where $L$ is the constructible universe (see [Je]). We didn’t have such an embedding in mind in declaring our axiom, but clearly a $j: L \to L$ is a possible interpretation. By observing what makes this latter type of embedding so weak, we can see what else needs to be added to elementarity of the embedding to capture our intention.
becomes apparent is that the action of a \( j : L \to L \) is dis-coordinated from the structure of the model. For instance, if we try to form the canonical ultrafilter\(^\text{11}\) \( U = \{ X \subseteq \kappa : \kappa \in j(X) \} \) using such a \( j \), we find that relative to \( L \), \( U \) is not a set. Not only is \( j \) globally unrecognizable to \( L \) (being highly undefinable), but it is also locally unrecognizable (since sets cannot always be defined using formulas—even bounded formulas—that refer to \( j \)).

Now from our Vedic perspective, we would not expect our embedding to be locally out of synchrony with the universe. Our final Vedic principle, Local Existence, implies that the dynamics of wholeness, represented by \( j \), while being highly undefinable, should nevertheless be well-coordinated with the universe; indeed the dynamics are supposed to be “present at every point.” One natural way to implement Local Existence is to require not only that \( j \) not be weakly definable, but, for each set \( A \), that \( j \restriction A \) also be a set.\(^\text{12}\) This eliminates situations that are like embeddings from \( L \) to \( L \), and seems very much in accord with our fifth Vedic principle.

We are now ready to state our Wholeness Axiom:

**Wholeness Axiom**

There is a nontrivial elementary embedding \( j : V \to V \) that is not weakly definable in \( V \), such that for all sets \( A \), \( j \restriction A \) is a set.

Here, by “nontrivial,” we mean that \( j \) is not the identity map, that it moves at least one set. It can be shown, as we observe in Proposition 3.2, that such an embedding must actually move some ordinal. The least such ordinal is called the critical point of the embedding.

We will call an embedding \( j \) given by the Wholeness Axiom a WA-embedding. Given a nontrivial elementary embedding \( j : V \to V \) that is

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\(^{11}\) When a collection \( U \) is defined in this way from an elementary embedding \( j : V \to M \) with critical point \( \kappa \), \( U \) is certainly a set and has the properties of a normal measure on \( \kappa \), which makes \( \kappa \) a measurable cardinal. Embeddings of the form \( j : L \to L \) are therefore too weak to give rise to a normal measure on \( \kappa \); in fact it can be shown that \( L \) does not contain a measurable cardinal.

\(^{12}\) In the literature on this subject (developed subsequent to the present paper), the Wholeness Axiom is defined so that Local Existence is translated into an even stronger mathematical requirement: that Separation holds for all \( j \)-formulas. See[Co2].
not weakly definable, if $V$ has the property that for every set $A, j \upharpoonright A$ is a set, we will say that $V$ is $j$-closed.

We should observe here that WA is not formally expressible in the language of ZFC set theory. In Theorem 4.10, we show how to formalize WA in a language in which there is, in addition to the usual binary relation symbol $\epsilon$, one unary function symbol $j$.

Before systematically examining some mathematical consequences of the Wholeness Axiom, which we do in the next section, we pause here to prove some claims that we made earlier to motivate WA. In particular, we need to show that

1. The critical point $\kappa$ of the embedding $j$ has virtually all large cardinal properties.
2. $V_\kappa \prec V$.
3. $j$ gives rise to a Laver sequence in a fairly natural way.

We prove (2) and (3) here; we will also prove (1) up through supercompactness and reserve the proof of the stronger assertion (that $\kappa$ is super-$n$-huge for every $n$) for the next section.

To begin, we dispense with an easy but essential observation, which is proved using the standard argument (see [Je, Lemma 28.5]):

3.2 Proposition. If $j: V \rightarrow V$ is a WA-embedding, there is an ordinal $\kappa$ such that $j(\kappa) \neq \kappa$. ❑

Given a WA embedding $j$, as we mentioned before, the sequence $\kappa, j(\kappa), j^2(\kappa), \ldots$ is not weakly definable, for, if it were, by Replacement, its range would have a supremum $\lambda$, and we could reproduce Kunen’s inconsistency argument (see [KM, p. 202]) for embeddings $i: V_{\lambda+2} \rightarrow V_{\lambda+2}$. Thus we have the following:

3.3 Proposition. If $j: V \rightarrow V$ is a WA-embedding, then the sequence $\kappa, j(\kappa), j^2(\kappa), \ldots$ is not weakly definable in $V$ and is unbounded in ON. ❑
Using this $\omega$-cofinal sequence, we can reason\(^{13}\) as though we had an $I_3$-embedding $V_\alpha \rightarrow V_\alpha$ to show that
\[
V_k \prec V_{j^*(k)} \prec \ldots \prec V_{j^{n}(k)} \prec \ldots
\]
and hence that for each $n$, $V_{j^n(\kappa)} \prec V$, since in this case $V$ is the union of an elementary chain (see [KM, p. 203]). Thus, the following proposition takes care of (2):

3.4 Proposition. If $j : V \rightarrow V$ is a WA-embedding, then for each $n$, $V_{j^n(\kappa)} \prec V$. □

Our last proposition in this section establishes the existence of Laver sequences in the presence of WA (taking care of (3)); it also tells us that the critical point of a WA-embedding is supercompact. We begin with a review of the definition of a Laver sequence at $\kappa$:

3.5 Definition. Suppose $\kappa$ is an infinite cardinal. A Laver sequence at $\kappa$ is a function $f : \kappa \rightarrow V_\kappa$ such that for every cardinal $\lambda \geq \kappa$ and every set $A$, if $|\text{TC}(A)| \leq \lambda$, then there is a canonical supercompact embedding $i : V \rightarrow M$ (where $M$ is a transitive class model of ZFC) defined from a supercompact ultrafilter over $P_\kappa \lambda$ such that $A = i(f)(\kappa)$.

3.6 Proposition. If $j$ is a WA-embedding with critical point $\kappa$, then there is a Laver sequence at $\kappa$.

Proof. By [La, p. 386], it suffices to show that $\kappa$ is $\lambda$-supercompact for every $\lambda \geq \kappa$. Given $\lambda \geq \kappa$, let $n_\lambda$ be the least integer $n$ such that $j^n(\kappa) > \lambda$ (using Proposition 3.3). Let
\[
U = \{A \subseteq P_\kappa \lambda : (j^{n})'^\lambda \in j^{n}(A)\}.
\]

\(^{13}\) The remarks here mask certain subtleties that need to be handled carefully in the formal proof of Proposition 3.4. Though the reasoning, at a high level, is like that used in the context of $I_3$ embeddings, there are technical issues that arise in the context of WA (for instance, it is possible that $j^n(\kappa)$ fails to exist for certain $n$), which make the proof considerably more difficult. See [Co1, Proposition 8.13ff.] for a detailed treatment of these issues. A similar point applies to other applications of the $j$-closed property in subsequent proofs.
Set $n = n_\lambda$. We use the fact that $V$ is $j$-closed to verify that $U$ is a set. First, notice that $j^n$ may be replaced by $j^n | PP_\kappa \lambda$. We also have

$$(j^n)'' \lambda \subset j^{2n}(\kappa).$$

Therefore,

$$(j^n)'' \lambda = \{ \beta \in i(\kappa) : (\exists \alpha < \lambda) i(\alpha) = \beta \},$$

where $i = j^{2n}$.

Now since $n_\lambda$ was chosen large enough, $P_\kappa \lambda \in U$, and the usual arguments [Je, Chapter 33] show that $U$ is a $\lambda$-supercompact ultrafilter over $P_\kappa \lambda$. □

4. The Theory ZFC + WA

In this section we prove several mathematical consequences of assuming the Wholeness Axiom. In the context of the general theme of our paper, the most important of these is that the critical point of a WA-embedding has virtually all large cardinal properties. We also show that the large cardinal axiom $I_\alpha$ is strictly stronger than WA and discuss a formalization of ZFC + WA.

We begin with a review of the definition of some of the strongest large cardinal axioms. For a thorough treatment of other large cardinal axioms and related notions, see [Je] or [KM].

4.1 Definition. Suppose $\kappa$ is an infinite cardinal and $n \in \omega$. Then $\kappa$ is $n$-huge if there is a transitive model $M$ of ZFC containing all the ordinals and an elementary embedding $i: V \rightarrow M$ with critical point $\kappa$ such that every function $g: i^n(\kappa) \rightarrow M$ belongs to $M$. Such a map $i$ is called an $n$-huge embedding.

4.2 Definition. [Ba] Suppose $\kappa$ is an infinite cardinal, $n \in \omega$, and $\alpha$ is an ordinal. Then $\kappa$ is said to be $n$-huge $\alpha$ times if there exist a sequence $\langle M_\beta : \beta < \alpha \rangle$ of transitive models of ZFC containing all the ordinals and a sequence of elementary embeddings $\langle i_\beta : V \rightarrow M | \beta < \alpha \rangle$, each with critical point $\kappa$, such that each $i_\beta$ is an $n$-huge embedding, and the
sequence of targets \( \langle i_\beta(\kappa) : \beta < \alpha \rangle \) is strictly increasing. The cardinal \( \kappa \) is called super-\( n \)-huge if \( \kappa \) is \( n \)-huge \( \alpha \) times for every \( \alpha \).

4.3 Definition. The axiom \( I_\alpha \) is the statement that there is a limit ordinal \( \alpha \) and a nontrivial elementary embedding \( i : V_\alpha \rightarrow V_\alpha \). The axiom \( I_{n} \) is the statement that there is a limit ordinal \( \alpha \) and a nontrivial elementary embedding \( i : V_{\alpha+1} \rightarrow V_{\alpha+1} \).

4.4 Proposition. [Ba] If \( \kappa \) is super-\( n \)-huge for every \( n \), then \( \kappa \) is supercompact, extendible, and \( n \)-huge for every \( n \).

The property of being super-\( n \)-huge for every \( n \) is the strongest among the well-known large cardinal properties that are weaker than \( I_3 \). (The fact that \( I_3 \) is consistency-wise stronger than super-\( n \)-huge for every \( n \) follows from the proofs of Theorems 4.5 and 4.6 below.)

4.5 Theorem. Suppose \( j : V \rightarrow V \) is a wa-embedding with critical point \( \kappa \). Then \( \kappa \) is super-\( n \)-huge for every natural number \( n \).

Proof. We first verify that \( \kappa \) is \( n \)-huge for every \( n \). To see that \( \kappa \) is huge, let \( U = \{ A \subseteq P(j(\kappa)) : j''(j(\kappa)) \in j(A) \} \). Because \( V \) is \( j \)-closed, \( U \) is a set. The usual arguments (see [Je] or [KM]) show that \( [j(\kappa)]^\omega = \{ B \subseteq j(\kappa) : \text{ordertype}(B) = \kappa \} \in U \), that \( U \) is a normal, fine ultrafilter over \( P(j(\kappa)) \), and hence that \( \kappa \) is huge. To see that \( \kappa \) is \( n \)-huge for every \( n \), reason as above, replacing \( j(\kappa) \) with \( j^n(\kappa) \).

Now, to prove super-\( n \)-hugeness for every \( n \), it suffices to show that for all \( m, n \in \omega \), \( \kappa \) is \( n \)-huge \( \kappa_m \) times, where \( \kappa_m = j^m(\kappa) \). We will first show that there is a stationary subset \( S \) of \( j(\kappa) \) each of whose elements is the target of an \( n \)-huge embedding with critical point \( \kappa \); then we apply a suitable elementary embedding repeatedly to \( S \) to show that similar stationary sets exist below each \( \kappa_m \).

Let

\[
U = \{ X \subseteq j(\kappa) : j(\kappa) \in j(V_{j(\kappa)})(X) \}.
\]

\( U \) is a set because \( V \) is \( j \)-closed. The usual arguments show that \( U \) is a normal, nonprincipal ultrafilter over \( j(\kappa) \). Note that the critical point of \( j(V_{j(\kappa)}) \) is \( j(\kappa) \).
Now, setting
\[ S_1 = \{ a < j(k) : a \text{ is a target of some } n\text{-huge} \]
embedding having critical point \( k \}, \]
it is easy to see that \( S_1 \in U \) since \( j(\kappa) \) is a target of an \( n\)-huge embedding having critical point \( \kappa \), as we showed in the first paragraph. Hence, \( S_1 \)
is stationary.

For each \( m > 0 \), let \( i_m = j(j(V_{\kappa_{m+1}})) \). Now for each \( m \), inductively define
\[ S_{m+1} = i_m(S_m). \]

By elementarity, \( S_m \) is a stationary subset of \( \kappa_m \) each of whose elements is a target of an \( n\)-huge embedding with critical point \( \kappa \). □

In order for the last result to be significant, we need to know that
\( \text{WA} \) is consistent. On the one hand, our hope is that our intuitive model is compelling enough to make it “obvious” that a universe with such an embedding exists—at least, perhaps, as compelling as Cantor’s Absolute Infinite as an intuitive model to justify the existence of a model \( V \) of ZFC. On the other hand, from a strictly mathematical point of view, we can at least show that the relative consistency of ZFC + \( \text{WA} \) is implied by a known large cardinal axiom, namely \( I_3 \); for, suppose \( \alpha \) is a limit and \( j : V_{\alpha} \rightarrow V_{\alpha} \) is an elementary embedding with critical point \( \kappa \). Then as in [KM, p. 203], \( V_{\alpha} \) is a model of ZFC. But it is also clear that \( V_{\alpha} \) is \( j \)-closed, and, by Kunen’s Theorem, that \( j \) is not weakly definable in \( V_{\alpha} \). We have therefore:

4.6 Theorem. Con(ZFC + \( I_3 \)) implies Con(ZFC + WA).

What about the converse? Notice that if \( \alpha \) is the least limit for which there is a nontrivial elementary embedding \( V_{\alpha} \rightarrow V_{\alpha} \), then we have the following:
\[ V_{\alpha} \models \text{WA} + \neg I_3. \]

Thus, \( I_3 \) bears the same relationship to the ZFC + \( \text{WA} \) universe as an inaccessible bears to the ZFC universe, and it is not even possible to prove from ZFC + \( \text{WA} \) the consistency of ZFC + \( I_3 \).
4.7 Theorem. \( \text{Con}(\text{ZFC} + \text{WA}) \) does not imply \( \text{Con}(\text{ZFC} + I) \).

From our Vedic perspective, how is the axiom \( I \) to be viewed, since it isn’t derivable from \( \text{ZFC} + \text{WA} \)? Since our Vedic model for the universe is compatible with Cantor’s, we may still legitimately apply the Reflection Principle; doing so tells us that if there is a WA-embedding \( j: V \rightarrow V \), the same ought to be true relative to some stage \( V_\alpha \) of the universe.

Nevertheless, even with Reflection, we do not have a way, using our Vedic model, to justify the existence of the large cardinals given by any of the axioms \( I_2 \) to \( I_0 \). The strongest of these, \( I_0 \), arose in Woodin’s work to prove the consistency of AD relative to large cardinals (see [Ma2] for a discussion). The fact that no inconsistency arose in the rather involved uses to which Woodin put this axiom has offered some hope that even this strong assertion may be consistent. In the present context, however, it is not obvious how to motivate \( I_0 \) using the Vedic model described here. We leave this problem as an open question.

4.8 Open Question #1. Can the Vedic perspective given here be deepened (in a natural way) so that the truth of any of the axioms \( I_2 – I_0 \) becomes “obvious”?

Our applications of WA so far have necessarily been metatheoretic since WA cannot be expressed in the language of ZFC. We now offer a more formal way of presenting the Wholeness Axiom and indicate why the formal version matches our intended intuition for the axiom; see [Co2] for a detailed account. To formulate our axiom, we will work in the usual language of set theory, with one additional (unary) function symbol \( j \), intended to stand for an elementary embedding of the universe. The introduction of this new function symbol expands the range of formulas that are now used in the formal theory to represent mathematical assertions. Formulas in which the new symbol \( j \) occurs will be called \( j \)-formulas (or \( j \)-sentences if there is no occurrence of a free variable). Formulas in which \( j \) does not occur are called \( \in \)-formulas. The Wholeness Axiom (WA) is now formally defined as a schema of axioms consisting of the following:
Elementarity Schema. Each of the following \( j \)-sentences is an axiom, where \( \phi(x_1, x_2, \ldots, x_m) \) denotes an \( \epsilon \)-formula:

\[
\forall x_1, x_2, \ldots, x_m \left( \phi(x_1, x_2, \ldots, x_m) \iff \phi(j(x_1), j(x_2), \ldots, j(x_m)) \right).
\]

Nontriviality. For some set \( x \), \( j(x) \neq x \).

Amenability. For every set \( x \), \( j \upharpoonright x \) is a set.

The expanded set theory we are proposing is ZFC + WA. We note that if \( i: V_\alpha \rightarrow V_\alpha \) is an \( I_3 \) embedding, then \( \langle V_\alpha, \epsilon, i \rangle \) is a model of ZFC + WA, so the formal theory is consistent relative to the axiom \( I_3 \), exactly as we described above. Also, we note that in any model of ZFC + WA, the interpretation \( j \) of \( j \) in the model can never be weakly definable, because of Kunen’s Theorem, so the formally defined \( j \) must fail to be weakly undefinable in any model in which it is realized.

5. Laver Sequences

Our application of the Vedic paradigm to set theory has placed the concept of a Laver sequence in a prominent role. In this section we study certain properties of Laver sequences more closely, as they relate to our Vedic model. This analysis will lead to the observation that a cardinal \( \kappa \) admits a generalized Laver sequence if and only if \( \kappa \) is strong, and will highlight several simple properties of Laver sequences that have clear parallels to features of the structure of the Veda.

We begin by recalling the significance of Laver sequences in our application of the Vedic model. In applying the Blueprint Principle, our goal was to locate some compact collection that would code up the essential information about every set in the universe, and that would emerge in some natural way from the interaction of a WA-embedding \( j \) and its critical point \( \kappa \). What made the notion of a Laver sequence suitable in this context was that, first, we could obtain such a sequence \( f \) from \( j \) and \( \kappa \), and second, that every set occurs as the \( \kappa \)th term of some image of \( f \) by a suitable elementary embedding.\(^{14}\) Actually, the

\(^{14}\) We make a few observations here about elementary embeddings and Laver sequences. Nontrivial elementary embeddings of the form \( i: V \rightarrow M \) (where \( M \) may or may not be equal to \( V \)
sort of Laver sequence that was defined by Laver imposed a more stringent requirement: namely, the only elementary embeddings that one is allowed to apply to \( f \) to satisfy the condition are canonical embeddings that are defined from \( \lambda \)-supercompact ultrafilters for various \( \lambda \geq \kappa \). While this requirement is essential in certain applications (for example, in the proof of the consistency of the Proper Forcing Axiom), it isn’t necessary to meet the more general philosophical requirement we are seeking. These considerations lead us to define generalized Laver sequences:

5.1 Definition. Suppose \( \kappa \) is an infinite cardinal. Then a generalized Laver sequence at \( \kappa \) is a function \( f: \kappa \to V_\kappa \) such that for every set \( A \), there is an elementary embedding \( i: V \to M \), where \( M \) is a transitive model of ZFC containing all the ordinals and \( \kappa \) is the critical point of \( i \), so that \( A = i(f)(\kappa) \). In this case, we say that \( \kappa \) admits a generalized Laver sequence.

The definition suggests the following question: What is the consistency strength of the statement, “\( \kappa \) admits a generalized Laver sequence”? If we replace “generalized Laver sequence” with “Laver sequence,” it is obvious that the supercompactness of \( \kappa \) is both necessary and sufficient because of the dependence of the concept of Laver sequence on supercompact embeddings (and because of Laver’s theorem [La, p. 386]). For the generalized case, we first observe that if \( \kappa \) admits a generalized Laver sequence, then \( \kappa \) must at least be a strong cardinal. (Recall that \( \kappa \) is strong if for each \( \lambda \geq \kappa \) there is an elementary embedding \( i: V \to M \) such that \( M \) is a transitive model of ZFC containing all the ordinals, \( \kappa \) is the critical point of \( i \), and \( V_\lambda \subseteq M \). See [MS].) This holds because every \( V_\lambda \) must be an \( i(f)(\kappa) \) for some \( i: V \to M \) with critical point \( \kappa \), and hence \( V_\lambda \in M \). The proof of the converse requires more work; details may be found in [Co2].

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itself) with critical point \( \kappa \) always have the property that \( \overline{i}(\kappa) \supset \kappa \). By elementarity of such an \( i \), one can show that for any function \( f: A \to B \), \( i(f) \) is also a function having domain \( i(A) \) and codomain \( i(B) \). In particular, if \( f: \kappa \to V_\kappa \) is a Laver function, whenever \( i: V \to M \) is elementary with critical point \( \kappa \), \( i(f) \) is a function with domain \( i(\kappa) \). Since \( \kappa \in i(\kappa) \), we can apply \( i(f) \) to \( \kappa \). The key property of a Laver sequence is that, for any set \( A \), we can find \( i \) so that \( A = i(f)(\kappa) \).
5.2 Theorem. An infinite cardinal $\kappa$ admits a generalized Laver sequence if and only if $\kappa$ is strong.

If Laver sequences—especially those defined from a WA-embedding—are supposed to be an analogue to the Veda, how far does the analogy go? To what extent does the structure of a Laver sequence mirror that of the Veda? We make some observations here as a starting point for deeper research to appear in later work.

For our discussion, the commutative diagram pictured below will be useful. In the diagram, $\lambda \geq \kappa$, $U_\lambda$ is a fine, normal measure on $P_\kappa \lambda$, $i_\lambda$ is the canonical embedding, $U = \{X: \kappa \in i_\lambda(X)\}$ is the measurable ultrafilter defined from $i_\lambda$, $i$ is the canonical embedding defined from $U$, and $k$ is defined by $k([g]) = i_\lambda(g)(\kappa)$. (See [Je, Chapter 28].)

A. The Veda “contains everything.” Maharishi has explained that every item in the manifest universe occurs in seed form in the Veda. It is interesting to note that in order for a Laver sequence $f$ at $\kappa$ to “capture” every set, it must be the case that each member of $V_\kappa$ occurs stationarily often in $f$. Moreover, any set $A$ must be represented by a function $g: P_\kappa \lambda \to V$ and stationarily often, $g$ “agrees with” $f$. We state this more precisely in the following proposition:

5.3 Proposition. Suppose $f: \kappa \to V_\kappa$ is a Laver sequence at $\kappa$, $A$ is a set, and $\lambda \geq \max(\kappa, |TC(A)|)$. Then there are $U_1, i_1, U, i$ as in the diagram and a function $g: P_\kappa \lambda \to V$ representing $A$ in the ultrapower by $U_\lambda$ such that

$$\{P \in P_\kappa \lambda : g(P) = f(P \cap \kappa)\} \in U_\lambda$$

and

$$\{\alpha < \kappa : \exists P (\alpha = P \cap \kappa \land g(P) = f(\alpha))\} \in U.$$
Proof. Recall (see [Je, p. 409]) that for all $\alpha \leq \lambda$, $\alpha$ is represented in $M_\lambda$ by the function $t_\alpha$ defined by

$$t_\alpha(P) = \text{ordertype}(P \cap \alpha),$$

and $i_\lambda''(\alpha)$ is represented by $r_\alpha$ where

$$r_\alpha(P) = P \cap \alpha.$$

Note also that because $B \in U$ iff $i_\lambda''(\lambda) \in i_\lambda(B)$, we have

$$\{P \in P_\lambda: \text{ordertype}(P \cap \kappa) = P \cap \kappa\} \in U_\lambda.$$

Now, given $A$ and $\lambda \geq \max(\kappa, |\text{TC}(A)|)$, let $i_\lambda$ be such that

$$(*) \quad i_\lambda(f)(\kappa) = A.$$

Clearly $A \in M_\lambda$. Let $g: P_\lambda \rightarrow V$ represent $A$ in $M_\lambda$. Since the constant function $c_f$ represents $f$, and $t_\kappa$ represents $\kappa$, we can rewrite $(*)$ as

$$[c_f][t_\kappa] = [g].$$

By Łoś’ Theorem and the fact that $\text{ordertype}(P \cap \kappa) = P \cap \kappa$ on a set in $U_\lambda$,

$$\{P \in P_\lambda: f(P \cap \kappa) = g(P)\} \in U_\lambda.$$

Next, observe that

1. $i_\lambda(f)(\kappa) = [g] = i_\lambda(g)(i_\lambda''\lambda)$;
2. $\kappa = i_\lambda''\lambda \cap i_\lambda(\kappa)$; and
3. $i_\lambda'' \in P_{i_\lambda(\kappa)}i_\lambda = i_\lambda(P_\lambda)$.

Hence,

$$\kappa \in \{\alpha < i_\lambda(\kappa): \exists P (\alpha = P \cap i_\lambda(\kappa) \land g(P) = f(\alpha))\}.$$

It follows that $\{\alpha < \kappa: \exists P (\alpha = P \cap \kappa \land g(P) = f(\alpha))\} \in U$. □
B. **Infinite dynamism/infinite silence.** The structure of the Veda is said to embrace a wide range of opposite values; in particular, its structure is said to be at once infinitely dynamic and infinitely silent. The coexistence of these opposite values is said to make possible the full range of activity in the manifest universe. If we wanted to formulate the notion of dynamism as a property of a mathematical structure, we might think of fast-growing functions; and the most nonactive or “silent” behavior a function could exhibit would probably be that of being the identity map. We might conjecture that for any Laver sequence \( f \), the function \( \alpha \mapsto [f(\alpha)] \) dominates, on a stationary set, any function in \( ^{\kappa}\kappa \) that is definable in \( V_\kappa \) and that \( f \) is the identity on another stationary set.\(^{15}\) Note that the first half of the conjecture would imply that Laver sequences at \( \kappa \) are not definable in \( V_\kappa \). This accords with our Vedic view, since the Veda, like the dynamics of wholeness corresponding to \( j \), is to be understood as entirely unmanifest in nature.

In spirit, our conjecture does indeed turn out to be true. Although it is possible for a Laver sequence to be definable in \( V_\kappa \), this can only happen (assuming WA) if a significant restriction is placed on the structure of the universe itself: Assuming WA, there is such a definable Laver sequence if and only if there is a definable well-ordering of the universe. Intuitively, we may consider \( V_\kappa \) as a model of wholeness, to be “too ungraspable” to admit such a well-ordering; from this perspective, Laver sequences are always undefinable in \( V_\kappa \). Also, though we cannot show that Laver sequences always dominate definable functions in \( ^{\kappa}\kappa \),

---

\(^{15}\) In set theory, stationary sets represent “rather big” subsets of a regular cardinal \( \kappa \), much more substantial than an arbitrary subset of size \( \kappa \). Intuitively, we wish to demonstrate that “infinite silence” and “infinite dynamism” are both displayed in the behavior of a Laver function simultaneously on different big subsets of its domain, and we can achieve this if by “big” we mean “stationary.” A helpful analogy is to consider different kinds of subsets of the unit interval \([0,1]\) of size \( c \) (the cardinality of the continuum). The “thickest” among such sets would be a (Lebesgue) measure 1 set. Next would be sets that have outer measure 1 (it is possible for there to be two such sets that are disjoint, but this is not possible for measure 1 sets; note that if \( X \) has outer measure 1 and is disjoint from another outer measure 1 set, then \( X \) must meet every measure 1 set). Finally we have the “thinnest” such sets—sets of measure zero. The analogy to subsets of a regular cardinal \( \kappa \) is this: The thickest subsets of \( \kappa \) are those that contain a closed unbounded set (“closed” means that it contains the supremum of each of its subsets); these are so thick that intersecting fewer than \( \kappa \) of these yields another such set. Next are the stationary sets, which are those sets that meet every closed unbounded set; one can always build disjoint stationary sets. Finally, the “thinnest” subsets of \( \kappa \) are those that lie in the complement of a closed unbounded set.
we are able to provide (Proposition 5.10) a construction that produces Laver sequences that do have this property.

The next proposition verifies that Laver sequences exhibit “infinite silence” in the sense that they behave like the identity function on a large set.

5.4 Proposition. If \( f: \kappa \to V_\kappa \) is a Laver sequence at \( \kappa \), there is a stationary set \( S \subseteq \kappa \) such that for all \( \alpha \in S \), \( f(\alpha) = \alpha \).

Proof. Choose \( \lambda \) so that \( \iota_\lambda(f)(\kappa) = \kappa \). It follows immediately from Łoś’ Theorem that
\[
\{ \alpha < \kappa : f(\alpha) = \alpha \} \in U. \quad \square
\]

We turn to a canonical construction for Laver sequences; the construction, in the presence of WA, has enough flexibility to allow us to create very fast growing Laver sequences, as described earlier. The parameter \( t \) in the construction represents an arbitrary sequence of sets \( t: \kappa \to V_\kappa \). The function \( f \) defined in the construction will be shown to be Laver, under the assumption of WA.

5.5 Canonical Construction CC\((t)\). Define \( f: \kappa \to V_\kappa \) by recursion as follows:
\[
f(\alpha) = \begin{cases} 
some \, t_\alpha \in V_\kappa, \text{ if } f|\alpha \text{ is Laver at } \alpha \\
some \, x \in V_\kappa, \text{ where } x \text{ witnesses } \phi(f|\alpha, \lambda_{f, \alpha}) \end{cases}
\]

where \( \phi(g, \delta) \) denotes the following formula:

“There exists a cardinal \( \alpha \) such that \( g \) is a function \( \alpha \to V_\alpha \) and \( \delta \) is the least cardinal such that for some set \( y \) we have \( |TC(y)| \leq \delta \) and \( i(g)(\alpha) \neq y \) for all \( i \) canonically generated from a supercompact ultrafilter over \( P_\alpha^\delta \).”

5.6 Proposition. Assuming WA, the function \( f \) given by the construction CC\((t)\) is well-defined and is in fact a Laver sequence at \( \kappa \). Moreover, the sequence \( t \) of sets may be defined so that the function \( \alpha \mapsto |f(\alpha)| \) dominates, on a stationary set, every function \( \kappa \to \kappa \) that is definable in \( V_\kappa \).
Proof. To see $f$ is well-defined, first recall that by WA we have $V_\kappa \prec V$. Note that if there is a pair $(x, \lambda)$ witnessing $\phi(f|\alpha, \lambda_{f|\alpha})$ at all, then, since $f|\alpha \in V_\kappa$, there is, by elementarity of $V_\kappa$ in $V$, such a pair in $V_\kappa$ itself.

Now, suppose $j: V \to V$ is a WA-embedding with critical point $\kappa$. Let $D = \{ A \subseteq \kappa : \kappa \in j(A) \}$ and let $i: V \to V_\kappa / D \cong M$ be the canonical embedding. We observe first that if $\{ \alpha : f|\alpha \text{ is Laver at } \alpha \} \not\in D$ then, as $\kappa \in j(\{ \alpha : f|\alpha \text{ is Laver at } \alpha \}), f$ is Laver at $\kappa$. So, to complete the proof, it suffices to show that this set is indeed in $D$.

Working toward a contradiction, assume that $\{ \alpha : f|\alpha \text{ is Laver at } \alpha \} \not\in D$. Then we have $\{ \alpha : f(\alpha) \text{ witnesses } \phi(f|\alpha, \lambda_{f|\alpha}) \} \not\in D$, whence $j(f)(\kappa)$ witnesses $\phi(f, \lambda_f)$. To obtain a contradiction, we find an embedding $i$ canonically derived from a supercompact ultrafilter $U$ for which

$$i(f)(\kappa) = j(f)(\kappa).$$

Obtain $i$ by setting $U = \{ X \subseteq P, \lambda : j''\lambda \in j(X) \}$ and letting $i$ be the canonical embedding $i: V \to V_{\kappa}^{\text{H}} / U \cong N$. Let $k: N \to V$ be defined by $k([g]) = j(g)(j''\lambda_f)$. By the usual argument, $k$ is elementary, $j = k \circ i$, and it is straightforward to verify that $k(x) = x$ whenever $|\text{TC}(x)| \leq \lambda_f$. Hence, setting $x = j(f)(\kappa)$ and recalling that $x$ is a witness to $\phi(f, \lambda_f)$, we conclude from the previous observation that $k(x) = x$. Thus,

$$k(x) = x = j(f)(\kappa) = k(j(f))(\kappa) = k(i(f))(k) = k(i(f))(\kappa).$$

whence $x = i(f)(\kappa)$, a contradiction.

Now, to ensure that $\alpha \mapsto |f(\alpha)|$ dominates, on a stationary set, every function $\kappa \to \kappa$ that is definable in $V_\kappa$, we define a sequence $t$ of sets, to be used in CC($i$), as follows: Let $\{ h : \xi < \kappa \}$ enumerate the members of $^\kappa \kappa$ that are definable in $V_\kappa$. Then, whenever $f|\alpha$ is a Laver
sequence at $\alpha$, we let $t_\alpha = \sup \{ h_\xi(\alpha) : \xi < \alpha \} + 1$. Since the set of $\alpha$ for which $f|\alpha$ is Laver at $\alpha$ has $D$-measure 1, $f$ dominates each $h_\xi$ on a $D$-measure 1 set, and hence on a stationary set. □

Using ideas from the proposition, we may now show that, assuming WA, unless there happens to be a definable well-ordering of the universe, no Laver sequence at $\kappa$ is definable in $V_\kappa$.

5.7 Corollary. Assume WA. Then the following are equivalent:

1. There is a Laver sequence at $\kappa$ that is definable in $V_\kappa$.
2. There is a definable well-ordering of the universe.

Proof. (1 \implies 2) Suppose $f: \kappa \to V_\kappa$ is a Laver sequence definable in $V_\kappa$. Define $g: V_\kappa \to \kappa$ by

$$g(x) = \text{least } \alpha \text{ such that } f(\alpha) = x.$$ 

Let $b: \kappa \to g^"V_\kappa$ be the increasing enumeration of $g^"V_\kappa$. Define $k: \kappa \to V_\kappa$ by $k = g^{-1} \circ b$. Now $k$ is a well-ordering of $V_\kappa$, and since $f$ is definable in $V_\kappa$, so is $k$. But now because $V_\kappa \prec V$, the defining formula for $k$ defines a well-ordering of $V$ as well.

(2 \implies 1) Assume $<$ is a definable well-ordering of $V$; since $V_\kappa \prec V$, the induced well-ordering $<^\downarrow V_\kappa \times V_\kappa$ is definable in $V_\kappa$ with the same formula. Use the construction $\text{CC}(t)$ to define a Laver sequence $f: \kappa \to V_\kappa$, with $t_\alpha = \emptyset$ whenever $f|\alpha$ is Laver at $\alpha$, and with the additional refinement that, in case $f|\alpha$ is not Laver at $\alpha$, $f(\alpha)$ is chosen to be the $<^\downarrow$-least set satisfying the second condition of the construction. Clearly, $f$ is the required Laver sequence. □

C. Apaurusheya Bhāshya—uncreated commentary. As was mentioned in the Introduction, according to Maharishi, the Veda is structured in such a way that it provides its own commentary on the knowledge it contains; this self-commentary is called Maharishi’s Apaurusheya Bhāshya.\footnote{This feature of the structure of the Veda, discovered by Maharishi Mahesh Yogi, is discussed in [Ch], [Wa], and [Co3].} The way that this self-commentary takes shape is as follows:
The totality of knowledge contained in the Veda is considered present in seed form even in the first letter of the Veda, “A”; it is present in progressively more “elaborated” forms in the first syllable “AK,” the first word “AGNIM,” the first verse, the first hymn, and the first ṛmaṇḍa. Successively larger “packets” of this kind express the same totality of knowledge as earlier stages, but in more elaborated form.

Assuming WA, we can observe a similar flavor in the structure of Laver sequences: stationarily many initial segments of $f$ are themselves Laver sequences. This means that the global property of being a Laver sequence can be located at progressively smaller “micro” scales within the structure of $f$ itself, very much like finding the totality of the Veda in progressively smaller “packets” within its own structure. We state the result in the following proposition:

**5.8 Proposition.** If $j$ is a WA-embedding with critical point $\kappa$ and $f: \kappa \to V_\kappa$ is a Laver sequence at $\kappa$, then the set

$$\{\alpha < \kappa : f \upharpoonright \alpha : \alpha \to V_\alpha \text{ is a Laver sequence at } \alpha\}$$

is stationary.

**Proof.** Let $D = \{A \subseteq \kappa : \kappa \in j(A)\}$. By the usual argument, $D$ is a measurable ultrafilter, and, by $j$-closedness, $D$ is a set in $V$. Since $j(f) \upharpoonright \kappa = f$, we have

$$\{\alpha < \kappa : f \upharpoonright \alpha : \alpha \to V_\alpha \text{ is a Laver sequence at } \alpha\} \in D,$$

as required. □

Our analogy here between Maharishi’s *Apaurusheya Bhāṣya* of the Veda and the property of Laver sequences demonstrated in Proposition 5.8 has its limitations. Perhaps the most evident of these is the fact that the obvious analogue to the first letter of the Veda is the first term of a Laver sequence $f$; however, the latter has no special significance. Indeed, the first term could be replaced by any other set in $V_\kappa$, and the sequence $f'$ so obtained is indistinguishable from $f$ in that
for all canonical supercompact $i$, $i(f) = i(f')$.\textsuperscript{17} We could modify the canonical construction defined above by artificially requiring the first term of the Laver sequence to be the first $V_\alpha$ for which $V_\alpha \prec V_i$ and this would capture a certain sense of wholeness at the outset; but a Laver sequence of this kind does not emerge in any natural or canonical way from WA and so we consider the question to remain open.

5.9 Open Question #2. Is there a canonical way to define a Laver sequence from WA that even more closely reflects the Apaurusheya Bhāshya characteristic of the Veda?

6. Conclusion

Intuitive models have always been used to guide mathematical research; a good model often suggests the truth of important conjectures and methods for proving them. And the usefulness of such models has clearly played a role in the history of research in large cardinals. Many set theorists have taken Cantor’s Absolute Infinite quite seriously as a tool to motivate large cardinals; the unified perspective offered by this intuitive model adds cogency to arguments in favor of accepting the large cardinals suggested by the model. On the other hand, \textit{ad hoc} intuitive principles for justifying other large cardinals lose considerable cogency precisely because they do not emerge from a compelling conceptual framework. Based on these considerations alone, it has seemed reasonable to attempt to broaden Cantor’s intuitive model in a philosophically natural and mathematically useful way.

Though largely unfamiliar to the mathematical community, the Vedic model of wholeness has seemed a particularly natural philosophical extension of Cantor’s Absolute Infinite because, on the one hand, it agrees with Cantor’s view as far as the latter goes, and, on the other hand, it arose as a fundamental detailed insight—by an entire culture—into the nature of the infinite, without the ulterior motive of justifying a pet mathematical theory. The fact that such extensive non-mathematical research offers structural principles that seem, even in detail, to be closely related to the modern notion of elementary embeddings of the universe argues well for the suitability of the Vedic model.

\textsuperscript{17} In fact, this is also true for any $f': \kappa \to V_\alpha$ that disagrees with $f'$ only on a nonstationary set.
as a “natural” extension of Cantor’s model with genuine mathematical utility.

It is often argued that the ultimate criterion for deciding among candidates for large cardinal axioms is the reasonableness of their mathematical consequences, and not some philosophical viewpoint imposed from the outside and divorced from mathematical practice. The fact is, however, that the enormous range of appealing mathematical consequences of the dozens of known large cardinal axioms has not been sufficient to establish the latter among the basic axioms of set theory. This diversity of results has served more to legitimize large cardinals as a worthy set of supplemental notions to be used as sparingly as possible. Our view is that a compelling conceptual framework, like Cantor’s Absolute Infinite, contributes a key ingredient in the determination of the primitive notions of mathematics. Moreover, when such a conceptual framework is sufficiently natural, it serves to unify diverse notions and to stimulate deeper insight into the mathematical entities under investigation—an effect that can hardly be classified as originating from an artificial imposition of philosophical dogma.

In actual mathematical practice, set theorists freely use large cardinals as the need arises; moreover, a perusal of applications of large cardinals in recent years reveals that most authors no longer even feel the need to apologize for their use of these axioms. It seems reasonable to the author that there be an overarching mathematical context in which all mathematical research takes place instead of a long sequence of viewpoints that decrease in their plausibility as more large cardinal axioms are included. The Wholeness Axiom, based on the Vedic model, is, we feel, one worthy attempt at providing such a context.

In some circles it is argued that the view that there ought to be such an “absolute” underlying mathematical framework is old-fashioned; we should rather seek to build different mathematical universes to suit the requirements of the particular needs of different areas of mathematics, rather than insist on “stuffing” all mathematics into a single universe (see for example [Be, pp. 235–239]). Our view, again motivated by the Vedic model, is that within the context of “wholeness” there is room for all particularized mathematical models and universes. Each “world” of mathematics has its own validity and its own range of applications. But the diversity of such models does not conflict with the possibility of an
underlying unity in which these diverse world views coexist. Certainly, for example, the world of Choice and the world of Determinacy are quite different; yet in a large enough universe (if there exists a super-compact cardinal for instance) these two worlds coexist (AC holds in $V$ while AD holds in $L(R)$).

Our view that all foundational models ought to have a place within an absolute foundation also explains our lack of concern to resolve undecidable propositions like CH or statements having large cardinal strength like the existence of saturated ideals. It seems to the author quite reasonable, and in accord with our Vedic model, that different “portions” of the universe ought to reflect different possibilities concerning the size of the continuum, the existence of saturated ideals, and other such questions. On the other hand, if one adopts the view that certain large cardinals should be banned from mathematics (not for mathematical, but rather, philosophical reasons), this situation would seem to be a matter of genuine concern, for then some significant portion of mathematics is supposed to be understood as taking place, rather artificially, outside the realm of “legitimate mathematics.” Thus, our efforts in supplementing ZFC with an axiom schema such as WA have been concerned with providing a context for all large cardinals, that is, all possible consistency strengths, and not at all with resolving any of the well-known propositions that are undecidable by ZFC alone.

On a practical note, because of its richness, the Vedic model suggests many principles and properties for which one might expect to find natural mathematical analogues in the structure of the universe. We have pursued some of these here and others in [Co3]. Pursuing these analogies results in interesting mathematical conjectures and a stream of research that might otherwise never have emerged. Moreover, the results so obtained contribute to the general program of erecting a unified foundational framework that encompasses all of mathematics. It is the author’s belief that there is a great need to restore a sense of purpose and direction to foundational studies; we hope our work here will provide a step toward meeting this need.

We have found the research here compelling because of the striking relevance of the Vedic paradigm to deep structural questions of the modern set-theoretic universe. Why should insights arising from ancient methods of contemplating the ultimate structure of real-
ity exhibit strong parallels with modern-day large cardinal research? An intuition about it, again suggested by our Vedic model, is that the “infinity” that has been studied in mathematics for more than a century may well be a conceptualization of the “Infinity” that is pointed to in the ancient Vedic wisdom. From this point of view, the relevance of the ancient wisdom no longer seems surprising but instead, just what one would expect. Our hope is that, as the field of mathematics continues to evolve, Maharishi Vedic Science and the wisdom of the ancients will find their place among the resources and tools upon which mathematical researchers rely as they continue to advance their discipline.

References


THE WHOLENESS AXIOM


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Section II

Modern Mathematics in the Light of Maharishi Vedic Mathematics
Maharishi’s Absolute Number: 
The Mathematical Theory and 
Technology of Everything

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MAHARISHI’S ABSOLUTE NUMBER

ABSTRACT

The Absolute Number, recently discovered and described by Maharishi Mahesh Yogi, fulfills the historical goals of mathematics by giving complete understanding of the entire range of orderliness and precision studied by mathematics. It provides a means of fathoming and perfectly quantifying the entire field of all the laws of nature governing the universe. Even more importantly, it provides a technology whereby everyone can live in full accord with these laws so that every aspect of life, individual and national, is lived mistake-free with complete coherence and harmony throughout the world.

As well as providing an overview of Maharishi’s Absolute Number and outlining some of its special features, this article will introduce Maharishi Vedic Mathematics, which Maharishi describes as the structuring dynamics of natural law. Before doing this, however, we will briefly introduce two defining and central historical themes of mathematics, which we have called sensory-based mathematics and intellect-based mathematics.

This article will show how Maharishi Vedic Mathematics and the Absolute Number are both a timely response to the weaknesses of these approaches to mathematics and a natural completion of the direction of knowledge initiated and developed by them. In addition, it will indicate how Maharishi’s Absolute Number has the precision and the organizing power to make possible, rapidly and effortlessly, such achievements as invincible defense for all nations and perfect management for every level of society, thereby ushering in a new era for life everywhere—Heaven on Earth.

Introduction

Mathematics provides the language and the tools to study precision and orderliness through the careful definition and analysis of abstract mathematical concepts and to apply the results of this analysis to understand and quantify the orderliness of the physical world. Mathematicians usually explain that the goal of mathematics is to increase our understanding of this orderliness in its theoretical and its applied values. This article introduces Maharishi Vedic Mathematics and the Absolute Number and shows that they do not just provide another step towards this goal through an increment in our understanding of orderliness. Rather, together they complete all areas of mathematics through providing full knowledge of the entire range of orderliness and hence of the entire range of all the laws of nature gov-
erning the universe. Even more importantly, Maharishi Vedic Mathematics and the Absolute Number provide a technology for everyone to live in full accord with these laws and to gain their support for every aspect of life, individual and collective, to become mistake-free and to live in complete coherence and harmony.

In order to see the profundity of Maharishi Vedic Mathematics and the Absolute Number, it will be necessary to consider two defining and central historical themes of mathematics; we call these sensory-based mathematics and intellect-based mathematics. Roughly speaking, sensory-based mathematics is mathematics that uses the senses to count and measure physical objects to generate and validate its statements. In contrast, intellect-based mathematics or modern mathematics uses the logical abilities of the intellect to form chains of deductions leading to universally accepted mathematical results, the theorems of mathematics.

Inherent weaknesses in these two types of mathematics limit their power to benefit mankind, but these weaknesses can be overcome through Maharishi Vedic Mathematics and the Absolute Number. The major weakness of sensory-based mathematics is that it cannot deal with the abstract concepts essential for today’s mathematics and its applications. For instance, this type of mathematics cannot deal with the concepts of infinity and infinite processes, concepts which run throughout modern mathematics. The weaknesses of intellect-based mathematics are more subtle. One problem is that it must proceed using discrete steps, whereas the laws of nature that govern the universe act instantaneously and are all-encompassing. Another problem is that intellect-based mathematics can only passively observe and describe the laws of nature and therefore can never fulfill the goal of providing a technique for everyone to gain the spontaneous support of all the laws of nature by living in full accord with them.

When examined from the perspective of Maharishi Vedic Science, sensory-based and intellect-based mathematics can be seen as steps leading to the fulfillment of mathematics in Maharishi Vedic Mathematics and its central component, Maharishi’s Absolute Number. Maharishi Vedic Science presents the essential insight that the entire universe is the expression of a fundamental, unified field of pure intelligence, of pure consciousness. The laws governing the eternal self-referral or self-interacting dynamics of this field Maharishi has called
the Constitution of the Universe. (See Maharishi Mahesh Yogi, 1996, pp. 77–149 for a full discussion of the discovery of the Constitution of the Universe within the structure of this unified field.) These dynamics give rise to all the laws of nature that structure creation and guide its evolution. In his Vedic Science, Maharishi explains how evolution for human life incorporates the development of higher states of consciousness, in which the experience of self-referral, pure consciousness is increasingly integrated into daily life so that life becomes in harmony with the laws of nature and mistake-free. “Life according to natural law,” Maharishi (1986) explains, “will always be orderly, evolutionary, and nourishing to everyone” (p. 98). In the highest state of consciousness, Unity Consciousness, the field of pure consciousness is experienced to be the underlying reality not only of one’s subjective nature, but of every specific area of activity (Maharishi Mahesh Yogi, 1995a, p. 506). Maharishi Vedic Mathematics is the mathematics of this field of pure consciousness. “Vedic Mathematics,” Maharishi (1996) points out, “is the mathematics of the absolute, self-referral field of pure consciousness, where everything is simultaneous, where everything is simultaneously administered on the level of perfect order” (p. 372).

Maharishi Vedic Mathematics is not a branch of mathematics such as geometry or analysis, but a totally new level of mathematics. It provides the means to describe the totality of all the laws of nature and is, at the same time, the structure of pure consciousness actually responsible for their harmony and orderliness. “Vedic Mathematics is the structuring dynamics of natural law; it spontaneously designs the source, course, and goal of natural law—the orderly theme of evolution” (Maharishi Mahesh Yogi, 1996, p. 338).

Maharishi’s Absolute Number is a central feature of his Vedic Mathematics. Its defining role is that it is a common basis and source for all aspects and concepts of modern mathematics.

The mathematical precision and order maintained in the universe and calculable through the utilization of mathematical structures and number systems that serve to express the precise theories of different disciplines of modern science can now be seen as having their common basis in the field of the Absolute Number—the unmanifest catalyst of all numbers, number systems, and mathematical structures that quietly initiates (from the unmanifest field of intelligence—the field of
Maharishi’s Absolute Number is, however, more than just another mathematical concept. “The most important characteristic of the Absolute Number in Vedic Mathematics,” Maharishi (1996) explains, “is that it is a meaningful living reality, not just a notion or concept, and therefore does not depend on the intellect” (p. 625). This means that Maharishi’s Absolute Number is a field that supports all aspects of modern, abstract mathematics, but which is, at the same time, able to be directly experienced as the field of pure intelligence. The qualities of Maharishi’s Absolute Number of orderliness and mistake-free precision can spontaneously develop in each individual through the regular direct experience of this field.

In this way, Maharishi’s Absolute Number is both a timely answer to the weaknesses of modern mathematics described above and a natural completion of the knowledge initiated and developed by modern mathematics. It also has qualities leading to benefits that go far beyond anything possible through modern mathematics. For example, Maharishi’s Absolute Number has the organizing power necessary to be the foundation for a totally new concept in the defense of a nation which creates a level of invincibility so powerful that it even prevents the creation of an enemy. Maharishi (1996) explains, “My Absolute Theory of Defence has its basis in invincibility, which is characterized by the self-referral dynamics of consciousness, the eternal, unbounded, unmanifest, never-changing state of eternal Unity, which stands for the Absolute Number” (p. 634). Maharishi’s Absolute Number also has the precision and order necessary to form the basis for Maharishi Master Management which provides perfect management by engaging the managing intelligence of natural law. These applications are further discussed below.

This article presents a brief overview of Maharishi Vedic Mathematics and Absolute Number and might be considered as a map of some of the main features of a vast territory which Maharishi is charting. Other important features are the relationship of Maharishi’s Absolute Number with the first syllable, Ak, of the Rk Veda, the starting point of the Vedic literature and Maharishi’s Apaurusheya Bhāṣya of Rk Veda, which, Maharishi explains, is the science of the Absolute Number. Details
are given in Maharishi Mahesh Yogi (1996). Because Maharishi Vedic Mathematics and the Absolute Number are such new and profound additions to mathematics, much of this article will consist of recent quotations from Maharishi that discuss them. Most of these quotations will be from Maharishi’s Absolute Theory of Defence (1996). It is also planned that this article will form the basis for a later, more complete presentation of Maharishi Vedic Mathematics and the Absolute Number.

**Sensory-Based Mathematics**

Earliest records of human history indicate that mathematics started as a means of counting and keeping records of quantities related to daily life—how many people in a tribe, how many animals in a herd, and so on. Later, mathematics served the practical needs of agriculture, business, and industry in civilizations such as Egypt, Mesopotamia, India, and China. Initially, the counting numbers were the only components of mathematics and they were used to tally numbers of objects. Records were frequently kept by the simple strategy of assembling piles of stones or using marks in a one-to-one correspondence with the objects they represented. These marks, lines on cave walls drawn with charcoal or, later, impressions in clay produced by a stylus, eventually evolved into more sophisticated number symbols. Menninger (1969) describes the steps in this evolution.

This counting, describing, and testing of mathematical facts done by identifying mathematical concepts with physical objects and marks we shall refer to as sensory-based mathematics. Apart from the ancient examples mentioned above, there are many present day instances of this approach; members of certain tribes have been observed drawing lines on the ground with a stick to count or to test numerical statements, and young children do the same thing with their fingers. Finding the area of an actual geometrical shape by representing it as a drawing on graph paper and counting the squares within the shape is another more complex example of this same sensory-based approach to mathematics.

**Intellect-Based Mathematics**

So long as mathematics was tied to physical objects to describe mathematical facts and test their validity, its applications were severely limited. In the history of western mathematics, the Greeks were the
first to cut this bond and treat mathematics as an abstract, intellectual endeavor—a momentous step which took place between 600 and 300 B.C. The Greeks recognized that even simple mathematical objects such as triangles really are completely abstract and only have existence within the mind and the intellect. Such objects cannot be a part of sensory-based mathematics. For example, the very definition of a triangle requires that it have three straight sides. But we can’t draw such a side since the implication is that it is perfectly straight and infinitely thin. Any “triangle” drawn in a book is, as Plato wrote in *The Republic*, Book X, a “shadow” of the essential form of a triangle. Also, results such that the sum of the angles in a triangle add exactly to 180 degrees is a statement which can only have validity for these abstract triangles. Empirical statements about drawings or constructions of triangles can only be approximate.

Furthermore, during the period just described, the Greeks began to give rigorous proofs of mathematical facts. They recognized that these proofs were completely abstract and not dependent on sensory properties of objects. In other words, mathematics became an endeavor of the intellect, rather than an empirical fact-finding technology using specific objects. This view that mathematics should not be tied to the realm of objects was held so strongly by Plato that he criticized those who considered using mathematics for applications. “Nothing can be more ridiculous,” he wrote in *The Republic*. For Plato, mathematics had “divine necessity” and its students “must carry on [its] study until they see the nature of numbers with the mind only” (Book VII Sec 525; see Cornford, 1941).

Most concepts in modern mathematics, however, do not even have “shadows” in the physical world. Among the most important of these are the ideas of infinity and infinite processes which run throughout every area of mathematics. The most widespread use of the concept of infinity is in calculus, which requires the notion of an infinite limit for its basic definitions. Mathematicians also routinely use infinite-dimensional spaces and families of sets whose sizes are represented by hierarchies of infinities. Clearly mathematics involving infinity cannot be sensory-based since there are only a finite number of objects in the physical universe, even if considering objects down to the level of atomic particles. Yet the concepts of infinity and infinite processes are
vital not only within the abstract realm of mathematics; their applications can be found in the analysis of phenomena, ranging from satellite motion to the financial markets.

All mathematicians now practice what we have called intellect-based mathematics, or simply modern mathematics. This is mathematics in which its concepts and objects are abstract ideas, and its facts, called theorems, depend on chains of logical implications linking them together. On a simple level, the transition from sensory-based to intellect-based mathematics can be seen, at least at a preliminary stage, in the activities of children. When they are very young they count and add by using their fingers or manipulating objects; later they perform these same arithmetical processes using their intellects. Two apples plus three apples equals five apples becomes $2 + 3 = 5$. Thus the seeds of intellect-based mathematics are contained within sensory-based mathematics.

In the opposite direction, modern mathematicians frequently use sensory-based mathematics, such as analysis of diagrams or the construction of physical models, to generate and test hypotheses. But no matter how much sensory-based physical or experimental evidence there is for a particular conclusion, for it to be accepted as a part of the canon of mathematics, it must have its “signature” authenticated, and this can only be achieved by showing that it has a final proof conforming to the rigorous intellectual standards of modern mathematics. Modern or intellect-based mathematics raises sensory-based mathematics to new levels of permanence, universality, and, for many, aesthetic beauty.

The successes of modern mathematics are of such quantity and quality that many commentators believe we are now in the “golden age of mathematics” (Stewart, 1996). In this scientific age, the techniques of intellect-based mathematics have been applied to all areas of scientific knowledge. This success of modern mathematics is due to treating it as a creative intellectual discipline that deals, as G.H. Hardy, the great English mathematician, wrote, with ideas rather than objects (1940, p. 84). Yet here lies a puzzle. Mathematics relies on careful intellectual argument to validate its claims, while the testing of scientific claims requires precise measurements of the physical world. The sweeping universality of the scientific applications of mathematics has led mathematicians and scientists to speak of “the unreasonable effectiveness of mathematics in the natural sciences,” to quote the title of a landmark
paper by E.P. Wigner, a physics Nobel Laureate (Wigner, 1967). Gorini (1997, this volume) has analyzed this problem, offering a solution based on Maharishi Vedic Science. She explains that both mathematics and science study the same expressions of order and intelligence founded on the field of pure consciousness, the ground state of existence, but from different vantage points. The language and procedures of mathematics provide the best intellectual tools for the scientist to study the physical world leading to the “unreasonable effectiveness” of mathematics.

Despite its successes, there are two major weaknesses in intellect-based or modern mathematics. The first is that since its arguments are constructions of the human intellect, they have to proceed temporally step-by-step. So long as mathematics proceeds in this way, it can never fully capture the laws of nature governing our universe, laws that are spontaneous and all-encompassing. Consider, for example, the motion of the earth through space. Even if we used Newton’s law of gravitation to predict the earth’s motion as it happens, we would instantly have to solve equations involving an almost infinite number of components representing the positions of every other physical object in the universe.

The second weakness of modern mathematics involves the compelling order and precision of our universe and humanity’s efforts to understand and even master the laws of nature that give rise to them. Intellect-based mathematics, that is, modern mathematics, became the primary language and means of trying to unlock the secrets of these laws. Yet, all that mathematics can do is passively describe the laws of nature governing our universe. It can neither provide a direct way to harness the support of these laws nor keep us from continually breaking them.

Maharishi (1995a) makes it clear that both these pitfalls of modern mathematics are inherent in the very approach that gave it its initial successes, namely the approach via the intellect, when he writes, “Modern Mathematics, as a tool of performance (logic as a means to conclude sequentially developing, progressive steps) traps the mathematician in a web of his own creation” (pp. 295–296). Maharishi indicates here that symbolic expressions or mathematical formulas keep the mathematician’s awareness “trapped” at the level of the intellect and do not by themselves allow this awareness to evolve into its self-referral, absolute value.
This separation of the concepts of mathematics and the actual self-referral consciousness of the mathematician is a necessary consequence of the intellect-based approach of modern mathematics because, as Maharishi (1995a) explains,

The field of the intellect is the field of duality, the field where the subject and the object are separate. In the field of the intellect, diversity dominates. Only in the state of enlightenment, where the intellect becomes intelligence, are the subject and object unified, and the field of duality begins to breathe the freshness of Unity. (p. 399)

Just as the seeds of intellect-based mathematics were contained within sensory based mathematics, Maharishi is explaining that it is necessary for the intellect to become intelligence so that, as we shall see in the next section, intellect-based mathematics will be raised to the level of Maharishi Vedic Mathematics.

*pMaharishi Vedic Mathematics*

The previous section discussed the underlying order of the universe, glimpsed through the language and techniques of intellect-based mathematics describing and quantifying the laws of nature. In fact, progress in this area has been so great in the past decade that today’s physicists frequently talk about establishing the TOE or “theory of everything.”

Steven Weinberg, a physics Nobel Laureate, recently wrote:

We certainly do not have a final theory yet . . . but from time to time we catch hints that it is not far off. Sometimes in discussions among physicists, when it turns out that mathematically beautiful ideas are actually relevant to the real world, we get the feeling that there is something behind the blackboard, some deeper truth foreshadowing a final theory that makes our ideas turn out so well. (1992, p. 6)

Other scientists believe that the “TOE is hovering right around the corner” (Mukerjee, 1996). On a closer reading, however, we see that “theories of everything” make no attempt to incorporate the consciousness of observers into their models; therefore, they still maintain the duality between the observer and the objects of observation. More precisely, they separate the observer, the process of observation, and the objects of observation.
Maharishi Vedic Science unifies these three qualities of knowledge by extending the objective approach of modern science to incorporate the observer and the process of observation into the field of investigation. (See, for example, Maharishi Mahesh Yogi, 1994, pp. 153–169.) This complete science originates from the Veda, which, Maharishi (1994) explains, means pure knowledge or complete knowledge in both structure and function; it perfectly organizes and promotes all the processes of creation and evolution.

It encompasses the whole range of science and technology; it is theory and practice at the same time; it is the structure of total knowledge—Samhitā of Rishi, Devatā, Chhandas—the togetherness of the observer, process of observation, and object of observation. (p. 5)

The essential insight of Maharishi Vedic Science is that the entire universe is a sequential expression of a fundamental field of pure intelligence, pure consciousness. Maharishi describes the fundamental mechanics of creation as arising from the unified field of pure intelligence diversifying itself and thereby creating the observer, process of observation, and observed. As indicated in the preceding quotation, these are known as Rishi, Devatā, Chhandas, while Samhitā refers to their unified value as pure consciousness or pure intelligence. Maharishi explains that the Veda and the Vedic literature are the impulses of intelligence expressed as the sounds of the self-interacting dynamics of Rishi, Devatā, and Chhandas within the structure of Samhitā and that this is the administering intelligence of the universe. (See Maharishi Mahesh Yogi, 1994, p. 36 and Maharishi Mahesh Yogi, 1995a, p. 16.) The dynamic, ordering principle that gives precision to this sequential unfoldment is Maharishi Vedic Mathematics. “Vedic Mathematics,” Maharishi (1996) explains, “is the tool that structures different laws of nature from the holistic value of natural law in self-referral consciousness” (p. 355). It also creates the material universe from this same field (pp. 354–5) via the Veda and the Vedic literature:

Vedic Mathematics is the self-sufficient tool of self-referral consciousness that designs the structuring dynamics of the Veda, and materializes the design in the orderly structure of the Vedic literature, and from there, the orderly structure of the ever-expanding universe. (p. 358)
Maharishi also expresses this role of his Vedic Mathematics in terms of the Constitution of the Universe introduced in the Introduction and discussed in detail in Maharishi Mahesh Yogi (1996, pp. 77–149). For example, Maharishi (1996, p. 363) explains that his Vedic Mathematics constitutes the systems and procedures that are available in their entirety within the structure of the Constitution of the Universe, which in turn is the source of all the laws of nature.

The transformation and unfolding of the Veda into the particular physical form of the human physiology and its functioning has recently been made very clear by Tony Nader, a medical doctor and international expert in the area of brain and cognitive science. Working under the guidance of Maharishi, he discovered that the laws that construct the human mind and body are the same as those that give structure to the syllables, verses, chapters, and books of the Vedic literature. Details are given in Nader (1995).

Maharishi Vedic Mathematics is not only responsible for the precision of the laws of nature upholding the physical universe including the human physiology, but it also manages the laws that govern the behavior of all life:

Vedic Mathematics manages all the activity of everyone and everything, handling the innumerable tendencies in Nature with perfect order—maintaining the ever-expanding universe in perfect order. Every move of every ant and elephant, the stars and galaxies at every moment are in perfect precision in space and time, managed by Vedic Mathematics—the mathematics of pure knowledge. (Maharishi Mahesh Yogi, 1996, p. 375)

On this basis, Maharishi (1996) explains that it is “the sovereign ruler of the universe” and “the commander of natural law” (p. 335).

Maharishi (1996) points out that Vedic Mathematics, being structured within the field of pure consciousness, is available within the consciousness of everyone.

Vedic Mathematics, being the mathematics of the order-generating principle of pure consciousness, is itself the mathematician, the process of deriving results, and the conclusion; whatever consciousness is and wherever consciousness is, there is the structure of Vedic Mathematics, the source of perfect order. (p. 339)
The previous section of this paper ended by raising the need for the intellect of the mathematician to expand to the level of pure intelligence so that the expressed values of the formulas and laws are experienced as arising from the field of pure intelligence. This expansion is accomplished through the techniques described in the last section of this paper to become a master in Maharishi Vedic Mathematics. The result, Maharishi (1995a) explains, is that knowledge that was once binding is transformed into knowledge that is liberating:

This remarkable transformation of the objectively available mathematical formulas to the reality of the subjective quality of the mathematician’s intelligence is the most intelligent and most enjoyable display of the reality of the supreme sovereignty of knowledge, which raises knowledge from the quality of slavery to the most exalted state of freedom in mastery—the self-sufficient total potential of knowledge. (p. 298)

The second weakness of modern mathematics explained in the previous section is that, since it is based within the realm of the intellect, it must by its very nature proceed sequentially. In contrast, “Vedic Mathematics is the mathematics of the absolute, self-referral field of pure consciousness, where everything is simultaneous, where everything is simultaneously administered on the level of perfect order” (Maharishi Mahesh Yogi, 1996, p. 372). Maharishi continues:

Modern Mathematics is the field of steps, whereas Vedic Mathematics is the field of pure intelligence that gets what it wants instantly without steps. It is the field of infinite correlation, the field of simultaneity of steps, because it functions in the frictionless field of infinite correlation—the field of self-referral intelligence.

In Vedic Mathematics all steps are synthesized to promote the result without the need for going through the steps and stages to arrive at the goal. Vedic Mathematics is a spontaneous revelation, it is not a step-by-step derivation. (p. 389)

One of the fascinating characteristics of Maharishi Vedic Mathematics is the way that it completes the correspondence of levels of mathematics with the levels of consciousness or intelligence described in the Vedic literature. The first three levels of consciousness are, starting with the grossest level and moving to the subtler levels, senses (Indriyas), mind (Manas), and intellect (Buddhi) (Maharishi Mahesh
Yogi, 1995b, p. 32). The correspondence of each of these levels with a particular level or approach to mathematics is straightforward. Starting with the grossest level, the level of the senses, its natural correspondence is with sensory-based mathematics. The next level, one step deeper, corresponds to the first stage of modern mathematics where the concepts of mathematics are recognized to be abstract, but prior to the full rigor of intellectual analysis. The level of the intellect corresponds with intellect-based mathematics.

There is, however, a fourth level of consciousness referred to as Ātmā. This is the level of the Self, a unified level of natural law. On this point, Maharishi (1995b) explains,

It is interesting to observe that the holistic value of natural law has its seat in the Ātmā (Self) of everyone, the self-referral intelligence of everyone—the unified level of natural law, the level of that holistic value of natural law which is more than the collected value of all of the (thirty-six) parts of natural law. (p. 31)

This is the level of Maharishi Vedic Mathematics since, as Maharishi (1996) writes, “The field of modern Mathematics is the field of the intellect, whereas the field of Vedic Mathematics is the field of pure intelligence—Ātmā” (p. 577).

We shall now look at an even more recent discovery by Maharishi, his Absolute Number, to examine how it actually initiates and maintains order throughout the entire universe.

**Maharishi’s Absolute Number**

“Without the Absolute Number, modern Mathematics (logic guided by natural numbers) cannot explain the supreme level of reality—the world of wholeness or many wholenesses.” With this opening sentence to the *Discovery of the Absolute Number in Maharishi’s Absolute Theory of Defence* (Maharishi Mahesh Yogi, 1996), Maharishi sets the stage for the introduction of a profound and fundamental element within Vedic Mathematics, Maharishi’s Absolute Number, which, he explains, is the starting point for Vedic Mathematics (p. 627) and the common source for all numbers (p. 613).

Maharishi’s Absolute Number brings to completion the process of doing intellect-based mathematics since, as Maharishi (1996) points
out, “Without the use of the Absolute Number, the intellect of the mathematician will always be questioning and searching for the ultimate on the ground of logic” (p. 612). More than this, it guides the orderly evolution from the unified field of pure consciousness to the process of its expression in the full diversity of the universe. Maharishi notes that this entire evolution can be experienced as self-interacting dynamics within our own consciousness:

Without the Absolute Number the order that prevails in the universe cannot be explained, it cannot be sustained; and we must admit that the order that is witnessed in the infinite diversity of creation and evolution has to be not only properly understood and accounted for on the intellectual level with mathematical accuracy, but also must be enlivened on the level of experience, so that life can be lived on the level of order—the eternal Cosmic Order, which upholds the total creative process in its ever-evolving quality. (p. 615)

When seeking a definition of Maharishi’s Absolute Number it is necessary to step outside the framework of intellect-based mathematics just as it is necessary to step outside of the framework of sensory-based mathematics to define key concepts of intellect-based mathematics such as infinity and infinite processes. More specifically, since Maharishi’s Absolute Number accounts for the order and evolutionary progress throughout the entire universe, it must lie beyond ordinary intellect-based numbers and mathematical concepts. In fact, as Maharishi (1996) explains,

It is a world on its own; it is a world of the enlightened. It is this world of the infinite number of wholenesses, which is expressed in the field of law, the world of natural law, where everything is permeated by wholeness—where every single Law of Nature is lively in terms of the total potential of natural law. (p. 611)

It is not just another mathematical concept to be defined using the language of modern mathematics, but something that can be directly experienced.

The most important characteristic of the Absolute Number in Vedic Mathematics is that it is a meaningful living reality, not just a notion or a concept, and therefore does not depend on the intellect. It is its own reality which functions within itself and gives a structure to knowledge
and its infinite organizing power, and therefore is the basis of all numbers and mathematical structures just as the unified field of natural law is the basis of all the force and matter fields, the common source of all the laws of nature. (p. 625–626)

In order to provide a clearer understanding of the Absolute Number, Maharishi (1996) gives a number of definitions, but always on the basis that it is its own reality. For instance, Maharishi writes that the definition of the Absolute Number is “infinite and unlimited” (p. 625) and that it is the “total unified field of natural law” (p. 625). Later it is defined “as that which functions from within itself” (p. 626), but ultimately “everything in the universe offers a definition of the Absolute Number” (p. 626).

As a basis and a catalyst for all numbers, Maharishi’s Absolute Number gives ordinary numbers an absolute or cosmic status. This is the mathematical counterpart of Unity Consciousness in which everything in the universe is experienced in terms of the underlying reality of the field of pure consciousness (Alexander et al., 1997). Further, as this field is recognized as the field of one’s own Self, everything takes on a cosmic status. Maharishi explains that by circling each number we are reminded of this status.

By circling any number, the number begins to indicate that it is a part and parcel of the Absolute Number—that its boundaries are unmanifest or, in spite of its boundaries, it is a continuum—it plays its part in explaining the eternal order that sustains the evolution of the universe. Its individual status has become Cosmic—as an individual it has been elected to be a ruler—the full potential of its creativity has blossomed. (p. 614)

Earlier we mentioned a weakness of modern mathematics that it cannot give more than a passive description of the laws of nature. In contrast, Maharishi’s Absolute Number has direct application to all practical areas of life since it maintains the orderly evolution of the individual and the universe. Maharishi (1996) explains the mechanics of this ordering of all aspects of life in the following way:

The mechanics of ordering have to be mathematically derived in order for the knowledge to be really complete, and also for the infinite organizing power of knowledge to be precisely, properly, and thoroughly applied so that life can be naturally lived on the ground of orderly evo-
lution, so that nothing shadows life—nothing shadows the immortal, eternal continuum of bliss, which is the nature of the self-sufficient, self-referral quality of the Absolute Number, from where everything emerges, through which everything is sustained, and to which everything evolves. (p. 616)

Maharishi continues by explaining that this process of ordering to be lived in everyday life requires that his Absolute Number be enlivened in conscious awareness. In the next section, we will discuss how the Maharishi Transcendental Meditation program is the key method for achieving this.

Some indication of the depth and scope of the practical applications can be obtained from carefully reviewing the two principal sources where Maharishi discusses his Absolute Number. The first, Maharishi Mahesh Yogi (1996), is dedicated to creating invincibility for every nation and a permanent state of world peace. In this book, Maharishi writes, “My Absolute Theory of Defence has its basis in invincibility, which is characterized by the self-referral dynamics of consciousness, the eternal, unbounded, unmanifest, never-changing state of eternal Unity, which stands for the Absolute Number” (p. 634). The mastery over natural law provided by Maharishi’s Absolute Number is so complete that the question of defeating enemies does not even arise, for it disallows the birth of an enemy (p. 630).

The second principal source, Maharishi Mahesh Yogi (1995b), is a book designed to create the most effective managers because they will manage from the level of self-referral, pure consciousness. Describing these managers, Maharishi writes, “They will be the embodiment of positivity and harmony, in whose presence nothing can go wrong, and will raise management to a new, enlightened level of performance, which will nourish everyone and everything” (p. 3). Such management requires a stable, secure basis at a level that supports and gives rise to the laws that manage all levels of life. This basis is provided by the complete precision of Maharishi’s Absolute Number. “My Absolute Number provides a reliable basis to that system of management that can achieve any objective and place management on a stable level of fulfillment” (p. 259). Furthermore, the essential role of the Absolute Number in perfect management cannot be played by ordinary intellect-based mathematics:
The Mathematics of natural numbers is not competent to explain the absolute precision and order that prevails in the field of perfect management, or absolute management; that is why I had to introduce the Absolute Number and evolve the Mathematics of the Absolute Number to account for the absolute precision and absolute order that perpetually prevails in the field of perfect management—the absolute management through the agency of the absolute value of natural law. (p. 326)

Mastery of *Maharishi Vedic Mathematics* and the Absolute Number

The questions now arise: What is the proper and fastest way to become a master of Maharishi Vedic Mathematics and the Absolute Number? How can one quickly live all that they promise in everyday life? Maharishi (1996) makes it clear that first and foremost the answer to both questions is the practice of his Transcendental Meditation technique since it enlivens the qualities of his Absolute Number in one’s awareness:

It is a joy to mention here that Transcendental Meditation is the process of maintaining connectedness with the Absolute Number—the source of the creative process—and through this programme, the precision of evolution and order in the process of creation is enlivened in human awareness, and is expressed in all thought, speech, and action.

Through Transcendental Meditation, the Absolute Number has the spontaneous opportunity to infuse its holistic nature in every step of the creative process, and this process of maintaining connectedness with the absolute source in the absolute alertness of the Absolute Number is the theme of creation of the ever-expanding universe. (pp. 616–617)

Yogic Flying is also important for developing full expertise in Maharishi Vedic Mathematics. “Transcendental Meditation,” Maharishi (1996) writes, “is the blissful exercise of Vedic Mathematics, and Yogic Flying habituates the mind to float in the bubbles of bliss, developing invincibility—the ability to accomplish anything—the characteristic quality of a Vedic Mathematician” (p. 395).

The Transcendental Meditation technique is a simple technique that allows the mind to settle and directly experience the unbounded field of pure consciousness, which we have seen is the field of Maharishi’s Abso-
lute Number. The more advanced techniques comprising the Maharishi TM-Sidhi program, of which Yogic Flying is a central part, stabilize the experience of pure consciousness and develop the ability to act from the level of this field. Complete descriptions of the Transcendental Meditation program, Yogic Flying, and the TM-Sidhi program are given in Maharishi Mahesh Yogi (1994, pp. 260–262, 283–288). A summary of their benefits is given in Orme-Johnson (1995). These benefits are found in all spheres of life—physiological, psychological, sociological, and ecological—and have been documented in more than 500 studies conducted at 200 independent universities and institutions in 33 countries and published in over 100 leading scientific journals.

Maharishi also adds reading the Vedic literature as another procedure for becoming a master of Maharishi Vedic Mathematics and the Absolute Number. The Vedic literature is the elaborated form of the Veda which was described earlier as the field of pure knowledge or complete knowledge. Writing on the role of the Vedic literature in uncovering the total mathematical precision and order of life, Maharishi (1996) explains:

The Vedic Literature, as I have organized it in the form of a perfect science, displays the mathematics of order in creation, and this it does not only on the intellectual level, for the sake of understanding, but also on the practical level, in the field of application, by providing the technology to practically enliven, initiate, and sustain this Cosmic Order in individual life. (p. 617)

Reading the Vedic literature helps to identify the awareness of the reader with the qualities of this field of pure consciousness, which is the field of Vedic Mathematics and the Absolute Number. As Maharishi (1994) writes,

Every aspect of the Vedic Literature expresses a specific quality of consciousness. Reading every aspect of the Vedic Literature as it flows and progresses in perfect sequential order has the effect of regulating and balancing the functioning of the brain physiology and training consciousness, the mind, always to flow in perfect accordance with the evolutionary direction of natural law. (pp. 144–145) ¹

After explaining that the Transcendental Meditation program, the TM-Sidhi program, particularly Yogic Flying, and reading the Vedic

¹ For further details on the value of reading the Vedic literature, see Sands (1997).
literature form the basis for becoming a master of Vedic Mathematics, Maharishi (1996) writes, “This is the only way to gain the knowledge that will make education fulfilling so that everyone attains perfection in life, and is able to accomplish everything with the full support of natural law” (p. 634).

Maharishi gives authenticity to this program by quoting from the Bhagavad-Gītā, an area of the Vedic literature he refers to as the essence of the Veda. Maharishi (1995a, p. 304) lists three verses in the Bhagavad-Gītā that explain the requirements for becoming an expert in Vedic Mathematics. The first of these verses is:

*Nistrai-gunyo bhav-Arjuna

Be without the three Guna, O Arjuna! (2.45)

The three gunas are the forces responsible for the process of evolution. Since they belong to the relative or expressed levels of existence, Maharishi explains that this verse is the instruction to practice the Transcendental Meditation technique in order to experience the source of the gunas in pure consciousness, “Gain Transcendental Consciousness—self-referral consciousness. For this, practice Transcendental Meditation twice daily” (p. 305).

The second of the three verses is:

*Yogasthab kuru karmāni

Established in the Self, perform action. (2.48)

The instruction from this verse, Maharishi (p. 305) explains, is “Act from the most settled, silent, coherent state of mind—act from the peaceful state of mind, the level of Transcendental Consciousness.” Maharishi makes clear that this is best accomplished through the daily practice of the TM–Sidhi program to optimize brain functioning and derive the support of natural law.

The last of the three key verses is:

*Sahajam karma kaunteya

Perform natural duty because unfathomable is the course of action. (18.48)
This verse compactly describes the benefits of having the support of natural law, a spontaneous feature of the state of enlightenment. Commenting on this verse, Maharishi writes: “Support of natural law will render all thought, speech, and action free from stress and strain—life will naturally progress to greater levels of achievement and fulfillment; life will naturally be easy, without problems or failures” (p. 305). Research shows that support of nature systematically and spontaneously increases with the regular practice of the Maharishi Transcendental Meditation and TM-Sidhi programs (Alexander et al., 1997).

By including these simple, yet completely powerful, exercises in Maharishi Vedic Mathematics and the Absolute Number in every school, college and university, mathematics will become a procedure of revelation, rather than one of step-by-step derivation. Mathematics will be transformed from a series of problems to a welcome part of a program for growth to enlightenment. Through the regular practice of these exercises, students will directly experience the source of all mathematics in their own awareness and come to recognize that it is also the source of all the orderliness and precision guiding everything from their own actions to the evolution of the universe. Maharishi Vedic Mathematics and the Absolute Number, the mathematical theory and technology of everything, will nourish the full blossoming of natural law in the daily life of every individual and every nation so that the world enjoys, in Maharishi’s words, Heaven on Earth.

References


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Maharishi Vedic Mathematics:
The Fulfillment of Modern Mathematics

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ABSTRACT

The discipline of mathematics has a long history of uncovering ever deeper and more unifying patterns and relationships. The fulfillment of mathematics, however, does not lie within the intellectual level of modern mathematics but rather in the transcendental level of Maharishi Vedic Mathematics. This paper describes how Maharishi Vedic Mathematics is the fulfillment of modern mathematics and presents Maharishi’s discovery of the Absolute Number, the transcendental basis for all numbers. The Transcendental Meditation program is seen to be a procedure that anyone can use to realize the fruits of Maharishi Vedic Mathematics and to live a fulfilled life free from mistakes.

Introduction

Mathematics is a highly praised discipline. Carl Friedrich Gauss, the great German scientist and mathematician, held mathematics to be the “queen of the sciences.” The Pythagoreans felt that “all is number” and the ancient Greeks included arithmetic and geometry as two of the four parts of the quadrivium, the core of their educational system. Jyotish Vedanga says, “Like the crest of a peacock, like the gem on the head of a snake, so is mathematics at the head of all knowledge.”

The purpose of this paper is to show how the discipline of modern mathematics finds its fulfillment in the Vedic Mathematics of Maharishi Mahesh Yogi and to invite every individual to gain the full benefit of Maharishi Vedic Mathematics.

The Nature of Mathematics

Mathematics is as old as civilization itself; every culture that has left written records has also left indications of mathematical activity. Mathematics is validated by reasoning and logic and is eternally true. The mathematics known to the ancients is still correct and valid today. Results stated long ago without proof, for example formulas discovered by the ancient Egyptians or given in the Sulba Sutras, have been shown to be correct later. Even the gaps in Euclid’s reasoning that were first identified only toward the end of the 19th century have been filled without invalidating any of his conclusions.
Other sciences, however, are continually being reformulated and refined. It is not unusual for a scientific theory to be completely repudiated on the basis of new experimental results. Even older results that are still used today, such as Newton’s Universal Law of Gravitation, are at best only approximations of reality. Mathematics, like the sciences, studies patterns and relationships, but it is the nature of the patterns and relationships studied by mathematics that sets mathematics apart from the sciences and ensures that mathematical results are enduring.

Physics describes regularities in the behavior of physical phenomena; chemistry studies how atoms and molecules interact with one another; and biology investigates the laws governing living systems. Mathematics, in contrast, has more abstract objects of study—numbers and their operations in arithmetic and algebra, geometrical shapes and their properties in geometry. Numbers, circles, and squares are not concrete physical objects the way organisms, cells, and molecules are. They are more abstract even than the unseen forces of physics, which nevertheless influence matter in a direct and measurable way. Mathematical objects are purely conceptual; Plato (Calinger, 1982, p. 65) describes them as “those absolute objects which cannot be seen otherwise than by thought.”

The procedures for doing mathematics are also very different from procedures used in the sciences, which constantly refer to physical observations and measurements. Mathematics depends on the functioning of the researcher’s mind and intellect. The French mathematician Henri Poincaré describes this special feature of mathematics (Calinger, 1982, p. 645):

[Mathematical creation] is the activity in which the human mind seems to take least from the outside world, in which it acts or seems to act only of itself and on itself, so that in studying the procedure of geometric thought we may hope to reach what is most essential in man’s mind.

Mathematics thus has a unique role in the realm of human knowledge. Like the sciences, mathematics requires precision, rigor, and logic. More importantly, however, mathematics has a subjective aspect, depending as it does on the activity of the human mind and the functioning of intelligence itself.
Maharishi Mahesh Yogi (1997, vol. 3, p. 160), the world’s foremost Vedic scholar and expert in the science of consciousness, locates the seat of mathematics in the consciousness of the mathematician:

Mathematical knowledge deals directly with the functioning of the field of intelligence—consciousness. The principles of Mathematics are universally valid principles of knowledge that describe the dynamics of the field of intelligence—the functioning of the mathematician’s own consciousness.

A scientist examines the physical world through the intermediate channels of the senses, scientific instruments, and measuring devices, but a mathematician is directly observing the functioning of his or her own field of intelligence. Poincaré speaks of “this feeling, this intuition of mathematical order, that makes us divine hidden harmonies and relations” (Calinger, 1982, p. 646) as essential to the process of creating, or “divining,” mathematics. In mathematics, there is no distance between the observer and the observed; the mathematician is an observer using feelings and intuition to observe the patterns and functioning of his or her own consciousness. The patterns thus located are necessarily subjective, but become objectively valid when expressed in the precise language of mathematics and systematically verified using logical proof.

Identifying the source of mathematics as the consciousness of the mathematician provides us with an explanation of what has sometimes been called “the unreasonable effectiveness of mathematics” (see Gorini, 1997). The principles of functioning of our own human intelligence are precisely the principles of functioning of nature’s intelligence and these principles are captured in a pure and compact form in mathematics. Once captured, they can be applied to all areas of life. As Maharishi explains (1996, pp. 304–305):

This universality of application [of mathematics] can be traced back to the fact that all aspects of Nature and areas of life are governed by the same principles of order and intelligence that have been discovered subjectively by mathematicians by referring back to the principles of intelligence in their own consciousness.

The wealth of applications of mathematics in the sciences shows the value of mathematics, but the ultimate goal and fulfillment of math-
mathematics must naturally be closely linked to its source, intelligence itself, independent of its applications in the sciences. We see progress towards this goal in the fundamental theorems of mathematics. These theorems show a drive towards unification of diverse structures. For example, we can look at the fundamental theorem of arithmetic and the fundamental theorem of algebra.

The fundamental theorem of arithmetic says that every integer can be written uniquely as a product of prime integers. The fundamental theorem of algebra says that every polynomial with complex coefficients can be completely factored over the complex numbers into linear factors; this theorem implies that every polynomial equation can be completely solved using complex numbers. Both of these theorems show that all objects of a particular class, integers in the case of arithmetic and polynomials in the case of algebra, can be expressed in terms of a specific collection of very simple objects in the same class. These simple objects—prime numbers in the case of arithmetic and factors of the form \((x - a)\) in the case of algebra—are capable of generating all other objects—numbers and polynomials—through the self-interacting dynamics of multiplication.

Another example, the fundamental theorem of calculus, shows how two very different processes, differentiation and integration, are in reality very closely related as opposites of each other. An example from geometry, the Pythagorean theorem, tells exactly how the lengths of the three sides of any right triangle are related to one another.

These and other fundamental theorems in mathematics have a unifying value; they show common characteristics belonging to all the diverse objects in a particular class (integers, polynomials, derivatives, or triangles). In fact, every theorem or equation in mathematics has this property of locating how objects that appear different from a superficial point of view can be related, unified, or seen to be the same. The most significant theorems of mathematics unify collections of different objects or show connections between objects that initially seem to be completely unrelated to one another. From this perspective, we see that the fulfillment of mathematics lies in the direction of finding unifying properties and relationships for a greater and greater diversity of mathematical objects. The ultimate fulfillment of mathematics is in locating the unified source of all mathematics.
**Maharishi Vedic Mathematics**

To understand the fulfillment of mathematics, we look to the Vedic Mathematics of His Holiness Maharishi Mahesh Yogi. Modern mathematics starts from the diversified values of intelligence and locates progressively more unified values of intelligence in the form of equations, formulas, and theorems. In contrast, Maharishi Vedic Mathematics begins from the unified value of intelligence. Maharishi (1996, p. 384) describes this difference:

> It is interesting to see that modern Mathematics starts from the field of diversity and locates its source in the field of unity; whereas Vedic Mathematics remains in the state of unity and deals with the whole structure of unmanifest diversity from the state of unity; and as the basis of all diversity is unity, Vedic Mathematics has its influence on every level of diversity.

The nature of Maharishi Vedic Mathematics is quite different from that of modern mathematics. The diversity of modern mathematics is upheld by the intellect, which discriminates and reasons in an objective way. Maharishi Vedic Mathematics, on the other hand, depends on the subjective nature of consciousness. This is not the variable and mutable subjectivity of the senses, feelings, and emotions, but the pure subjectivity of fully alert consciousness, that unified value of awareness that is precise and orderly and recognizes only the truth. The Absolute Number, discussed in the next section, is the experiential basis for Maharishi Vedic Mathematics and is structured within the consciousness of the mathematician.

Maharishi characterizes Maharishi Vedic Mathematics as the science of relationship. The most fundamental relationship is that between unity, pure intelligence or pure consciousness, and diversity. Maharishi identifies unity as Samhitā, the field of the Veda, and makes it clear that this can be directly experienced on the level of one’s own awareness during the practice of the Transcendental Meditation technique. Maharishi describes diversity in terms of its three roles as the knower, the process of gaining knowledge, and the known. Thus, according to Maharishi, the seat of Maharishi Vedic Mathematics is that point where diversity—the field of knower, known, and process of knowing—emerges from Samhitā—the completely unified field of consciousness.
This means that Maharishi Vedic Mathematics maintains the integrity of the unified value of life and at the same time upholds the eternally evolving field of diversity. According to Maharishi (1996, p. 345), “Maharishi Vedic Mathematics is the balancing power between two opposite qualities of its own nature—unifying and diversifying.”

Modern mathematics is valuable to scientists because it provides the tools they need to express the natural laws they have observed and to make predictions from these laws. Maharishi Vedic Mathematics is more fundamental, because, as Maharishi (1996, p. 355) explains, “Vedic Mathematics is the tool that structures different laws of nature from the holistic value of natural law in self-referral consciousness.” Maharishi Vedic Mathematics is more complete than modern mathematics. Maharishi (1996, p. 352) makes it clear that knowledge of Maharishi Vedic Mathematics is of great benefit to anyone because “to really know the whole range of Maharishi Vedic Mathematics amounts to knowing the entire process of creation and evolution and gaining mastery over natural law.”

Maharishi (1996, p. 366) further discusses the role of Maharishi Vedic Mathematics in its role of supporting the manifest universe:

Eternal absolute order available in the universe is the expression of Vedic Mathematics. Vedic Mathematics, emerging in terms of Veda, spontaneously upholds the ever-expanding universe from its basis in self-referral consciousness and thereby maintains order everywhere.

Vedic Mathematics upholds the ever-expanding universe in perfect balance within the structure of the eternal silence of pure singularity, within the structure of the Samhitā level of consciousness.

Maharishi Vedic Mathematics is the fulfillment of modern mathematics because it is capable of unifying all the diversified threads located by modern mathematics. Moreover, the fulfillment of Maharishi Vedic Mathematics is available to everyone because Maharishi Vedic Mathematics is structured in the consciousness of everyone, unlike modern mathematics, which is available only to those who have been highly trained. As Maharishi (1996, pp. 341–342) explains:

The cognition of Vedic Mathematics is most delightful. It is available to fully alert consciousness—Ritam-bharā-pragyā; it is available within the Ātmā of everyone, in the self-referral consciousness of everyone; it
is available as the structuring dynamics of each Śūtra of the Veda and Vedic Literature, particularly in the Darshan Śūtra of the Veda, and most vividly in the structuring dynamics of the Vedānt Śūtra.

This Samhitā level of self-referral consciousness is the source of thought—pure subjectivity or pure awareness—that belongs to us all. However, it is not possible to understand wholeness, the Samhitā level of reality, using logic guided by the intellect. The intellect must be transcended and pure consciousness must be gained in order to comprehend the unified wholeness of Samhitā.

Fortunately, anyone can experience pure consciousness through the practice of the Maharishi Transcendental Meditation program, a simple, natural procedure that is practiced twice a day for 15–20 minutes. With regular practice of the Transcendental Meditation technique, the Samhitā level of consciousness becomes established in one’s awareness; then, one can gain the full benefit of Maharishi Vedic Mathematics. Maharishi (1996, p. 366) makes it clear that this is possible for everyone:

Anyone, through Maharishi’s Course on Vedic Mathematics, can identify his consciousness with this absolute level of precision and order and become custodian of Vedic Management and gain the ability to actualize automation in administration—perfect administration through the support of natural law—administration through the Principle of Least Action.

This reality of mastery over natural law is validated by the Veda, as Maharishi (1996, pp. 365–366) affirms:

Ṛk Veda certifies that the individual is capable of functioning from this level of self-referral consciousness, and the individual can become controller of this level—

\[ \text{Brahmā bhavati sārathib.} \]

(Ṛk Veda, 1.158.6)

\[ \text{Brahman is the charioteer.} \]
Maharishi’s Absolute Number

To give a deeper understanding of Maharishi Vedic Mathematics, we now consider the Absolute Number. Maharishi (1995, p. 326) identifies the need for the Absolute Number in the lack of completion of the number systems of modern mathematics:

The Mathematics of natural numbers is not competent to explain the absolute precision and order that prevails in the field of perfect management, or absolute management; that is why I had to introduce the Absolute Number and evolve the Mathematics of the Absolute Number to account for the absolute precision and absolute order that perpetually prevails in the field of perfect management—the absolute management through the agency of the absolute value of natural law.

Maharishi locates the Absolute Number as the starting point of Maharishi Vedic Mathematics, just as numbers are the starting point of modern mathematics. The diversity of numbers in mathematics is handled by the intellect. In contrast, the Absolute Number transcends the intellect and must be experienced on its own transcendental level. As Maharishi (1996, pp. 625–626) explains:

The most important characteristic of the Absolute Number in Vedic Mathematics is that it is a meaningful living reality, not just a notion or a concept, and therefore does not depend on the intellect. It is its own reality which functions within itself and gives a structure to knowledge and its infinite organizing power, and therefore it is the basis of all numbers and mathematical structures—just as the Unified Field of Natural Law is the basis of all the force and matter fields (Physics)—the common source of all the Laws of Nature.

The Absolute Number is the unified source for the diversity of numbers, and as John Price (1997, p. 174) points out, “As a basis and a catalyst for all numbers, Maharishi’s Absolute Number gives ordinary numbers a cosmic status.” It provides an expression of the connection between unity, represented by a circle, and diversity, represented by a specific number. Thus, \( \Theta \) is an Absolute Number. Maharishi (1996, p. 614) describes this construction:

By circling any number, the number begins to indicate that it is part and parcel of the Absolute Number—that its boundaries are unmanifest or, in spite of its boundaries, it is a continuum—it plays its part in
explaining the eternal order that sustains the evolution of the universe. Its individual status has become Cosmic—as an individual it has been elected to be a ruler—the full potential of its creativity has blossomed.

The construction of the Absolute Number, according to Maharishi, brings modern mathematics to its fulfillment in Maharishi Vedic Mathematics, which has the supremely practical value of removing disorder from life. This practical value is available to those practicing the Transcendental Meditation technique, described by Maharishi (1996, pp. 616–617) in this way:

It is a joy to mention here that Transcendental Meditation is the process of maintaining connectedness with the Absolute Number—the source of the creative process—and through this programme, the precision of evolution and order in the process of creation is enlivened in human awareness, and is expressed in all thought, speech, and action.

The validity of this assertion is supported by over 600 scientific research studies on the Transcendental Meditation program (see Orme-Johnson and Farrow, 1976 and subsequent volumes) conducted at over 200 independent research institutions in 30 countries throughout the world. These studies show holistic benefits in all fields of life—physiological, psychological, sociological, and ecological.

The supreme value of Maharishi Vedic Mathematics based on the Absolute Number is summarized by Maharishi (1996, pp. 633–634):

Without the support of the Absolute Number the study of modern Mathematics can never account for the total reality, which is infinite, because it proceeds in steps. This is the time for the mathematics of finite numbers, of finite steps, to peep into the reality of the Absolute, and this Absolute Number represents complete knowledge, the Veda. It is the complete disclosure of Natural Law which presents two worlds of wholeness: 1) the unmanifest world of the Absolute Number, and 2) the manifest world of the finite numbers, which now have been raised to the unmanifest wholeness under the influence of the Absolute Number.

**Conclusion**

In this brief introduction to Maharishi Vedic Mathematics, we have seen how modern mathematics, based on the intellect, is fulfilled by Maharishi Vedic Mathematics, which is the structuring dynamics of natural law, the mathematics of consciousness. The Absolute Number,
discovered by Maharishi, is a reality available at the Saṁhitā level of everyone’s consciousness and offers fulfillment to modern mathematics. Maharishi Vedic Mathematics and the Absolute Number can be lived by anyone through the Maharishi Transcendental Meditation program and will guide human life to be mistake-free, evolutionary, and eternally fulfilling.

References

Section III

Self-Referral Dynamics and Mathematics
Self-Referral
in the Foundations of Mathematics

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Michael Weinless, Ph.D., received his B.S. from M.I.T. in 1964 and his Ph.D. in mathematics from M.I.T. in 1968. He went on to Harvard University where he held the positions of Benjamin Pierce Lecturer and Assistant Professor of Mathematics from 1968 to 1971. In 1972, Dr. Weinless became one of the founding faculty members of Maharishi International University (renamed Maharishi University of Management in 1995), where he pioneered the development of a unified field-based mathematics curriculum, integrating principles of the Science of Creative Intelligence and Maharishi Vedic Science with the traditional mathematical content of the courses. Dr. Weinless was chairman of the Department of Mathematics from 1972 to 1990.
Maharishi Vedic Science reveals that the structure of knowledge is, at its basis, self-referral. Pure consciousness, being pure wakefulness, is awake to itself, and therefore knows itself. In that flow of knowledge, pure consciousness simultaneously assumes the role of knower, object of knowledge, and process of knowing. A deep investigation into any field of knowledge should therefore reveal profound self-referral principles and dynamics at its basis.

In this paper, we study many of the self-referral characteristics of mathematical foundations. We discuss an alternative, broader formulation of sets—the theory of non-well-founded sets—in which sets may consistently be members of themselves. We study logical and mathematical ramifications of the Liar Paradox (“this sentence is false”), including Gödel’s incompleteness theorems. We discuss Berry’s Paradox and the impossibility of formalizing the notion of “describability.” We will see how the extreme value of the self-referral notion of impredicativity gives rise to Conway’s surreal numbers, which provides a unified context for studying all the familiar number systems, including infinitesimals and infinite ordinals. We discuss the self-referral dynamics underlying the modern notion of computation and see how these dynamics are naturally represented in the \( \lambda \)-calculus. And we discuss high points of an alternative foundation of mathematics, topos theory, which provides highly flexible modeling of self-referral phenomena in mathematics, and which provides a concrete realization of the three-in-one structure of consciousness by way of sheaf semantics, which naturally gives rise to a model of the polymorphic lambda calculus.

**Introduction**

Maharishi Vedic Science is a complete science of life, which locates the basis of all expressions of nature in the unified field of natural law, the field of pure intelligence or pure consciousness. Research reported in this volume and elsewhere has documented the way the disciplines of modern science, through their objective approach to knowledge, have begun to glimpse the qualities of the unified field of pure consciousness at their deepest level of understanding of the laws of nature. Perhaps the most striking development in this direction has been the recent development of totally unified quantum field theories in physics; as analyzed in Hagelin (1987), the unified field of modern physics displays the fundamental qualities of
the unified field of intelligence described by Maharishi Vedic Science and can on this basis be equated with the latter.

The situation of mathematics is rather unique among the scientific disciplines. Mathematical knowledge, by its very nature, is completely abstract and is thereby immediately grounded in the subjective field of intelligence. In the case of a field of objective science such as physics, the role of intelligence at the basis of the phenomena described by science is something hidden that must be brought to light through steps of experimentation and inference, but in the case of mathematics the role of intelligence at the basis of the discipline is apparent. What needs to be done, in the case of mathematics, is to examine the nature of the field of intelligence at the basis of the discipline, and see whether it can indeed be equated with the unified field of pure intelligence as described in Maharishi Vedic Science.

Maharishi has given a very precise and bold description of the nature and fundamental qualities of the unified field of natural law, the field of pure intelligence. Many of these qualities transcend the ordinary experience of human intelligence in its localized expressions in “waking state” awareness. These include the qualities of absolute unboundedness, infinite dynamism, and a purely self-referral structure of knowledge. The main theme of the author’s approach to the examination of the foundations of mathematics in the light of Maharishi Vedic Science has been to identify the role of these transcendental aspects of the nature of intelligence at the foundation of modern mathematics.

What is perhaps most apparent in this context is the role of infinity at the foundation of mathematics. This has been elaborated in Weinless (1987), in which it is argued that the ultimate mathematical infinity, the infinite wholeness of the universe of sets, can be viewed as a mathematical expression of the absolute infinite wholeness of consciousness, the Samhitā of Maharishi Vedic Science.

In this article we shall focus on what Maharishi calls the principle of self-referral in the foundations of mathematics. Maharishi has emphasized the self-referral nature of the field of pure intelligence, the unified field of natural law. The self-referral nature of the field of consciousness is expressed in the structure of pure knowledge, in which consciousness knows itself: the knower and the known are one and the same. This ultimate expression of self-referral dynamics is pure knowledge...
and nothing else; it is not ordinary mathematical knowledge. Yet this self-referral structure has its reflection in the structure of mathematical thought at its deepest levels, and the analysis of these reflections of self-referral in mathematics can serve to illuminate the way in which mathematical knowledge is ultimately grounded in the self-referral nature of the transcendental field of intelligence.

We shall consider in this article a number of examples of self-referral in different foundational areas of mathematics. We shall examine the relationship between these expressions of self-referral and the description of the self-referral nature and dynamics of consciousness contained in Maharishi Vedic Science. In the process, we shall identify several themes common to a number of these mathematical examples.

One such theme is *impredicativity*. An impredicative definition is a definition of an object that makes essential reference to a totality that already contains the object being defined (see Wang, 1974). That is, an impredicative definition defines a point value in terms of a wholeness that already contains the point being defined. From the usual predicative viewpoint, we think of creating first points, and then collecting these points together into wholes. Impredicative situations give expression to a structure of knowledge in which the point is inseparable from the wholeness that contains the point. This type of self-referral situation has a parallel in Maharishi’s analysis of the internal dynamics of pure intelligence. This is described in terms of the relationship between wholeness, infinity, “A,” and its extreme opposite—the point value within that wholeness, “K.” In this most fundamental relation between infinity and the point, the two values are inseparably linked, as they eternally coexist at the unmanifest source of creation. (see Weinless, 1987; Maharishi Vedic University, 1985).

A second theme concerns the connection of expressions of self-referral to logical paradoxes and other expressions of mathematical impossibilities. A logical paradox is a situation in which some statement is found to be both true and false. Because of the way the formal development of mathematics is grounded in logic, it is essential that logical paradoxes be strictly avoided. Nevertheless, in the informal analysis of fundamental mathematical and logical concepts, such as truth, describability, and set formation, paradoxes emerge in an innocent and natural way. For example, the notion of truth gives rise to the liar paradox,
Describability gives rise to the Berry paradox and Richard’s paradox as well as paradoxes connected with the reflection principle, and set formation gives rise to Russell’s paradox. These paradoxes generally involve impredicative definitions where the wholeness involved is a transcendental, all-inclusive, mathematical totality of some sort.

The paradoxes of self-referral have traditionally been avoided by formalizing the relevant intuitions in a partial way that is logically consistent, albeit incomplete. The self-referral aspect is then reflected in the existence of a hierarchy of formal expressions of the intuition, which are in some sense more and more complete, and which point towards a transcendental wholeness that transcends the formalism. From this perspective, the paradoxes reveal how the consistent formalization of mathematics can never capture the wholeness of the subjective faculty of mathematical intuition; that in fact the most fundamental mathematical intuitions have a fundamentally transcendental character, which reflects the transcendental nature of the field of intelligence underlying these intuitions.

Related to this theme of paradox are other mathematical expressions of self-referral that are simply mathematical impossibilities. For example, the lambda calculus provides a self-referral formalization of the concept of function that is based upon an intuitive model that violates fundamental principles of set theory. Such mathematical “impossibilities” provide a challenge to the mathematician to somehow expand his own conceptual framework so that it can consistently model these extreme expressions of self-referral.

There have been several recent foundational developments in mathematics in this direction, which have made it possible to model self-referral situations that were classically “impossible.” These involve modifications of logic, modifications of the set concept, or a combination of both. The most important of these developments involve either topos theory or non-well-founded sets. We shall see how both these developments involve rather striking mathematical expressions of the self-referral structure of knowledge, and how Maharishi Vedic Science provides a natural motivation for this direction of development in the foundations of mathematics.
This article is organized in 11 sections as follows:

Section 1 describes the recent development of the theory of non-well-founded sets, in which sets can be elements of themselves. This will provide a natural model for the singular, self-referral structure of pure knowledge as described in Maharishi Vedic Science. Maharishi Vedic Science will further be found to provide natural (and needed) motivation for the adoption of this framework as an expression of an intuitively meaningful new concept of a set.

Section 2 discusses the concept of a set of all sets. This concept provides a natural mathematical expression of the self-referral nature of the transcendental wholeness of life, called Brahman in Maharishi Vedic Science, which is described as arising through the synthesis of all expressions of natural law into a transcendental, self-referral wholeness, characterized as being a “whole greater than the sum of its parts.” Two formulations of set theory are discussed that permit one to describe such a self-referral set of all sets in a logically consistent way.

Section 3 describes a new approach to semantics that provides a meaningful interpretation of self-referral sentences involving the notion of truth. This approach assigns meaning, in particular, to the liar sentence, “This sentence is false.” From a conventional perspective, the liar sentence cannot be consistently assigned a meaning because of its self-contradictory nature. We shall examine how this breakthrough in modelling self-referral expressions of knowledge is based upon the self-referral structure of non-well-founded sets, together with an Austenian approach to semantics, which takes explicit account of the Rishi value, the value of the knower, in the structure of knowledge.

Section 4 examines the self-referral approach to the study of the symbolic language of mathematics introduced by Gödel in his arithmetization of metamathematics, through which the symbolic structure of a mathematical theory can be described within the symbolic language of the theory itself. On this basis, Gödel constructed a self-referral symbolic formula that asserted its own unprovability; from this he derived his famous incompleteness theorems, which establish the intrinsic incompleteness of any formalization of mathematical knowledge. We shall discuss the way this self-referral approach to metamathematics highlights the distinction between formal logical inference and the natural, informal structure of mathematical reasoning that is
directly grounded in the self-referral dynamics of intelligence. From the perspective of Maharishi Vedic Science, mathematical intuition is seen to have its roots in the transcendental infinite reality of pure intelligence, and the wholeness of this intuitive faculty at the basis of mathematical knowledge can never be captured within the localized boundaries of any formal mathematical system.

Section 5 discusses the paradoxes arising from the self-referral application of the notion of describability. This theme is expressed in one of the deepest principles of set theory, the reflection principle, which provides a rather paradoxical description of the ultimate mathematical infinity, the universe of sets, as being “indescribable.” We shall glimpse the rich mathematical implications that unfold from this “paradoxical” description, and note the parallel to the description, in Maharishi Vedic Science, of the transcendental reality of pure consciousness, the ultimate infinity, as being “indescribable.”

Section 6 discusses the foundations of analysis, that area of mathematics that provides the language of modern science. At the foundation of analysis is the theory of the continuum, the continuous line of real numbers. We shall see that the fundamental principle characterizing the continuous mathematical structure of the number line, the completeness principle, has a self-referral character, involving an impredicative definition. As a consequence of the self-referral character of the completeness principle, the continuum turns out to be a vast, “uncountably” infinite set, a transcendental mathematical wholeness from the viewpoint of set theory. We shall discuss the way the mathematical continuum thereby provides a natural model for the transcendental continuum of consciousness, the self-referral reality located by Maharishi Vedic Science at the unified basis of creation. We shall further discuss two new approaches to analysis based upon the self-referral concept of infinitely small numbers (infinitesimals) or infinitely close points: non-standard analysis and synthetic differential geometry.

Section 7 discusses the surreal numbers, introduced by Conway. These provide the ultimate extension of the self-referral theme of creating numbers by “cuts,” a concept expressed in one formulation of the completeness principle for the real numbers. We shall examine the remarkable way all the surreal numbers sequentially unfold, starting from nothing, through a mechanism whereby at each unfolding stage
one simply adds the “gaps” located at the preceding stage. This will provide a striking mathematical parallel to the theme of Maharishi’s Apaurusheya Bhāṣya of Rk Veda, in which the Veda is seen to sequentially unfold, starting from the unmanifest point value expressed by “K,” on the basis of a self-referral mechanics whereby the gaps at each level become elaborated in the text at the later stages.

Section 8 discusses recursive functions, the most fundamental values of transformation in mathematics. We shall see how these functions are described by self-referral formulas that sequentially unfold the values of the function. This will be parallel to the description, in Maharishi Vedic Science, of the sequential emergence of creation from the self-referral dynamics of consciousness. We shall see further how the self-referral description of recursive functions can be translated into a characterization in terms of fixed points of certain mathematical transformations. Fixed points, which identify non-change in the midst of change, will be seen to connect quite generally with self-referral phenomena in mathematics, and will be discussed in the context of Maharishi Vedic Science, which provides a perspective on reality in which the value of non-change is found to permeate all values of transformation in nature.

Section 9 examines the untyped or “type-free” lambda calculus. This is a formalism that treats the mathematical concept of a function in a purely self-referral way. We shall identify two basic expressions of self-referral in the lambda calculus: the self-referral concept of applying a function to itself as an argument, and the principle of abstraction, which provides an unrestricted scheme of impredicative definition. The intuitive model for the lambda calculus will be seen to reflect fundamental qualities of Maharishi’s description of the relationship between wholeness and its point value in the dynamic structure of pure consciousness. The fixed-point principle for the lambda calculus will be discussed, as well as its application to allow explicit self-referral definitions. It will be seen how the meaning of such self-referral definitions can be sequentially unfolded through a dynamic process starting from ⊥, the element corresponding to the point value “K” in Maharishi’s description of the phenomenon of Akshara at the source of the sequential unfoldment of the self-interacting dynamics of consciousness. Historically, the intuitive model of the lambda calculus turned out to be difficult to actualize; the obvious choice (which would be a set $S$ whose
elements are precisely the functions from $S$ to $S$) cannot be realized in a standard set theory universe. We will discuss two approaches to actualizing the intuitive model that circumvent the limitations encountered in the naive approach. One of these makes use of the rich self-referral structure of non-well-founded sets; the other, which provides an even more complete realization of the underlying intuition, makes use of a relatively new area of mathematics, topos theory. Topos theory provides a way of creating mathematical universes displaying a wide range of extraordinary properties. One striking feature of these universes is that they are governed by a self-referral structure of knowledge, as formalized in sheaf semantics. In this self-referral structure of knowledge, the values of the knower, called stages of knowing, and the values of the known, the sets, are one and the same. Topos theory thereby provides a natural mathematical model for the self-referral structure of pure knowledge, which Maharishi Vedic Science locates at the ultimate basis of all expressions of knowledge. The self-referral structure of knowledge in topos theory has been used as a basis for modelling self-referral situations in a number of foundational areas. An instance of this rich supply of models from topos theory is the surprisingly natural models that it provides for the lambda calculus.

Section 10 discusses denotational semantics, the mathematical study of the semantics, or meaning, of programming languages. The semantics is described by a mathematical transformation from the symbolic expressions of the language to abstract mathematical entities called denotations. We shall see that, because of the way high level programming languages are built upon self-referral constructs, the relevant mathematical functions involved in defining the semantics have a highly recursive, self-referral character; the abstract, self-referral structure of the lambda calculus has in fact proved to be necessary to successfully define these mathematical transformations in a rigorous way. The indispensible role of the self-referral structure of the lambda calculus at the foundation of the theory of programming languages will be taken to be an indication of the deep expression of the self-referral nature of intelligence in the structure and function of computer languages.

Finally, Section 11 discusses the polymorphic lambda calculus, which provides a further expression of self-referral through abstraction on types. The polymorphic lambda calculus provides a natural tool for
modeling the semantics of polymorphic programming languages, in which one can have variable types. Because of the extreme value of self-referral available in the polymorphic lambda calculus, it cannot be consistently modeled using classical methods; the only known models are provided by topos theory and are governed by intuitionistic logic. In examining the relationship between polymorphic programming languages, the polymorphic lambda calculus, and topos theory, we shall find a striking identity of two extreme values of Rishi and Chhandas, both of which will be represented by the recursive functions: on the Chhandas side, the recursive functions represent the actual numerical computation of the computer; on the Rishi side, the recursive functions are the basis for constructing the internal logic of a topos, the effective topos, which provides the abstract model for the polymorphic lambda calculus. In this we shall find a beautiful synthesis and integration of three fundamental expressions of self-referral in the foundations of mathematics—recursive functions, the lambda calculus, and topos theory—into one self-referral wholeness, which embraces, at one extreme, the concrete phenomenon of numerical computation and at the other extreme, the most abstract values of mathematical thought.

§1. Non-Well-Founded Sets

The primitive relation of set theory is the membership relation $\in$, the relation of a set to its elements. As analyzed in Weinless (1987), the membership relation can be viewed as a mathematical expression of the knower-known relationship in the field of intelligence.

The self-referral value of the membership relation will be expressed most directly and explicitly by the concept of a set being an element of itself, that is, a set $A$ having the property that $A \in A$. In the usual formulation of set theory, a set can never be an element of itself. One of the axioms of set theory, the axiom of foundation, explicitly prohibits this. In a new formulation of set theory introduced by Aczel (1988), the axiom of foundation is replaced by the anti-foundation axiom, which implies the existence of sets that are elements of themselves.

The self-referral concept of a set being an element of itself is closely related to several paradoxes of set theory. For example, the naive concept of a “set of all sets,” $U$, would be an element of itself, $U \in U$, and such a universal set leads to logical paradoxes. For example, it violates
Cantor’s theorem that states that the power set of any set is always a strictly larger set. (The power set of U would be U itself.) Russell’s paradox is also intimately connected to the concept of a set being an element of itself; Russell’s paradox arises from considering the set R of all sets that are not elements of themselves.

However, Aczel (1988) has shown that the anti-foundation axiom is consistent with the other axioms of set theory (provided the other axioms are themselves consistent). This means that the self-referral concept of a set being an element of itself does not in itself lead to logical inconsistencies.

In the ordinary universe of sets, all sets are sequentially unfolded from the null set. These sets are said to be well-founded. In the theory of non-well-founded sets, the ordinary universe of sets is extended to contain additional sets, which cannot be sequentially created from the null set. The simplest of these non-well-founded sets is a set designated Ω, which is characterized by the equation: Ω = {Ω}. This means that the set Ω contains a single element, which is Ω itself, so Ω ∈ Ω.

Non-well-founded sets provide a natural framework for modelling a variety of self-referral phenomena. Aczel’s original motivation came from computer science, in providing a model for Milnor’s synchronous calculus of communicating systems (see Aczel 1988). Non-well-founded sets provide natural models for the type-free lambda calculus, and also for the semantics of self-referential propositions.

The technical formulation of the anti-foundation axiom asserts that there is a correspondence between sets and graphs of a particular type. In the usual way of presenting such a graph, the node at the top represents the set itself; there is an arrow emerging from this node for each element of the set. At the end of each arrow, there is a node representing that element. From that node, there is an arrow emerging representing each of its elements, and so on.
Graph 1 contains a single node and no arrows. The node therefore represents a set containing no elements. Graph 1 therefore represents the null set or empty set, $\emptyset$.

Graph 2 represents a set containing a single element, where that element is the null set. Thus Graph 2 represents the set $\{\emptyset\}$.

In similar fashion, Graph 3 represents the two-element set $\{\emptyset, \{\emptyset\}\}$. Likewise, every ordinary, well-founded set can be depicted by a graph. The axiom of foundation tells us that each branch of the graph must ultimately end in the null set; that is, if we choose an element of a set, then an element of the element, then an element of the element of the element, and so on, after a finite number of steps we must end with the null set. This means that the graph representing the set must contain no closed loops. Well-founded sets thus have graphs that contain no closed loops.

The anti-foundation axiom asserts that every graph represents a set, even graphs containing closed loops. The graphs with closed loops represent non-well-founded sets. The simplest of these is the set $\Omega$, which is represented by Graph 4; it is a set containing a single element, where that element is the set itself. It can be shown in fact that every graph that contains only closed loops, that is, every graph that contains no branch that ends in a terminal node (which must represent the null set) will represent the same set $\Omega$, with every node of the graph representing $\Omega$.

Thus the two extreme values of sets described by the anti-foundation axiom are:

1. the ordinary well-founded sets, that correspond to tree graphs, containing no loops; and
2. the purely self-referral set $\Omega$, described by all graphs containing only loops, and having no terminal nodes.

Between these two extremes lie all kinds of intermediate possibilities, for example the set corresponding to Graph 5. This graph represents a set $A$ with the property that

$$A = \{[A], \emptyset\}.$$
How “natural” is the anti-foundation axiom? The usual axioms of set theory describe a reasonably well-defined intuitive concept of a set, namely the iterative concept of a set, in which all sets are sequentially created from the null set. Is there some intuitive concept of set that is described by the anti-foundation axiom?

Let us first ask: can we find an intuitive concept corresponding to the purely self-referral set Ω? Let us examine the intuitive meaning of the defining relationship for Ω:

\[ \Omega = \{\Omega\}. \]

As analyzed in Weinless (1987), the membership relation can be viewed as a mathematical expression of the knower-known relationship in the field of intelligence. In this context, the relation \( \Omega \in \Omega \) is an expression of a self-referral value of the membership relation in which knower and known are identical. Further, since \( \Omega \) is the only element of \( \Omega \), we can think of \( \Omega \) as expressing a totally unified structure of knowledge in which there is singularity, a single object of knowledge, which turns out to be identical to the subject and to the structure of knowledge itself. This corresponds precisely to the structure of pure knowledge, as described in Maharishi Vedic Science, in which Rishi, Devatā, and Chhandas, the knower, process of knowing, and the known, are in a singular, totally unified state. This unified structure of pure knowledge can be directly experienced in the simplest state of human awareness, the state of Transcendental Consciousness. It is this experience alone that can provide the needed intuitive insight into the meaning of the purely self-referral set \( \Omega \).

The familiar iterative concept of a set is based on a sequential viewpoint that sees the elements of a set as logically existing prior to the formation of the set. The set is then created through the collection, or synthesis, of these elements into a conceptual wholeness. This sequential viewpoint is expressed in Cantor’s definition of a set: “By a set we shall understand any collection into a whole \( M \) of definite, distinct objects \( m \) (which will be called the elements of \( M \)) of our intuition or thought.”
This sequential viewpoint fails when confronted with the self-referral set $\Omega$. We cannot view the elements of $\Omega$ as existing prior to the formation of the set $\Omega$, for the simple reason that the only element of $\Omega$ is the set $\Omega$ itself. In this sense, the self-referral set $\Omega$ must be viewed as expressing an uncreated structure of knowledge. Maharishi Vedic Science locates this uncreated value of knowledge in the self-referral structure of pure knowledge.

Maharishi Vedic Science thus locates the “intuitive” meaning of $\Omega$ in the uncreated, singular self-referral structure of knowledge in the transcendental field of pure consciousness. The familiar well-founded sets also have their intuitive meaning in the abstract field of intelligence: they are totally abstract conceptual entities that are sequentially unfolded from nothing based on the internal dynamics of intelligence. These well-founded sets, however, have a totally different character as intellectually created entities. The more comprehensive universe of non-well-founded sets extends to the extreme value of the purely uncreated, singular reality of $\Omega$. All the intermediate sets lying in the gap between the well-founded universe and $\Omega$ can be viewed as the mathematical expression of those values lying in the gap between the uncreated and created structures of knowledge.

In discussing the anti-foundation axiom, Moss (1989) writes, “What is needed most is a clear and persuasive conception of a set (or something else) under which AFA [the anti-foundation axiom] would be not only true but obviously true. The best possible genesis of such a conception would be some compelling new understanding of the physical and social worlds.”

We suggest that the “best possible genesis of such a conception” should be rather a compelling new understanding of the abstract world of intelligence, and that this understanding is provided by Maharishi Vedic Science.

§2. Set of All Sets

In Section 1, we saw how non-well-founded sets provide a mathematical expression of the self-referral value of the membership relation, whereby a set can be an element of itself. One extreme expression of this was found in the relation $\Omega \in \Omega$, which we saw to be an expression of the totally unmanifest, uncreated, singular structure of pure
knowledge. At the other extreme, there is strong intuitive appeal in
the concept of a set of all sets, which contains itself as an element. This
would express the self-referral nature of the transcendental wholeness
arising through the synthesis of all possible sets.

We discussed in Weinless (1987) the way the self-referral concept of
a set of all sets leads to logical paradoxes and therefore the transcen-
dental wholeness of set theory, the universe of sets, $V$, could not be itself
considered a set. Although $V$ could not express self-referral in the form
$V \in V$, it could still express its self-referral nature in the form of the
reflection principle, whereby it in some sense “approximated” the self-
referral relation $V \in V$. This was good enough for us to argue that the
universe of sets was a mathematical expression of the absolute infinite
wholeness of the Samhitā of Vedic Science, the transcendental whole-
ness of life. Nevertheless, there would be some additional aesthetic sat-
isfaction in being able to somehow extend the set concept to provide a
consistent description of a fully self-referral set of all sets, containing
itself as an element.

Although non-well-founded sets allow sets to be elements of them-
selves, one still cannot have a set of all sets. For example, Russell’s para-
dox would still arise. If there were a set of all sets $V$, we could form the
subset $R$ of $V$ consisting of all sets that are not elements of themselves:
$R = \{ x : x \notin x \}$. One sees then that $R \in R$ if, and only if, $R \notin R$, so we
have a logical inconsistency.

If one wishes to have a set of all sets, a different approach is needed.
We shall describe two such approaches below. But first we will take a
closer look at the source of the paradox.

The paradox can be seen to have its basis in the impredicative nature
of the definition of $R$: $R = \{ x : x \notin x \}$. The variable $x$ in the defini-
tion ranges over all possible sets. For every possible set $x$, we consider
whether or not $x \in x$, and collect together all those sets for which this
does not hold. The set $R$, which is being defined, is one of the sets the
variable $x$ ranges over, and we cannot decide whether or not to put $R$
in $R$ unless we already know if $R$ is in $R$, in which case we simply do
the opposite!

The principle used to form the set $R$ is called the principle of com-
prehension; it asserts that for any property $P$, one can form the set $A$ of
all sets having property $P$. The usual approach in set theory is to restrict
the range of $x$ to the elements of some set, and not allow it to range over all possible sets. In this way we obtain the axiom of subsets, which allows us to form the set $A = \{x \in B : P(x)\}$, where $B$ can be any set. This principle still allows impredicative definitions (as occur, for example, in the development of analysis), but does not lead to any known inconsistency. In this conventional framework, there is no set of all sets; the universe of sets is not itself a set (if it were, we would obtain Russell’s paradox again!).

There are, however, other possible approaches that avoid the paradox and that still allow one to form a set of all sets. We shall describe two such approaches.

The first is the approach introduced by Quine in his New Foundations (1937). Quine there introduced a very elegant axiomatization for set theory, whose key axiom is the comprehension principle restricted to stratified properties: for any stratified property $P$, one can form the set $A = \{x : P(x)\}$. “Stratification” is a technical restriction on the symbolic expression of the property $P$, which disallows such self-referral expressions as $x \in x$ or $x \not\in x$, so that the Russell set $R$ cannot be constructed, and Russell’s paradox is thereby avoided. One can, however, construct the set $\{x : x = x\}$, because the property $x = x$ is stratified; this means that one can construct a self-referral set of all sets.

A second approach is to permit the comprehension scheme in its most general form, but to modify logic in such a way that it doesn’t lead to a paradox. This is the approach taken in Maddy (1983), in which standard Zermelo-Fraenkel set theory is embedded in a larger class theory, in which there is a class of all classes. Here classes are governed by Kleene’s (1952) three-valued logic, in which, in addition to T (true) and F (false), there is a third truth value U (undefined). If one forms the class $R = \{x : x \not\in x\}$, that is, the class of all classes that are not elements of themselves, the problematic statement $R \in R$ turns out to have truth value U, so that it is neither true nor false, and no inconsistency arises.

We mentioned above the aesthetic appeal of a formulation of set theory containing a self-referral set of all sets. The two approaches we have described have their own limitations. Nevertheless, their success in consistently modelling this self-referral concept suggests that some development along these lines could provide a new and substantially
enriched foundation for set theory, in which the self-referral nature of consciousness is displayed in a more complete and holistic way.

§3. The Liar

The term “self-referral” is most commonly associated in the foundations of mathematics with the self-referential use of language; these instances are often associated with logical paradoxes. Perhaps the most fundamental of these is the liar paradox.

The liar paradox refers to variants of the self-referential statement: “This sentence is false.” The quoted sentence is easily seen to be true if and only if it is false. This means that the sentence cannot be consistently assigned a truth value (“true” or “false”).

As analyzed by Gödel (1934) and Tarski (1935/56), this situation implies that a formal mathematical language, if it is to be coherent, cannot contain a truth predicate; that is, the notion of truth for a formal language cannot be formalized within the formal language itself. (Other logical notions, such as provability, can be formalized within the language; see the discussion of truth and provability in Section 4.)

In the study of logic it is important to be able to formalize the notion of truth. Tarski showed how this could be achieved by means of a hierarchy of metalanguages: $L_0$, $L_1$, $L_2$ . . . . The first language, $L_0$, spoke only about the mathematical objects, e.g., numbers. $L_1$ provided an extension of $L_0$ that allowed one to speak additionally about truth for the sentences in $L_0$. The next metalanguage, $L_2$, allowed one to speak about truth for all sentences in $L_1$, and so on. In this way one obtains a sequence of languages having greater and greater expressive power, where the notion of truth for each language is expressible in the next language in the hierarchy.

In natural languages, such as English, the notion of truth is expressible within the language. The self-referential use of truth can be paradoxical, as in the liar paradox, but it also can be patently meaningful and unambiguous. There has been a natural interest in trying to model mathematically the semantics for languages that contain a truth predicate, to be able to do justice to this intuitive feature of natural languages.

Kripke (1975) provides such a semantics for a language with a truth predicate. The essential feature of Kripke’s approach, which avoids the
liar paradox, is that the semantics is partially defined; that is, not every sentence is assigned a truth value. This means that some sentences are true, some sentences are false, but some sentences are not assigned a truth value, so they are neither true nor false. The liar sentence, “This sentence is false,” in particular, is not assigned a truth value.

Kripke’s semantics can be viewed as based on a three-valued logic, in which there is a third truth value U (“undefined”) in addition to T (true) and F (false). However, with this approach the liar paradox reemerges in the form of the strengthened liar: “This sentence is either false or has undefined truth value.” This means that the semantical notion of having truth value U, or undefined truth value, cannot be expressed in the language being studied. So this resolution is not quite satisfactory.

A new approach to modelling self-referral in language has been recently developed by Barwise and Etchemendy (1987). This approach involves two innovative features—the use of non-well-founded sets in conjunction with an Austinian approach to semantics.

The essence of the Austinian approach (Austin, 1950) is that for every proposition there is implicit a hidden parameter, which is the “situation” the proposition is about. The situation is the relevant context of wholeness, which we can think of as representing the “viewpoint” of the individual asserting the proposition. Following are two simple examples taken from Barwise and Etchemendy (1987, pp. 171–2):

Whenever one encounters an apparent incoherence in the world, a natural thing to look for is some implicit parameter that is changing values. . . . Suppose we are looking at two people, and I say that \( A \) is to the left of \( B \), and you say \( B \) is to the left of \( A \). Can we both be right? Of course, since we can have different perspectives on \( A \) and \( B \). Here, what is generally expressed as a two-place relation is really a three- (or more) place relation, with one argument fixed by the location of the speaker. Examples of this sort abound, and some are not nearly so easy to see through. For instance, the so-called paradoxes of relativity are not really paradoxes, but show what seems like a two-place relation—that of two events being simultaneous—is actually a three-place relation: that of two events being simultaneous relative to an observer.

In Barwise’s adaptation of Austin’s semantics, the hidden parameter is taken to be the “portion of the world” the proposition is about. The “world” is the universe of “facts.” A proposition, however, can never
be about the universe at large; its context must be a proper subset of the world, called a situation. The liar sentence, “This sentence is false,” then expresses a different proposition about each possible situation \( S \). In Barwise’s analysis of the liar, for any situation \( S \), the liar proposition about \( S \), which we shall designate \( L(S) \), will be false. The falsity of \( L(S) \), however, will transcend the situation \( S \); that is, the fact that \( L(S) \) is false will not be one of the facts belonging to the situation \( S \). (This is what avoids the paradox.) The fact that \( L(S) \) is false will belong to the world of facts, which is the transcendental wholeness of the model.

The significance of the Austinian approach is that it makes explicit the value of \( \text{Rishi} \), the value of the knower, in the form of the relevant situation or context of wholeness. This value of wholeness can never be the all-inclusive wholeness of the totality of facts, the world. In expressing a proposition, there is implied a “collapse” of the totality to a localized value, the wholeness comprehended by the proposition. The self-referral quality of the liar proposition has a dynamic aspect that creates, for any localized situation \( S \), a fact transcending \( S \).

Intuitively, we can think of this in the following way. We consider the liar sentence, “This sentence is false,” in the context of some situation \( S \). \( S \) represents the totality of facts that constitute our present viewpoint. When we analyze the liar sentence in the context of the situation \( S \), we discover that it must be false. This fact, however, cannot belong to \( S \). Once we “know” the liar is false for \( S \), the situation has changed; we have learned a new fact. If we add this new fact to \( S \), we obtain a new situation \( S' \). We can now interpret the liar sentence in the context of \( S' \); this leads to new fact transcending \( S' \). This process can be continued indefinitely.

In the Austinian approach to semantics, a proposition has two constituents: a situation, and a type. The meaning of the proposition is that the situation is of the given type. As discussed above, the situation is a “hidden parameter” in the ordinary use of language representing the viewpoint of the speaker, and can be taken to represent the Rishi value. The type typically corresponds to the sentence used to express the proposition and can be viewed as an expression of the Chhandas value. We can illustrate these concepts with a simple example taken from Barwise.
Consider the sentence, “Claire has the ace of spades.” This sentence will in some contexts be true and in some false. This sentence itself characterizes a certain type of situation, namely the type of situation in which Claire has the ace of spades. Let us call this type $T$. We obtain a proposition, that is, a meaningful assertion, when the sentence is uttered in the context of some situation $S$. This means that the proposition has two components, the situation $S$ and the type $T$. The proposition will be true if the situation $S$ is of type $T$ and false otherwise. To determine whether or not a proposition is true, one must analyze the relation between the situation and the type. We can view this relationship as the Devatā value.

One of the central and innovative features of Barwise’s approach is that he uses non-well-founded sets to model the structure and semantics of language. In this approach, types, situations, and propositions are all represented by sets.

A situation is represented by a set whose elements represent “state of affairs,” the elementary facts constituting the situation. Types are also built up from states of affairs but in a somewhat more complicated way (using conjunction and disjunction).

A state of affairs is itself represented by a set whose elements represent the constituents of that state of affairs. A state of affairs may contain reference to a proposition. For example, if $P$ is any proposition, one possible state of affairs is that $P$ is true, and this state of affairs is represented by a set containing the (set representing the) proposition $P$ together with several additional elements indicating that it designates the state of affairs in which that proposition is true. Any situation containing this state of affairs will then be one in which $P$ is true. This same state of affairs also represents a type, namely the type of situation in which $P$ is true.

From this we see that a proposition is represented by a set containing a situation and a type, where the situation and type are themselves sets that can contain propositions as elements. If we think of the situation as the Rishi value and the type as the Chhandas value, with the proposition as the Samhitā value, the complete expression of knowledge, then we see that within both Rishi and Chhandas one can locate the Samhitā value.
If one were working with ordinary well-founded sets, then if one started with a given proposition $P$, the propositions referred to in its situation and type would always have to be “simpler” than $P$, and could never be $P$ itself. Using non-well-founded sets, however, one can model the self-referential situation in which a proposition refers explicitly to itself. Barwise shows in fact how, for any situation $S$, one can construct a proposition $P = \{S, T\}$ where the type $T$ of $P$ is the type “$P$ is false.” This means that the proposition $P$ is asserting: “The situation $S$ is of the type in which $P$ is false.” Thus $P$ is the liar proposition for the situation $S$. For such a self-referential proposition, we see that the wholeness of the proposition itself can be located within the Chhandas value, its type.

Maharishi has described the way the values of Rishi, Devatā, Chhandas, and Samhitā are completely unified in the self-referral structure of pure knowledge; he has elaborated how these values are nested one within another (see Maharishi Vedic University, 1985). Barwise’s approach to the semantics of self-referral propositions, based on the self-referral reality of non-well-founded sets together with the Rishi-oriented viewpoint of Austinian semantics, provides a beautiful mathematical expression of this self-referral structure of pure knowledge.

§4. Metamathematics

Metamathematics is the mathematical study of the symbolic structure of mathematical theories. In Weinless (1987) we discussed the self-referral theme underlying this development, introduced by Gödel in the arithmetization of metamathematics. This involved translating metamathematical statements into arithmetical statements that could then be expressed symbolically in the language of the theory being studied; this meant that the symbolic language could in essence talk about itself—it could describe its own symbolic structure and the mechanics of its sequential unfoldment in steps of logical proof.

Gödel exploited this self-referral feature in his proofs of the famous incompleteness theorems. His First Incompleteness Theorem asserts that no sufficiently rich theory $T$—such as set theory—is complete, in the sense that there must always be statements expressible in the language of the theory that are true but unprovable from the theory. He derived this theorem by showing how to construct a symbolic mathe-
mathematical formula that asserted its own unprovability. Shortly afterwards, he showed that the statement that \( T \) itself is consistent is, if true, also unprovable; his Second Incompleteness Theorem therefore asserts that no sufficiently rich consistent theory can prove its own consistency.

Gödel’s self-referral formula, which he used in the proof of his First Incompleteness Theorem, asserts: “This formula is not provable.” The formula bears a striking resemblance to the liar sentence: “This sentence is false.” Gödel’s formula, however, is not paradoxical, like the liar sentence, which can be neither true nor false. Gödel’s analysis of his self-referral formula showed that it must be true; as a consequence, it must also be unprovable.

Gödel attributed his discovery of his incompleteness theorems, as well as his other major discoveries, to his “epistemological attitude” (Wang, 1974). In the case of the incompleteness theorems, what was relevant was recognizing the distinction between the notions of truth and provability in a formal system. What was required was an epistemological attitude that recognized the reality of a transcendental notion of mathematical truth independent of any particular formalization.

Gödel showed how the notions of provability and consistency could be arithmetized and thereby expressed within the symbolic language of the theory being described. The transcendental notion of truth, however, could not in principle be so formalized. This was demonstrated through an adaptation of the construction of his self-referential formula. Gödel observed that if the notion of truth could be arithmetized, then one could construct a symbolic formula that asserts: “I am not true,” that is, one could construct a genuine liar. This would create a real paradox; the formula would have to be both true and false at the same time. This formula, being a formula of arithmetic, must necessarily be either true or false, but not both.

The Second Incompleteness Theorem provides an interesting commentary on the relationship between truth, provability in a formal system, and informal inference. The intuitive notion of truth is such that if \( P \) is any statement, asserting \( P \) and asserting “\( P \) is true” amount to essentially the same thing. But what happens when we consider a formal mathematical theory? Informally, asserting the axioms of the theory amounts to the same thing as asserting the truth of the axioms. But if we know the axioms are true, then we know they must be consistent.
Thus the consistency of the axioms is certainly an informal consequence of our belief in the axioms. However, from the axioms themselves, we cannot formally infer their consistency, as Gödel’s Second Incompleteness Theorem demonstrates. This brings out the limitation of formal logical inference. The informal flow of consciousness in mathematical reasoning, in which there is an epistemological commitment to the truth of what is being asserted, is not restricted by the boundaries of formal logic. This points out an essential limitation in the formalization of mathematics, which can never capture the wholeness of the dynamics of intelligence governing the sequential flow of the mind.

In general, the incompleteness phenomenon has had profound implications for the development of modern mathematics. It shows that the subjective faculty of mathematical intuition can never be captured in the boundaries of any single formal system. It has set the stage for an open-ended expansion of the axiomatic foundation of modern mathematics, in which deeper and deeper mathematical insights into the nature of the infinite are integrated into the formal structure of mathematics. This is a vertical expansion of mathematical knowledge, which progressively enlivens more and more of the self-referral value of the transcendental field of intelligence at the basis of mathematical intuition (see Weinless, 1987).

In analyzing the formal development of mathematics, we have made a distinction between the horizontal expansion of knowledge, in which formal logic is applied to sequentially unfold theorems from the axioms of a theory, and the vertical expansion of knowledge, in which new axioms are added, expressing progressively deeper values of mathematical intuition. The concept of logic, as described by Maharishi in the context of the Vedic literature, is much broader than the traditional western concept, embracing both the horizontal and vertical dimensions of expansion of knowledge. Maharishi (1971) has commented on the supreme role of logic in leading the awareness to the state of enlightenment. Maharishi has explained that Shankara’s commentary on the Bhagavad-Gîtâ, the Upanishads, and the Brahma Sûtra have precisely this value, in logically leading the intellect, step by step, to the ultimate realization of Brahman: *I am the totality*.

In terms of the structure of modern mathematics, this vertical dimension of logic might be taken to correspond to the informal steps
of reasoning that lead to the adoption of new, more powerful axioms. One such step, that follows directly from the self-referral nature of Gödel’s proof, is to add new axioms asserting that the former axioms are consistent. This, however, does not provide a very rich expansion of knowledge. In the next section we shall see how a very profound expansion of knowledge can be unfolded from the logical analysis of the self-referral expression of the concept of describability.

§5. Describability

The concept of a description of a number or of any mathematical object is certainly central to the development of mathematics and mathematical notation. The self-referral application of this concept leads, however, to logical paradoxes, for example, the Berry paradox (see Russell, 1908):

Consider all natural numbers describable in the English language using fewer than twenty words (for example, “two plus two,” “one million to the ten trillionth power,” etc.). Because there are only a finite number of words in the English language, there will be only a finite number of expressions in English containing fewer than twenty words, and only certain of these expressions will describe natural numbers. Therefore, there will only be finitely many different natural numbers that can be described in English using fewer than twenty words, and hence there must be a largest one. Consider now: “the largest natural number describable in English using fewer than twenty words, plus one.” We have just described, in fourteen words, a number greater than any number describable in fewer than twenty words!

The above paradox is easily seen to have its origin in the impredicative nature of the description of the number described in quotes at the end of the citation above. To construct the number, we must consider first the totality of all numbers describable in fewer than twenty words, but this totality must already contain the number being constructed, since it is described in fourteen words.

Just as the liar paradox is related to the fact that the notion of truth for a formal language cannot be expressed within that language, paradoxes of the present type, such as Richard’s (1905) paradox, imply that the notion of describability within a formal language cannot be expressed with that language. As a result, the notion of describability
must be formalized in a hierarchical way. One considers a hierarchy of levels of complexity of symbolic formulas. The concept of describability by a formula of a given level in this hierarchy can be formalized, but the symbolic formula expressing this concept necessarily belongs to a higher level of the hierarchy.

The paradoxical description in the Berry paradox is in essence describing a specific number as being indescribable. It may not seem that this type of description should have any direct relevance to the foundations of mathematics. It turns out, however, that one of the deepest principles in the foundations of set theory has precisely this character. This is the reflection principle, which asserts that the universe of sets, $V$, is structurally undefinable. That is, the reflection principle is in essence describing the universe of sets as having the property of being indescribable! (See Weinless, 1987, for a discussion).

The reflection principle specifically states that if the universe of sets, $V$, has some structural property $P$, then there must exist a set having property $P$. We analyzed in Weinless (1987) the way in which this principle describes the self-referral nature of the absolute infinite wholeness of $V$.

We further described the way the reflection principle is used to motivate and justify powerful set theory axioms, called large cardinal axioms, which assert the existence of large infinite sets displaying extraordinary qualities of unboundedness. The way this works is that one first recognizes that the universe of sets has some property $P$, expressing in some way its unbounded nature. Then one invokes the reflection principle to conclude that $P$ must hold true of some set $X$ in the universe; in this way, one can motivate an axiom that asserts the existence of a set having property $P$. The effectiveness of this approach comes from being able to start with a property $P$ that somehow captures as much as possible of the absolute unboundedness of $V$; the more unboundedness the property $P$ captures, the more powerful the corresponding large cardinal axiom will be. Since the reflection principle itself captures most completely the totality of $V$, the ideal application of this principle would be to reflect the reflection principle itself; that is, take $P$ to be the property of satisfying the reflection principle, and formulate an axiom asserting the existence of a set satisfying the reflection principle. This
self-referral application of the reflection principle to itself leads, however, to the following paradox (see Rucker, 1982).

If we apply the reflection principle to itself, we infer that there must exist a set $S$ that satisfies the reflection principle; that is, if $P$ is any property of $S$ then there must exist some element $S_1$ of $S$ that has property $P$. If we again apply the reflection principle to itself, we infer that there must exist an element $S_2$ of $S_1$ that satisfies the reflection principle. Proceeding in this way, we obtain an infinite sequence of sets, $S, S_1, S_2, \ldots$, where each set is an element of the preceding set, and each satisfies the reflection principle. This violates the axiom of foundation of set theory, which says that there can be no infinite chain of sets, $S, S_1, S_2, \ldots$, where each set is an element of the preceding set.

To avoid this paradox, the reflection principle must be restricted so that it does not apply to itself; we want the reflection principle to apply to as general properties as possible, but just not quite to itself.

Even though one cannot reflect the reflection principle itself, one can reflect some weakened version of the reflection principle, in which one restricts consideration to properties of some specific type. For example, if one restricts consideration to fixed-point properties, then one obtains the axioms for Mahlo cardinals (see Rucker, 1982). If one restricts consideration to properties expressible at some fixed level of the hierarchy of logical formulas, then one obtains the axioms for indescribable cardinals. Such an axiom has the form: There exists a partial universe $V_{\alpha}$ with the following property: if $P$ is any property of $V_{\alpha}$ described by a formula of type $\ldots$ then there exists a smaller partial universe $V_{\beta}$ (i.e., an element of $V_{\alpha}$) having property $P$. There is a whole hierarchy of such axioms, corresponding to the hierarchy of types of logical complexity of formulas. To this degree, the reflection principle, which describes the indescribable nature of the universe of sets, can be applied to itself, to unfold descriptions of “indescribable” sets.

It is interesting that the informal idea of simply describing the ultimate mathematical infinity as “indescribable” has such rich and profound consequences. This reminds one of Maharishi’s (1972) descriptions of the infinite nature of the mind as being “indescribable” in the Science of Creative Intelligence course. This same idea is expressed in the verse of the Bhagavad-Gītā (3.42), “That which is beyond even the intellect, is He” (Maharishi, 1969). This verse is intellectually delineat-
ing the ultimate reality as transcending the intellect! Set theory reveals the richness of content contained in such a paradoxical, self-referral description.

The Vedic literature contains a number of “paradoxical” descriptions of the ultimate reality, whereby it is described as displaying two contradictory properties, for example “smaller than the smallest, larger than the largest” (Katha Upanishad, 1.2.20). Maharishi, in his own systematic description of the self-referral nature and self-interacting dynamics of the transcendental field of pure consciousness, has likewise emphasized the “paradoxical” coexistence of contrasting mathematical values: the contrasting values of 3, trinity, and 1, unity, in the 3-in-1 structure of the Samhitā, and likewise the contrasting values of infinity and the point in his description of the phenomenon of Akshara. From a mathematical point of view, the transcendental field of consciousness would indeed appear to have a rather paradoxical nature.

In Weinless (1987) we discussed the universe of sets as a mathematical expression of the transcendental wholeness of life, the Samhitā of Maharishi Vedic Science. In terms of this correspondence, the paradoxical nature of the Samhitā seems to be most directly expressed by the reflection principle, which captures the indescribable nature of the ultimate mathematical reality.

§6. Foundations of Analysis

The field of analysis is that area of modern mathematics that provides the language of modern science; its fundamental theme is the quantitative analysis of continuous change. At the foundation of analysis is the theory of the continuum of real numbers. As discussed in Weinless (1987), the theory of the continuum is based on the abstract theory of infinite sets. When the continuum is analyzed as a set of points, it turns out to be a vast, uncountably infinite set, a transcendental mathematical wholeness.

The deep set-theoretic roots of the theory of the continuum has its origin in the self-referral nature of the completeness principle, the principle formalizing the continuous character of the real numbers. The completeness principle asserts that there are no “gaps” or “holes” in the real number system. There are several equivalent ways of formalizing the completeness principle, but they all have a self-referral character.
The simplest is perhaps the least upper bound principle: Every non-empty set $S$ of real numbers that is bounded above has a least upper bound $r$ (which is also a real number). This means that if a set of real numbers $S$ has the property that there is some number $m$ greater than or equal to all members of the set, then there is a smallest such real number $r$, which is called the least upper bound of the set $S$. The least upper bound principle allows one to define specific real numbers $r$ in terms of essentially arbitrary sets of real numbers $S$. This allows one to define real numbers in an impredicative way. For example, if we let $S$ be the “set of all real numbers having some property $P$,” then $S$ is itself formed by starting with the totality of real numbers and asking for each one whether it has property $P$. This means that we are defining a specific real number $r$ by reference to the totality of all real numbers, which already must contain $r$.

The impredicative character of the completeness principle has its roots in the nature of the geometrical intuition of a continuous line. We do not conceive of the line as being formed by first conceiving of individual points and then connecting them together into a line. Rather the line is conceived directly as a whole, as a direct mathematical expression of a transcendental continuum. The intellect then analyzes this reality in terms of point values. The precise intellectual analysis of the relationship between the point value and the transcendental, continuous wholeness gives rise to the various formulations of the completeness principle. In particular, the principle of nested intervals emerges from the intellectual analysis of the relationship of the continuum to the point in terms of the phenomenon of $Akh\text{\textbar}r\text{\textbar}$, collapse of infinity to a point, as analyzed in Weinless (1987).

In general, the relationship of the continuum to the point in mathematics is expressive of the relationship between the continuum of consciousness, or Being, to its own point value as analyzed in Maharishi Vedic Science; the impredicative, self-referral character of the completeness principle is a natural expression of the self-referral value of relationship between infinity and the point in the structure of pure knowledge, in which the two are inseparable, and the value of wholeness is primary.

The historical development of the mathematical concept of the continuum is consistent with this view. Bochner (1974) examines the
historical development of the concept of continuity in philosophy and mathematics; he traces the roots of the modern mathematical concept of the linear continuum to the ancient Greek philosophers, originating in Parmenides’ description of the transcendental continuum of Being.

Maharishi has described a quantitative aspect of the phenomenon of Akshara that illuminates the way the continuous, measuring aspect of the real numbers is expressed in the first steps of unfoldment of the self-interacting dynamics of consciousness, discussed in the context of his *Apaurusheya Bhāshya* of Ṛk Veda.

The theme of Maharishi’s *Apaurusheya Bhāshya* is that the Veda provides its own commentary in its sequential development. The first syllable of Ṛk Veda, AK, expresses the totality of knowledge in its most compact expression; the sequential unfoldment of the text of the Veda provides a sequential elaboration and commentary on this expression of complete knowledge.

The syllable AK expresses the phenomenon of Akshara—collapse of infinity, A, to its own point value, K. This is the primordial expression of the self-interacting dynamics of consciousness. Maharishi has elaborated that there are two converse processes implicit in the phenomenon of Akshara: the collapse of infinity to a point and the expansion of a point to infinity. These two aspects of Akshara are elaborated in the structure of the first mandala of Ṛk Veda, in terms of the quantities of wholeness expressed in the 192 Sūktas. Maharishi (1978) explains that the quantity of wholeness steadily decreases from its full value in the first Sūkta, to its nil value in the 97th Sūkta, the avyakta or unmanifest Sūkta, and then steadily increases to its full value again as one completes the circle of 192 Sūktas. Maharishi has elaborated how, in this mathematical quantification of wholeness, opposite Sūktas express complementary values in such a way that the sum of the two quantities of wholeness is always the same value, namely the wholeness of the first Sūkta.

From this perspective, the Ṛk Veda can be understood to provide a commentary on the relationship between infinity and the point in terms of a sequence of intermediate, quantitative values of wholeness. The principle of nested intervals provides a parallel mathematical theme of intellectually analyzing the relationship between the infinite continuum and its points in terms of a sequence of intermediate quantitative
values, namely subintervals whose lengths shrink to zero. The striking correspondence between these two approaches of modern mathematics and Maharishi Vedic Science highlights the deep and natural connection between the mathematical continuum and the transcendental continuum of consciousness.

The set-theoretic construction of the continuum of real numbers was developed in the 19th century to provide a foundation for analysis. It is interesting to note that the informal foundation for analysis, originally introduced by Newton and Liebniz in the 17th century, involved a rather different number concept, namely, a system of numbers containing infinitely small elements, called infinitesimals. The concept of such infinitesimal numbers was rejected when the theory of the continuum was developed as a foundation for analysis in the 19th century. It was shown, however, by Robinson (1966) that the real number system can be extended to a *hyperreal number system* containing infinitesimals, with which one can develop calculus in much the way that Newton and Liebniz originally did.

The infinitesimal approach to calculus has great intuitive appeal. This is reflected in the way physicists, since the time of Newton, have continued to use infinitesimals, even when the community of mathematicians had formally banished them from mathematics, as being a nonexistent fiction. One can think of infinitesimals as providing a mathematical metaphor to the self-referral structure of pure knowledge. In this self-referral structure of knowledge, consciousness knows itself: the knower and the known are one. Yet, in this structure of knowledge, a distinction is created between the knower and the known, so there is some “distance” created between them. To respect the identity of knower and known, this distance must be in some sense infinitely small. The mathematical concept of infinitesimals provides a way of quantifying this self-referral value of relationship between two points that are distinct and yet infinitely close.

In Maharishi Vedic Science, the self-interacting dynamics of consciousness is described as being lively at the point value in creation. This corresponds to the description of the unified field in modern physics, whose self-interacting dynamics is lively at the point-like value of the Planck scale of $10^{-33}$ cm or $10^{-44}$ sec. See Hagelin (1987). Even in classical physics, however, the mathematical description of the laws of nature
is formulated in terms of quantitative values of change at the point value in space and time. These physical laws are mathematically expressed by differential equations, equations relating derivatives of functions. The mathematical concept of the derivative quantifies the rate of change of a function at a point in space or time. The derivative thereby quantifies the dynamism of a function at the point scale.

It is illuminating to contrast the approaches to computing the derivative in classical analysis and in nonstandard analysis. In classical analysis, one computes the derivative using the limit process, an infinite sequential process of transcending. One first computes the average rate of change of the function over a finite, extended interval, and then studies the behavior of this quantity as the length of the interval shrinks to zero. The derivative is identified as the value approached in the limit. This is parallel to the mechanics of the TM-Sidhi program, in which the self-interacting dynamics of consciousness is enlivened in awareness through a transcending process, whereby the awareness converges to a state of singularity.

In non-standard analysis, one computes the derivative directly by simply comparing the values of the function at two infinitely close points. Here the whole computation takes place on the self-referral level of the point. This is parallel to the unfoldment of the self-interacting dynamics of consciousness from within the self-referral structure of pure knowledge itself, giving expression to the Richās of the Veda, the unmanifest structure of natural law.

The intuitive use of infinitesimals in some ways skirts dangerously close to logical paradox. In Newton’s calculation of the derivative, he first divided by some infinitesimal quantity $\alpha$ and then set $\alpha = 0$ to obtain his final answer. But if $\alpha = 0$, then one is not justified in dividing by $\alpha$ in the first place. Bishop Berkeley (cited in Moritz, 1914) in particular criticized Newton’s use of infinitesimals as logically unsound.

In the modern development of nonstandard analysis, the above paradox is avoided as follows. When one divides by $\alpha$, one doesn’t obtain the exact value of the derivative, but there remains an infinitesimal error. To obtain the exact value, one takes the “standard part” of the computed value.

We can ask whether there is some way to make the infinitesimal $\alpha$ so small that there is no error. The above simple argument makes it appear
that this is not possible, because the error term is typically proportional to $\alpha$ and will be 0 only if $\alpha = 0$. Nevertheless, there is a way this can be done, which involves a new approach to calculus and differential geometry called *synthetic differential geometry*. See Kock (1981).

Synthetic differential geometry utilizes *nilpotent infinitesimals*; in the case of the calculation of the derivative, these are infinitesimals $\alpha$ such that $\alpha^2 = 0$. In ordinary nonstandard analysis, the square of an infinitesimal is an infinitely smaller infinitesimal, but is still non-zero. In synthetic differential geometry, we can make the infinitesimal so “small” that its square is actually 0, and by using such infinitesimals, the error term can be made to vanish. The use of such nilpotent infinitesimals makes possible a purely algebraic development of calculus and differential geometry. In this development, however, the infinitesimals have properties that are paradoxical from the perspective of classical logic, and the theory must therefore be developed according to the rules of inference of intuitionistic logic. The best known and perhaps most natural models of synthetic differential geometry, are found in structures called *toposes*, which exhibit nearly all the closure properties of a universe of sets, but whose internal logic is naturally intuitionistic. It turns out that the self-referral structure of knowledge at the basis of topos theory is required to construct models for synthetic differential geometry. We introduce topos theory briefly at the end of Section 9. For a fuller introduction, see Weinless (1987); for a treatment of models of synthetic differential geometry, see Dubuc (1979).

Synthetic differential geometry provides one extreme expression of the self-referral concept of infinitely small numbers, or infinitely close points. A second extreme expression of the concept of infinitesimals, based on a quite different direction of approach, is provided by the surreal numbers.

§7. **Surreal Numbers**

The surreal numbers are an extraordinary system of numbers introduced by Conway (1976). There are a number of features that make this system of numbers so extraordinary.

The surreals, first of all, contain all the reals, but also contain infinitesimal elements as well as infinitely large elements. In fact, all the infinite ordinal numbers of set theory are contained in the surreals. See
Weinless (1987). In this way, the surreals integrate the ordinals and the reals into a single, all-inclusive number system.

The surreals display all the familiar algebraic properties of the real numbers: they are an ordered field. They also display the property of algebraic completeness of the reals: they are a real closed field. This means that any polynomial of odd degree whose coefficients are surreal numbers has a root that is a surreal number.

In addition to familiar properties of real numbers, the surreals have a number of extraordinary algebraic and analytic properties; details can be found in Alling (1987).

One of the striking features of the real number system is its vastness as a set—the set of real numbers is uncountably infinite. This makes the real number line “thick”: between any two numbers there are uncountably many points on the number line. The surreal number system is infinitely “thicker” than the reals, and in fact expresses the ultimate value of thickness—every interval of the surreals has cardinality \( \Omega \), the cardinality of the universe of sets. This means that there are as many surreal numbers between 0 and 1 as there are sets in the universe of sets, the ultimate mathematical wholeness.

The surreals can be characterized by a “completeness” principle that is similar to, and yet strikingly different from, the familiar completeness principle for the real numbers. One way of formulating the completeness principle for the reals is in terms of Dedekind cuts. A Dedekind cut of the reals \( \mathbb{R} \) is by definition a pair \((A, B)\) of subsets of \( \mathbb{R} \), such that:

1. \( A \) and \( B \) are non-empty;
2. \( A \cup B = \mathbb{R} \);
3. \( A < B \), that is, every element of \( A \) is less than every element of \( B \).

The completeness principle asserts that for every Dedekind cut \((A, B)\) there exists a unique real number \( r \) such that every number less than \( r \) is in \( A \) and every number greater than \( r \) is in \( B \). This means that \( r \) lies at the junction point of \( A \) and \( B \), and thus there cannot be a “hole” in the number system separating the pieces \( A \) and \( B \).

The surreal numbers are characterized by the following “completeness” principle: If \( A \) and \( B \) are any two sets of surreal numbers, such that \( A < B \), then there will always exist a surreal number \( r \) such that \( A <
\[ r < B, \text{ that is, } r \text{ is greater than every number in } A \text{ and } r \text{ is less than every number in } B. \] In fact, there will always exist a unique simplest surreal number having this property (Conway’s simplicity theorem).

Not only does every gap \( A < B \) define a number \( r \), but every surreal number \( r \) comes from a gap \( A < B \) between two sets of “simpler” surreal numbers. The process starts with the empty set \( \emptyset \). The gap between the empty set and itself (since \( \emptyset < \emptyset \)) gives rise to the first surreal number 0, which is the simplest surreal number.

At the next level we have two gaps: \( \emptyset < 0 \) and \( 0 < \emptyset \). The first gives rise to the number \( -1 \), which is the simplest number \( r \) such that \( r < 0 \), and the second gives rise to the number 1, which is the simplest number \( r \) such that \( r > 0 \).

At this point we have created three numbers: \( -1, 0, \) and 1. This gives rise to four new gaps: \( \emptyset < \{-1, 0, 1\} \), \( \{-1\} < \{0, 1\} \), \( \{-1, 0\} < \{1\} \) and \( \{-1, 0, 1\} < \emptyset \). These gaps give rise to the numbers \( -2, -1/2, 1/2 \) and 2: \( -2 \) is the simplest number less than \( -1 \); \( -1/2 \) is the simplest number \( r \) such that \( -1 < r < 0 \); \( 1/2 \) is the simplest number \( r \) such that \( 0 < r < 1 \); and 2 is the simplest number greater than 1.

Proceeding sequentially in this way, we obtain all the dyadic numbers, that is, all rational numbers expressible in the form \( n/2^m \), that is, all fractions whose denominator is a power of 2. When we conceptually complete this infinity of steps, we obtain the infinite set of dyadic numbers \( D \). Starting with \( D \), we can now resume the process of locating gaps.

At this point we obtain, among other things all the real numbers. For example, if \( A \) is the set of all dyadic numbers \( x \) such that \( x^2 < 2 \), and \( B \) is the set of all positive dyadic numbers such that \( x^2 > 2 \), then \( A < B \), and the gap between \( A \) and \( B \) gives rise to the real number \( \sqrt{2} \); \( \sqrt{2} \) is the simplest number \( r \) such that \( A < r < B \). Every real number that is not a dyadic number will be obtained in this way just as it would be by a Dedekind cut.

However, we obtain also numbers that are not familiar real numbers. For example, if we let \( A = D \) be the set of all dyadic numbers, and \( B = \emptyset \), then we obtain the simplest number that is greater than all the dyadic numbers, designated \( \omega \). \( \omega \) is the simplest infinite number.

If we take \( A \) to be the set of all dyadic numbers less than or equal to 0, and \( B \) to be the set of all positive dyadic numbers, we obtain the
The simplest number $r$ with the property that $0 < r$ but $r < 1/2^n$ for every natural number $n$. This means that $r$ is the simplest infinitely small positive number. The surreal number system thus contains infinitesimal elements.

We note that the concept of “gap” in the generation of the surreals, called Cuesta Dutari cuts, is rather different than that defined by the familiar Dedekind cuts. Dedekind cuts locate the gaps in the rational number system, which are then filled with the irrationals to obtain the real number system. If one applies Dedekind cuts to the reals, no new gaps are located; the real number system is complete.

From the point of view of Cuesta Dutari cuts, however, one can locate numerous gaps in the real number system. For example, take any real number, say 3. Let $A$ be the set of real numbers less than or equal to 3, and let $B$ be the set of real numbers greater than 3. Then $A < B$. The sets $A$ and $B$ constitute both a Dedekind cut and a Cuesta Dutari cut. As a Dedekind cut, they specify the real number 3, which lies at the junction point between the two sets $A$ and $B$. In this case, the number obtained already lies in the set $A$, so nothing new is created. As a Cuesta Dutari cut, however, the sets $A$ and $B$ specify a new number $r$ lying in the gap between $A$ and $B$, and having the property that $A < r < B$, that is, $r > 3$, and yet $r$ is less than any real number greater than 3, so that $r$ lies neither in $A$ nor in $B$. Thus the Cuesta Dutari cut intellectually locates a gap between the numbers 3 and all real numbers to the right of 3 on the number line; this gap is then filled by a new number.

Dedekind cuts and Cuesta Dutari cuts thus give mathematical expression to two different kinds of gaps. Dedekind cuts identify “apparent” gaps, as are found for example in the system of rational numbers. Cuesta Dutari cuts identify gaps even in the continuous structure of the real number system.

Maharishi (see Weinless, 1983) has commented on the significance of different kinds of gaps in the structure of the Veda:

The structure of the mathematical continuum exemplifies the gap between infinity and the point described in Vedic Science as Akshara. This is a gap where no gap is possible to imagine. Infinity is a continuum, but if one analyzes a continuous line one sees it is made of points, and this also is a reality. The gap between the two creates the value of relationship and this is the basis of all diversity. The nonexistent gap in
the field of the infinite continuum is the starting point for the emergence of an unending progression of gaps. The phenomenon of Akshara, the source of the Veda, tells the story of the gaps.

The significance of a nonexistent gap is given by the combined syllable ‘GNI.’ ‘G,’ ‘N,’ ‘I’ are all different, and therefore there must be a gap, but the combined syllable does not show the existence of a gap. This reveals two kinds of gaps—the apparent gap between ‘A’ and ‘G’ in ‘AG,’ and the unmanifest gap in ‘GNI.’ Between these two, all kinds of gaps emerge.1

Maharishi has elaborated how the Veda itself unfolds according to the theme of the Apaurusheya Bhāshya, whereby the gaps at each stage become elaborated by the text at the following stages (Maharishi Vedic University, 1985). This theme of sequential elaboration of the gaps is exemplified in the sequential creation of the surreal numbers. The gaps at each stage are intellectually located via Cuesta Dutari cuts. Each gap is then objectified into a surreal number that is adjoined to the number system at the following stage. In this process, every conceivable gap gives rise to a surreal number, and conversely, every surreal number arises from some gap.

The whole sequential process itself starts from nothing. The first surreal number, 0, emerges from the totally unmanifest gap between the empty set and itself: $0 < 0$. We can think of this as the primordial mathematical expression of the concept of a gap, at a yet unmanifest stage at which no elements are present. From the liveliness of this concept in the unmanifest field of consciousness, numbers then sequentially unfold. This whole sequential process is strikingly parallel to the sequential creation of sets, starting from the unmanifest point value of the null set.

The process of creating sets uses the dynamics of the power-set operation. This operation creates, from any set, a greater set, the power-set; the power-set emerges from within the original set on the basis of the dynamics of intelligence. See Weinless (1987). In the sequential creation of the surreal numbers, the power-set operation is replaced by Cuesta Dutari cuts; this likewise expresses an aspect of the dynamics of

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1 “AGNIM” is the first word of Rk Veda and therefore the first word of the entire Vedic literature.
intelligence, in this case the dynamics of intelligence involved in intellectually locating gaps in an ordered structure. In this case, likewise, the new, expanded wholeness arises from within the original system on the basis of the dynamics of intelligence.

§8. Recursive Functions and Fixed Points

In this section we shall examine the self-referral character of the recursive functions, the most fundamental transformations of numbers. We shall begin by reviewing the concept of a function, the mathematical abstraction that expresses the idea of transformation between sets.

A function from a set \( A \) to a set \( B \) is an assignment, to each member of \( A \), of a well-defined member of \( B \). For example, if \( A \) is the set \{1, 3, 5\} and \( B \) is the set \{2, 3, 4, 5, 6\}, then one example of a function from \( A \) to \( B \) is the function \( f \) that assigns 3 to 1, 6 to 3, and 3 to 5. One thinks of the function as expressing a value of transformation from \( A \) to \( B \), in this case the transformation that takes 1 to 3, 3 to 6, and 5 to 3.

One uses functional notation to describe functions; if we denote the function \( f \), and if \( a \) is an element of \( A \), then we use the notation \( f(a) \) to designate the corresponding element \( b \) of \( B \). The element \( b = f(a) \) is called the value of the function \( f \) applied to the argument \( a \). In our example, we have \( f(1) = 3, f(3) = 6, \) and \( f(5) = 3 \).

The recursive functions are all the possible functions from natural numbers to natural numbers whose values can be computed in a mechanical way; these are the functions that can in principle be evaluated by digital computers. The most general concept is that of partial recursive functions, for which the value of the function need not be defined for all possible natural numbers.

There are a number of different ways of formally defining the recursive functions. The most familiar way is in terms of self-referential formulas, which “curve back” on themselves. We shall illustrate how this works by considering one example, the factorial function.

The factorial function, \( \text{Fact}(n) \), is defined informally in the following way: \( \text{Fact}(0) = 1, \text{Fact}(1) = 1, \text{Fact}(2) = 2 \cdot 1 = 2, \text{Fact}(3) = 3 \cdot 2 \cdot 1 = 6, \text{Fact}(4) = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \), and so on. For every positive integer \( n \), \( \text{Fact}(n) \) is just the product of all the positive integers less than or equal to \( n \).

The factorial function can be described algebraically in a precise way by the two formulas:
(1) Fact(0) = 1, and  
(2) Fact(n + 1) = (n + 1) · Fact(n).

The second formula (2) contains the variable \( n \), for which one is free to substitute any natural number. If we substitute \( n = 0 \), we obtain Fact(0 + 1) = (0 + 1) · Fact(0) = 1 · 1 = 1; hence Fact(1) = 1. If we now substitute \( n = 1 \), we obtain, Fact(1 + 1) = (1 + 1) · Fact(1) = 2 · 1 = 2, so Fact(2) = 2. Proceeding in this way, the values of the factorial function for all natural numbers can be sequentially unfolded.

The two formulas (1) and (2) constitute a recursive definition of the factorial function. The formulas have a self-referral character; the factorial function is defined in terms of itself. This is expressed in the way the factorial function Fact, the function being defined, appears on the right hand side of the equation (2). It is the self-referral character of this formula that allows it to be applied over and over again, to sequentially unfold all values of the function from the value at 0.

We see that the self-referral nature of a recursive definition gives rise to a dynamic process that sequentially unfolds the values of the function. There is a second, complementary aspect of the self-referral nature of a recursive definition, developed by Kleene, that characterizes the same function in terms of non-change. This we shall now consider.

Let \( S \) designate the set of all possible functions from \( \mathbb{N} \) to \( \mathbb{N} \), where \( \mathbb{N} \) is the set of natural numbers. Consider the function \( B \) from \( S \) to \( S \) (that is, \( B \) is a function from functions to functions) defined in the following way: if \( f \) is any function from \( \mathbb{N} \) to \( \mathbb{N} \), we define \( B(f) \) to be the function \( g \) from \( \mathbb{N} \) to \( \mathbb{N} \) whose values are defined by the formulas:

\[
\begin{align*}
(3) & \quad g(0) = 1, \\
(4) & \quad g(n + 1) = (n + 1) \cdot f(n).
\end{align*}
\]

For any function \( f \), these two formulas will determine a well-defined function \( g \). (Since \( g \) occurs only on the left-hand side of these equations, they do not display the self-referral character of equations (1) and (2).)

The formulas (1) and (2) characterizing the factorial function Fact, just tell us that \( B(\text{Fact}) = \text{Fact} \), that is, the function \( B \) from \( S \) to \( S \) takes the element \( \text{Fact} \) of \( S \) into itself. We say that \( \text{Fact} \) is a fixed point of the function \( B \). The self-referral formulas defining a recursive function are
seen in this way to characterize the function as a fixed point of a certain transformation from $S$ to $S$. The function is thereby characterized, as a whole, in terms of its non-changing value.

Fixed points play a fundamental role in almost all areas of mathematics. Fixed points express the theme of non-change in the midst of change, a theme very much at the heart of Maharishi Vedic Science. Vedic Science locates the nonchanging field of pure consciousness at the fountainhead of natural law, lying at the source of all values of change in nature. In the state of enlightenment, the non-changing reality of consciousness is found permeating all expressions of change in phenomenal creation.

In the Vedic literature, the theme of fixed points is expressed in *Patanjali’s Yoga Sūtra*:

By *Sanyama* on the pole star, knowledge of the motion of the stars is gained.

*Sanyama* refers to a mental technique that lies at the basis of the TM-Sidhi program. Patanjali states that when the object of this technique of *Sanyama* is the pole star, then one gains knowledge of the motion of the stars.

Now the pole star represents the “fixed point” in the heavens. It is (apparently) the single non-changing point in the midst of the dynamic expression of the laws of nature governing the motion of all the celestial bodies (from the perspective of Earth); it is the fixed center round which the heavenly bodies (appear to) revolve. This suggests the following analysis on the pole star *Sūtra*.

The laws of nature governing celestial dynamics have their ultimate source in the non-moving transcendental reality of pure consciousness, the transcendental wholeness of the Samhitā. We can think of the pole star as the “representative” of this non-changing value of the wholeness of natural law in the field of change. When the pole star is the object of *Sanyama*, the awareness naturally awakens to the memory of the non-changing self-referral structure of natural law at the core of all expressions of change in nature, and on this basis one gains holistic knowledge of the whole range of dynamics of nature.

The pole-star *Sūtra* indicates the significance of fixed points in the analysis of transformations in Maharishi Vedic Science. In modern
mathematics, fixed points play a fundamental role in the study of mathematical transformations in a wide variety of contexts.

We have seen that a recursive function, such as Fact, can be characterized as a fixed point of a certain transformation. In the case of the factorial function, it is not difficult to see that the transformation $B$ corresponding to the equations (1) and (2) has a unique fixed point, so the self-referral definition of Fact by the equation \( \text{Fact} = B(\text{Fact}) \) is unambiguous. For other recursive formulas, however, there can be more than one fixed point of the associated transformation $B$; that is, there can be more than one function $F$ satisfying the equation $F = B(F)$. This means that there can be more than one function $F$ satisfying the defining formulas. Which of these should be understood as the meaning of the recursive definition?

A natural and elegant answer is provided by Kleene’s Recursion Theorem. Start with the empty function, $\perp$, the function whose domain is empty, that is, the function that is undefined for all natural numbers. Apply now $B$ to obtain a new function $B(\perp)$. Apply $B$ again to obtain a new function $B(B(\perp))$. Applying $B$ over and over again we obtain a sequence of partial functions

\[
\perp \leq B(\perp) \leq B(B(\perp)) \leq \ldots
\]

where the $\leq$ signs indicate that each function is an extension of the preceding function, that is, each function agrees with the preceding one except that it is defined for some additional values (see example below). This means that all these functions can be synthesized into a single function, $F$, called the limit of the sequence. The limit function $F$ will always be a fixed point of the transformation $B$ (namely, $F = B(F)$), and will further be the least fixed point of $B$; in other words, if $G$ is any fixed point, so that $G = B(G)$, then $F \leq G$, so that $G$ will be an extension of $F$. The least fixed point $F$ is taken to be the recursive function defined by the original recursive formulas.

What is the meaning of the increasing sequence (5) in the case when $B$ is the transformation defined by the formulas (3) and (4)? The function $\perp$ as indicated above, represents the everywhere undefined function, the partial function whose value is undefined for every natural number. The element $B(\perp)$ represents the function whose value at 0 is 1, but whose values everywhere else is undetermined. The element
$B(B(\perp))$ represents the function whose value at 0 is 1, whose value at 1 is 1, and whose values everywhere else are undetermined, and so on. The sequence of terms (5) thus represents a sequence of partial functions that contain more and more of the knowledge of the function Fact; the limit of the sequence, $F$, contains complete knowledge of the function Fact. This limit is attained through an infinite sequential process, starting from $\perp$, the everywhere undefined function, representing the state of pure potentiality. The sequential steps correspond precisely to the sequential steps of computation of the values of the factorial function starting from the recursive formulas (1) and (2). See Corazza (2011) for further discussion on these points. In general, the least-fixed-point characterization formalizes the “pattern-of-calls expansion” for a recursive function in computer science. See Manes & Arbib (1986).

In relation to Vedic Science, we can think of the element $\perp$ as representing the unmanifest state of pure potentiality, in which all possibilities are available but not yet expressed. This is the characteristic of the unmanifest point value of the “gap.” See Maharishi Vedic University (1985). The operator $B$ represents the dynamic principle inherent in the self-referral structure of knowledge, Fact $= B(\text{Fact})$. Through the dynamism expressed in the operator $B$, there sequentially unfold states expressing more and more precipitated values of knowledge culminating in $F$, in which all values of the function are determined. This corresponds to the theme of sequential emergence of creation from the self-interacting dynamics of consciousness in Maharishi Vedic Science, as we shall elaborate next, in the context of the $\lambda$-calculus.

§9. The Lambda Calculus

The lambda calculus, or $\lambda$-calculus as it is usually denoted, is a mathematical formalism that is based upon the concept of a function, the most fundamental concept of transformation in mathematics. This formalism was introduced by Church (1932/33) in an attempt to provide a new foundation for logic and mathematics. Church’s original system led to a paradox (Curry’s paradox) and had to be abandoned.

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2 In the literature one finds that “lambda calculus” and “$\lambda$-calculus” are interchangeable and bear the same relationship to each other as the English word for a number bears to the corresponding numeral (such as “fifteen” and “15”).
The purely functional part of the system was, however, demonstrably consistent and was found to have important connections with the theory of computation. Following several decades of limited activity in the $\lambda$-calculus, Scott (1969) showed how to construct a model for the $\lambda$-calculus based upon continuous lattices. This led to a great revival of interest and activity in the $\lambda$-calculus, largely connected with applications to the theory of programming languages in computer science.

The $\lambda$-calculus gives expression to a foundational viewpoint in mathematics complementary to the viewpoints of set theory and category theory discussed in Weinless (1987). The power and elegance of this viewpoint derive from the way it treats the function concept in a completely self-referral way. In this section we shall examine two fundamental expressions of self-referral in the $\lambda$-calculus in the context of Maharishi Vedic Science. These are: (1) the self-referral concept of applying a function to itself as an argument; and (2) the principle of abstraction, which provides an unrestricted scheme of impredicative definition. (For a detailed account of the $\lambda$-calculus, see Barendregt, 1981, Hindley & Seldin, 1986, or, for a more elementary treatment, see Stoy, 1977.)

The $\lambda$-calculus is a foundational approach to mathematics that takes as its primitive notion the concept of applying a function to an argument to obtain a value. When we deal with functional application in mathematics, we ordinarily have at least two different kinds of objects involved: functions and objects to which these functions are applied. For example, the objects may be numbers, and the functions are then transformations from numbers to numbers. One of the striking features of the $\lambda$-calculus is that it is formulated in terms of a single kind of object; there is no distinction between functions and arguments. Every argument is itself a function, and every function can have any function as an argument. This gives the $\lambda$-calculus an extraordinary self-referral structure, in which every function can be applied to itself as an argument. This situation is unheard of in ordinary mathematics; in fact, the way functions are described in the set-theoretic foundation specifically prohibits applying a function to itself as an argument. (This is because the set representing a function $f$ from a set $A$ to a set $B$ is always created at a later stage than the set representing any element of $A$.) This self-referral structure, which at first glance seems incompatible with
ordinary mathematics, lies at the very heart of the development of the \( \lambda \)-calculus. It was not until 1969, however, that a consistent interpretation for this self-referral structure was discovered (Scott, 1969). It will be necessary to elaborate further on the structure of the \( \lambda \)-calculus to describe this development.

We shall first briefly consider the intuitive conceptual model described by the \( \lambda \)-calculus. There is some domain of objects being described; these objects are the elements of some set \( S \). The set \( S \) is, at the same time:

1. the set of possible arguments for all functions being considered;
2. the set of all possible values of the functions;
3. the set of all possible functions.

This means that every element of \( S \) is actually a function from \( S \) to \( S \), and every function from \( S \) to \( S \) is an element of \( S \)! This conceptual universe thereby completely annihilates the distinction between elements and functions found in ordinary mathematics.

The set \( S \) represents the wholeness of this conceptual universe. The functions from \( S \) to \( S \) express the different possible values of transformation through which this wholeness moves within itself. These values of transformation are, at the same time, the elements of \( S \), that is, the point values within \( S \). These points are thus not inert, but contain the complete dynamics of the wholeness moving within itself.

This situation has a striking parallel in Maharishi Vedic Science in the analysis of the relationship between the wholeness of the unmanifest field of consciousness and the point value contained within it. This point value is described as having an infinitely dynamic nature; within the point one can locate the self-interacting dynamics of the unified field. This self-interacting dynamics of consciousness is at the same time described as “wholeness moving within itself,” the unbounded ocean of consciousness flowing within itself. In a parallel way, in the \( \lambda \)-calculus, each point in the field of wholeness \( S \) contains within itself the dynamism of the wholeness of \( S \) moving within itself.

A second parallel to Maharishi Vedic Science concerns the unification of the values of existence and dynamism. The elements of a set express the value of existence; they are the mathematical objects being
described. The functions express the value of transformation or dynamism. In our conceptual model, these two are one and the same: the values of existence are identical to the values of transformation. The values of existence and dynamism correspond to the two constituents of the transcendental reality of the Self in Maharishi Vedic Science: the silent, witnessing Being (existence) and the active, discriminating intelligence (dynamism). In the state of enlightenment these two values are fully integrated and unified in the individual’s awareness; each is seen to be an essential constituent of the other. See verses 6.5 and 6.6 of the Bhagavad-Gītā, Maharishi (1969).

Abstraction and \(\lambda\)-models
The idea of applying a function to itself as an argument is one expression of self-referral in the \(\lambda\)-calculus. A second expression is found in the way elements are defined in an impredicative way using a process called abstraction. This process has its basis in the way algebraic expressions using variables represent functions.

Consider as an example the expression \(3x + 7\). We assume the domain of mathematical discourse we are considering is the familiar system of integers. This symbolic expression does not name a specific number, because the variable \(x\) is not a name for a specific number. The variable \(x\) has the ability, however, to name any number; we can assign any possible value to \(x\); \(x = 1, x = 2, x = 3,\) and so on. Whenever we assign a value to the variable \(x\), then the expression \(3x + 7\) takes on a well-defined value. For example, if we assign the value 2 to \(x\), then \(3x + 7 = 3 \cdot 2 + 7 = 6 + 7 = 13,\) but if we assign the value 5 to \(x\) then \(3x + 7 = 3 \cdot 5 + 7 = 22.\) This means that the algebraic expression \(3x + 7\) gives rise to a function from numbers to numbers, the function \(f\) that takes 2 to 13, 5 to 22, and so on. Algebraic expressions, in this way, give rise to functions.

Lambda notation is a special notation to designate the functions arising from algebraic expressions. We use the notation \(\lambda x.3x + 7\) to designate the above function. The first \(x\) after the \(\lambda\) (lambda) is used to tell us that we are viewing the expression as a function of the variable \(x\).

We clearly obtain the same function if we start with the algebraic expression \(3y + 7,\) and view this expression as a function of the variable \(y.\) This means that \(\lambda y.3y + 7 = \lambda x.3x + 7.\)
In the pure $\lambda$-calculus, the only algebraic operation is the operation of application—applying a function to an argument to obtain a value. This operation is symbolically represented by juxtaposition, rather than the familiar functional notation using parentheses. Thus the expression $xy$ designates the result of applying the function $x$ to the argument $y$.

All the symbolic expressions of the $\lambda$-calculus, called $\lambda$-terms, are built up using just variables: $x, y, z, \ldots$, application, and $\lambda$-notation. The process of forming terms using $\lambda$-notation is called abstraction.

Here are several examples of terms and their meaning:

1. $\lambda x.x$ is the function that takes every element $x$ to itself.
2. $\lambda x.xx$ is the function that takes every element $x$ to $xx$, that is, to the result of applying $x$ to itself.
3. $(\lambda x.xx)(\lambda x.xx)$ is the result of applying the function defined in (2) to itself as an argument.
4. $\lambda x.\lambda y.xy$ is the function that takes every element $x$ to the function that takes every element $y$ to $xy$, that is, to the result of applying $x$ to $y$.

All the symbolic expressions of the $\lambda$-calculus are generated from variables through repeated use of application and abstraction.

To make the $\lambda$-calculus into a mathematically viable theory, one must do more than describe the symbolic structure of the language; one must describe the meaning of the symbolism. That is, one must show how the symbolism can be interpreted in terms of well-defined mathematical entities in a consistent way. Such an interpretation is called a model for the $\lambda$-calculus, or a $\lambda$-model. For a $\lambda$-model, one requires first of all a set of elements $S$; these are the elements referred to by the terms in the symbolic language. In the intuitive model discussed earlier, the elements of $S$ were at the same time the functions from $S$ to itself. Unfortunately, set theory makes this situation impossible. This comes from the way functions are represented as sets, as discussed above; this makes it, in principle, impossible for a function to have itself as an argument.

It turns out, however, that for the purpose of interpreting the symbolic expressions of the $\lambda$-calculus, it is not necessary for the elements of $S$ to be the same as the functions from $S$ to $S$. To interpret the application operation, it is enough that we have a binary operation defined on
S; to be able to interpret \( \lambda \)-expressions, however, the binary operation must have rather special properties.

We can clarify this by means of an example. Suppose we start with the set \( \mathbb{N} \) of natural numbers, and consider the binary operation of multiplication on the set \( \mathbb{N} \). We can interpret the symbolic expressions of the \( \lambda \)-calculus in the context of \( \mathbb{N} \) by simply interpreting application as multiplication. Thus, \( (3)(4) = 12 \), \( (5)(2) = 10 \), and so on. In this way, each element \( m \) of \( \mathbb{N} \) becomes associated with the function from \( \mathbb{N} \) to \( \mathbb{N} \), which just multiplies each number by \( m \).

While we have no difficulty interpreting application in this example, a problem arises when we try to interpret terms formed by abstraction. Consider, for example, the term \( \lambda x.xx \). This term represents the function that takes each number \( x \) to \( xx \), that is, to \( x \times x \). Call this function \( f \). Now the idea of a \( \lambda \)-model is that the terms should designate elements of the set \( S \), in this case, elements of \( \mathbb{N} \), that is, natural numbers. Thus, this term \( \lambda x.xx \) should designate a particular natural number \( m \). The property of \( m \) should be that application by \( m \) is the same as applying the function \( f \), which takes each \( x \) to \( x \times x \). This means that, for every natural number \( x \), we must have \( m \cdot x = x \cdot x \), but this is impossible. This means that the natural numbers with the operation of multiplication do not provide a model for the \( \lambda \)-calculus.

To find a \( \lambda \)-model, one must find a set \( S \) with a binary operation such that all possible \( \lambda \)-expressions represent elements of \( S \); this turns out to be quite a formidable mathematical problem. The first models were discovered after four decades by Scott (1969). Scott's models had the algebraic structure of continuous lattices. It is beyond the scope of this article to describe Scott's construction; the details can be found in Stoy (1977).

The difficulty in finding a model for the \( \lambda \)-calculus has its roots in the impredicative nature of the abstraction process described by \( \lambda \)-expressions. Consider, for example, the term \( \lambda x.xx \). Suppose we have in mind some model \( S \). The variable \( x \) in this expression is meant to range over all the elements of \( S \). When we allow \( x \) to range over all these elements, we obtain a certain function \( f \) from \( S \) to \( S \), the function that takes each element \( x \) to \( xx \). The term \( \lambda x.xx \) is meant to designate an element \( m \) of \( S \) that corresponds to this function, in the sense that application of \( m \) is the same as applying the function \( f \), that is,
\[ mx = f(x) = xx \text{ for all } x \text{ in } S. \] Thus, the element \( m \) being defined is required to be an element of the set over which the variables range in the definition; the definition is therefore impredicative. Because the use of abstraction is unrestricted in the \( \lambda \)-calculus, this principle allows impredicative definition in the most universal way.

We can thus identify the principle of abstraction as a fundamental expression of self-referral in the \( \lambda \)-calculus. This principle furthermore expresses self-referral in the context of the structure of knowledge: a \( \lambda \)-term is an expression of knowledge that defines a mathematical object in a self-referral way. The principle of abstraction thereby expresses the theme of self-referral structure of knowledge, which lies at the heart of Maharishi Vedic Science.

The Fixed-Point Principle

Earlier, we saw how the self-referral character of recursive definitions could be translated into characterizations of recursive functions as fixed points of certain transformations. Quite generally in mathematics, self-referral situations correspond to fixed points of transformations.

In ordinary mathematical contexts, one frequently encounters functions that have no fixed points. For example, the function \( f \) from the integers to the integers defined by \( f(x) = x + 1 \) has no fixed point; that is, there is no integer \( x \) such that \( x + 1 = x \). In the \( \lambda \)-calculus, the self-referral character of the principle of abstraction has as a consequence the rather striking result that every term of the \( \lambda \)-calculus has a fixed point; this is called the fixed-point principle. We give a simple argument to show why this is so:

Suppose we are given any term \( A \). Let \( C \) be the term defined by \( C = (\lambda x. A(xx))(\lambda x. A(xx)) \). Then \( C \) will be a fixed point of \( A \), that is, \( AC = C \).

We can see this as follows:

The term \( \lambda x. A(xx) \) represents the function that takes any element \( x \) into \( A(xx) \). This means that if we apply the term \( \lambda x. A(xx) \) to any term \( W \), we obtain \( (\lambda x. A(xx))W = A(WW) \). If we take now \( W \) to be \( \lambda x. A(xx) \) itself, we obtain

\[ (\lambda x. A(xx))(\lambda x. A(xx)) = A((\lambda x. A(xx))(\lambda x. A(xx))); \]

that is, \( C = AC \) where \( C = (\lambda x. A(xx))(\lambda x. A(xx)) \). This means that \( C \) is a fixed point for the term \( A \).
In the above derivation of the fixed point principle of the $\lambda$-calculus, we find a rich expression of self-referral. Not only is self-referral expressed in the impredicative nature of the $\lambda$-expressions, but also in the self-referral application of a function to itself as an argument. In the expression for the fixed point $C$, in fact, this second phenomenon occurs twice: first in the term $xx$, in which $x$ is applied to itself, and secondly in the complete expression for $C$, in which the term $\lambda x.\alpha(xx)$ is applied to itself as an argument.

The fixed point principle can be used to describe the recursive functions (Section 8) within the language of the $\lambda$-calculus; this description has important applications to the theory of computation as well as the theory of programming languages (see Section 10).

To describe the recursive functions in the $\lambda$-calculus, one first constructs specific terms to represent each of the natural numbers. The recursive description of a function can then be translated into a formal description of the function as a fixed point of a certain term in the $\lambda$-calculus (a more detailed treatment of the steps involved in this representation of the recursive functions can be found in Corazza, 2011).

In the application of the fixed point construction to define recursive functions, we find a second expression of the theme of self-referral definition in the $\lambda$-calculus, which we shall now analyze. Suppose we wish to define the function $\text{Fact}$. The property characterizing this function is that $\text{Fact} = \beta \text{Fact}$, that is, $\text{Fact}$ is to be a fixed point of a certain term $\beta$. We can think of the formula $\text{Fact} = \beta \text{Fact}$ as a definition of the function $\text{Fact}$. This type of definition is explicitly self-referral; the defined function $\text{Fact}$ appears explicitly on the right-hand side of the defining formula. The fixed point principle tells us that this type of explicit self-referral definition always makes sense.

If we think of the equation $\text{Fact} = \beta \text{Fact}$ as a definition of $\text{Fact}$, then there is possibly some ambiguity. This is because there may be more than one solution to this equation, that is, the term $\beta$ may have more than one fixed point. In the context of a lattice model, we can remove the ambiguity by requiring that $\text{Fact}$ be the least fixed point of the term $\beta$. This fixed point can always be obtained as the limit of an infinite sequential process, analogous to the Kleene-sequence construction considered below.
Specifically, suppose we have a $\lambda$-model $S$ that is a complete lattice.\(^3\) The structure of the lattice is described in terms of a partial-order relation, designated $\leq$. In terms of this partial ordering, the lattice will have a least element, designated $\bot$ (bottom); the property of $\bot$ is that $\bot \leq s$ for every element $s$ in $S$. Now suppose $P$ is any $\lambda$-term. Starting from $\bot$, we can generate a sequence of elements of $S$ by applying $P$ over and over again: $\bot, P(\bot), P(P(\bot)), \ldots$. One can show that this sequence will always be increasing:

$$(\ast) \quad \bot \leq P \bot \leq P(P\bot) \leq P(P(P\bot)) \ldots$$

This increasing sequence will always have a limit; that is, there will be an element $t$ in $S$ which this infinite sequence will approach as a limiting value. The formal property characterizing $t$ is that $t$ is the least element of the lattice such that every element in the sequence is $\leq t$. The element $t$ will always be a fixed point of $P$. In case $P$ has more than one fixed point, $t$ will be the least fixed point of $P$. The element $t$ will be the fixed point designated by the term $(\lambda x. P(xx))(\lambda x. P(xx))$.

When we view the recursive functions in the context of the $\lambda$-calculus, then the element $\bot$ in the lattice represents the everywhere undefined (empty) function. It also represents the undefined value: the element $\bot$ corresponds to the function from $S$ to $S$ that takes every element of $S$ to $\bot$. The element $\bot$ thus corresponds to the function that “collapses” the wholeness of $S$ to the point value $\bot$. This is parallel to the phenomenon of *Akshara* in Maharishi Vedic Science, as we shall now elaborate.

Maharishi has described how the self-interacting dynamics of consciousness is expressed in the first syllable of Rk Veda, AK, which presents the collapse of A, the expression of the unbounded wholeness of consciousness, to K, the expression of the value of a point. This collapse of A to K, called the *Akshara*, is the primordial expression of the self-

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\(^3\) A lattice is a partially ordered set $L$ having additional characteristics. Let $\leq$ denote the order relation on $L$. As a lattice, $L$ has the additional properties that, for any two elements $a, b$ of $L$, $L$ also contains a least upper bound ("join") and greatest lower bound ("meet") of $a$ and $b$. $L$ is said to be a complete lattice if for any subset $A$ of $L$, $L$ contains both a least upper bound and greatest lower bound of $A$. The least upper bound of $L$ itself is called the “top” element of $L$ and is denoted $\top$; the greatest lower bound of $L$ is called the “bottom” element of $L$ and is denoted $\bot$. Therefore, in a complete lattice, it follows that for every $a$ in $L$, $\bot \leq a \leq \top$. 

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interacting dynamics of consciousness. In the context of the $\lambda$-model $S$, the function that collapses the wholeness of $S$ to the point value of $\perp$ provides a mathematical expression of this phenomenon of Akshara.

Maharishi has further elaborated that the point value of $K$ is not inert, but contains within itself the infinitely dynamic self-interaction of the field of consciousness. This characteristic of the point $K$ is called Anyonyabhava, the infinite dynamism inherent in the point. In the $\lambda$-model $S$, the point value of $\perp$, contained within $S$, represents the dynamic principle that collapses $S$ to that very point value $\perp$, the perfect mathematical expression of the Anyonyabhava.

Maharishi has explained how the $\textbf{Richās}$ of $\text{Rk Veda}$ sequentially emerge from the point value of $K$, following the Akshara, AK; the $\textbf{Richās}$ present the hierarchical structure of the laws of nature sequentially unfolding from the self-interacting dynamics of the Samhitā. In the continuation of this sequential process, the creation itself emerges from the laws of nature. The full state of enlightenment, Brahman Consciousness, emerges from the synthesis of all these sequentially emerging values into the transcendental wholeness of the Samhitā, the self-referral structure of pure knowledge.

Now if we think of $\perp$ as the expression of the point value of $K$, and some term $P$ as expressing the dynamics of wholeness (as a function from $S$ to $S$), then $P$ promotes the sequential emergence of elements from $\perp$, the state of pure potentiality, as given by ($\ast$). This direction of unfoldment is the direction of more and more precipitated, localized expressions of natural law. This is the significance of the partial ordering $\leq$ in this context; it represents the direction of more complete knowledge of the specificity of a function. The limit $t$ of the sequence is obtained from all these partial expressions through a process of synthesis, which is at the same time a process of transcending. The value of transcending is expressed in the way an infinite process must be completed, because the sequence is infinite. Synthesis is expressed in the way that the knowledge in $t$ is just the synthesis of all the partial expressions of knowledge of the elements of the sequence. What gets structured through this synthesis is an element $t$ that is a fixed point of $P$: $t = Pt$. The element $t$ is the expression of the self-referral structure of knowledge derived from the function $P$; the self-referral expression for $t$ characterizes $t$ in terms of the value of non-change. The whole process
of sequential unfoldment of the recursive function from \( \bot \), culminating in the fixed point, \( t \), is thus seen to be parallel to the sequential emergence, in Maharishi Vedic Science, of natural law from the point value of \( K \), culminating in the self-referral structure of Brahman.

**Actualizing the Intuitive Model: Non-well-founded Sets**

In our discussion so far of models for the \( \lambda \)-calculus, we have been forced to compromise the initial intuitive model, in which the set \( S \) was identical to the set of all possible functions from \( S \) to \( S \). We shall consider here how the self-referral features of non-well-founded sets allow one to more completely actualize this intuitive model, and then in the subsection following, how this intuition finds an even better realization in the context of toposes.

In ordinary set theory, an element of a set \( S \) cannot at the same time be a function from \( S \) to itself, because a function cannot have itself as an argument. This violates the axiom of foundation, which prohibits a set from being an element of itself (or, in this context, an element of an element of itself!). In the self-referral world of non-well-founded sets, governed by the anti-foundation axiom, it is easy to construct functions that have themselves as arguments. In fact, one can start with any ordinary well-founded \( \lambda \)-model, such as one of Scott’s lattice models, and transform it into a non-well-founded model \( S \) in which each element of \( S \) is a function from \( S \) to itself. Such a model will capture much of the original intuitive model.

However, in these non-well-founded models, \( S \) will still not quite be the set of all possible functions from \( S \) to itself. This is a result of Cantor’s theorem, which tells us that the infinite size, or cardinality, of the set of functions from \( S \) to \( S \) will always be greater than the infinite size of \( S \). Cantor’s theorem is true for non-well-founded sets as well as the familiar well-founded sets, and therefore we cannot quite model the intuitive idea completely even in the self-referral world of non-well-founded sets. We consider next an approach that provides a full realization of our initial intuition about \( \lambda \)-models.

**Actualizing the Intuitive Model: Topos Theory**

A topos is a type of mathematical structure called a category. See Weinless (1987) for an introductory treatment of categories. It consists of two
types of structural elements: *objects* and *arrows* between the objects. Sheaf semantics provides a way each topos can be interpreted as a kind of universe of sets, governed in this context by intuitionistic logic. The essential feature of sheaf semantics is that the objects represent simultaneously the sets (Chhandas) and the stages of knowing (Rishi), while the arrows represent simultaneously the elements of the sets in the various stages of knowing (Devatā), as well as the functions (transformations between the Chhandas values) and passages between stages of knowing (transformations between the Rishi values). In this way, sheaf semantics provides a mathematical expression of a highly self-referral structure of knowledge, and for this reason, provides a more suitable context for building lambda-models.

Topos theory has had important applications in diverse areas of mathematics and logic. It has provided models for self-referral situations that could not in principle be modelled using conventional set theory based on classical logic. These include models for intuitionistic logic, and in particular, for synthetic differential geometry, as discussed in Section 6. In Section 3 we discussed the way the Austinian approach to semantics formalizes the value of Rishi, the knower, in the structure of knowledge. A similar theme is found in Kripke’s semantics for intuitionistic logic, in which Brouwer’s concept of the “creative subject” is formalized as “stages of knowing.” This theme is taken to its self-referral value in the sheaf semantics of topos theory; this topic is discussed in Weinless (1987) and more fully in Weinless (2011). Another fundamental area of application has been in providing generic models for geometric theories, in which one constructs a topos containing a specific model for a theory, such that the model embodies, in its own individuality, the total knowledge and infinite organizing power of the theory. See Weinless (1987). Maharishi Vedic Science locates the self-referral structure of pure knowledge at the fountainhead of all streams of knowledge and all expressions of organizing power in nature. From this perspective, we can identify the source of the extraordinary power of topos theory in modelling self-referral phenomena to be the way it captures mathematically the self-referral structure of pure knowledge, as formalized in sheaf semantics.

Topos theory, whose self-referral structure of knowledge is governed by intuitionistic logic rather than classical logic, also provides a way to
construct mathematical universes in which Cantor’s theorem does not apply, and therefore makes it possible to provide a full actualization of the intuition behind λ-models; in such models, one locates a “set” $S$ for which the elements of $S$ correspond to all possible functions from $S$ to $S$ (see Lambek and Scott, 1986). The construction of $S$ and of its ambient topos are in fact extremely natural. A certain type of category, called a cartesian closed category with a reflexive object, provides a natural model for the λ-calculus. Starting with such a category, one can apply a standard construction to obtain a topos in which the category is naturally embedded. This topos will be a kind of universe of sets, which contains among its objects a model $S$ for the λ-calculus in which all possible functions from $S$ to itself correspond to elements of $S$.

In this we see how the self-referral features of topos theory allow one to model most completely the intuitive, self-referral structure of the λ-calculus. A further expression of this theme will be discussed in Section 11, in which the self-referral structure of knowledge available in topos theory will be found to provide the only known models of the highly self-referral structure of the polymorphic λ-calculus.

§10. Denotational Semantics

Much of the interest in the λ-calculus during the past two decades has been centered round its connections with the theory of programming languages in computer science, in particular, its natural and unique ability to model the self-referral features of high-level programming languages. This aspect of programming language theory, called denotational semantics, is concerned with the description of the semantics, or “meaning,” of programming languages. The formal approach of denotational semantics was developed by Christopher Strachey’s Programming Research Group at Oxford University in the 1960s (see, for example, Strachey, 1966) and was made mathematically rigorous through integration with Scott’s work on models of the λ-calculus (Scott & Strachey, 1971).

The abstract approach of denotational semantics is to be contrasted with the more familiar and concrete approach of operational semantics. Operational semantics describes the meaning of a program as the sequential steps of calculation that are performed when the program is run on a computer. Denotational semantics describes the meaning of
a programming language in terms of abstract mathematical entities, called denotations. The idea is that every expression, command, procedure, or program in the language has an intrinsic, abstract meaning called its denotation, and that the denotational semantics can capture the wholeness of meaning of the language in a way that the operational semantics can never achieve.

For example, suppose we have a mathematical program that computes the factorial function. The program can take as its input any natural number \( n \). The program then goes through a sequence of computations and then gives as its output the number \( \text{Fact}(n) \). Operational semantics will describe the meaning of this program as a sequence of calculations. Denotational semantics will describe the meaning of the program as a function from \( \mathbb{N} \) to \( \mathbb{N} \). The function is the intended meaning of the program, but it is something abstract.

The main problem of denotational semantics is to find a suitable domain to represent the denotations of a programming language, and to construct a semantic function that maps symbolic expressions of the programming language into their denotations. This turns out to be quite a challenging problem. The main challenge comes from the way the straightforward definition of the semantic function is self-referential in a way that has no obvious, consistent mathematical interpretation. It was not until Scott’s construction of his lattice models for the \( \lambda \)-calculus that one could provide a consistent interpretation for such recursive descriptions of the semantics.

To appreciate the recursive nature of program semantics, we shall consider one simple example, the while . . . do . . . command, which is found in most imperative programming languages.

The syntax of the command is while \( A \) do \( B \), in which \( A \) is a boolean expression and \( B \) is a command. A boolean expression is a symbolic expression that evaluates to either \( T \) (true) or \( F \) (false). A command is an instruction to do something.

The command while \( A \) do \( B \) is executed on a computer in the following way. First, the boolean expression \( A \) is evaluated. If the resulting value is \( F \), then nothing is done; control is simply passed to the next command. If the value of \( A \) is \( T \), then command \( B \) is executed, and then the whole process is repeated. That is, one repeatedly executes \( B \) and
then evaluates $A$ until the evaluation of $A$ yields $F$ (if ever), and then control is passed to the next command.

This description can be recast as a recursive definition of while $A$ do $B$:

\[
\text{while } A \text{ do } B = (\text{if } A = F \text{ then do nothing, else do } B \text{ and then while } A \text{ do } B).
\]

The recursive character of the definition is displayed in the way the command being defined, while $A$ do $B$, is defined in terms of itself—it appears also on the right-hand side of the above equation.

It turns out that the $\lambda$-calculus provides a natural and elegant way of formalizing this type of recursive definition of program semantics. The denotation of the command is thereby characterized as the fixed point of a certain symbolic expression in the $\lambda$-calculus. Lambda models, such as complete lattices, can then be used to model the abstract level of meaning of the programming language. For a detailed account, see Stoy (1977).

More generally, the $\lambda$-calculus can be applied to define the complete semantics of high-level programming languages, defining in a mathematically rigorous way the denotations of all the symbolic expressions of the language: commands, procedures, functions, programs, and so on. Furthermore, the self-referral structure of knowledge available in the $\lambda$-calculus has been found to be necessary to provide a mathematically rigorous definition of the denotational semantics of programming languages.

It is quite striking that the highly abstract theory of the $\lambda$-calculus should play such an indispensable role in computer science. Computers, after all, are essentially finitistic in their behavior. This suggests that finitary mathematics should provide the theoretical foundation for computer science. How can one account for the need to take recourse to the much more abstract theory of the $\lambda$-calculus and continuous lattices?

The parallel to physical science is illuminating here. Physical measurements always give rise to numerical values that are rational; nevertheless, to describe the underlying laws of nature requires a theoretical model based on the continuum of real numbers, where the theory of the continuum is based on set theory, the mathematical theory of the infinite. We can think of the mathematical continuum as a mathemati-
cal expression of the transcendental continuum of consciousness, which Maharishi Vedic Science identifies as the home of all the laws of nature. It seems quite natural and fitting that the mathematical description of this transcendental level of reality should provide the basis for the mathematical quantification of the laws of nature that emerge from this field.

Now the field of computer science has its own obvious basis in the field of human intelligence. The behavior of a computer is just the most concrete, precipitated expression of the organizing power contained in a computer program. The program itself is a product of human intelligence, and its meaning must be sought in the abstract field of intelligence, not in the mechanical steps of sequential computing implemented on a computer. The need for the self-referral structure of the $\lambda$-calculus to model the meaning of programming languages just documents the fundamental role played by the self-referral nature of intelligence in creating and using these high-level languages.

In fact, the self-referral nature of intelligence is found displayed in an essential way in all levels of computer programming, from assembly language upwards. The use of loops in programming is an essentially self-referral phenomenon whereby programming units can refer to themselves. See Lester (1987). This has its basis, on the level of the hardware, in the way the program itself resides in memory, and programming instructions can refer to the addresses of other instructions in the program. This is something all programmers take for granted today, but was a great innovation when it was introduced by Burkes, Goldstine, and von Neumann (1944).

The seed for this idea is found in the concept of a universal computing machine, introduced by Turing (1936). Turing formalized the concept of a computer; this formalization is called a Turing machine. A Turing machine is an idealized type of computer that reads from, and writes to, an infinitely long tape. Turing showed how one could design a single Turing machine, the universal Turing machine, that could simulate all possible Turing machines. Each machine was systematically assigned a code number $n$; if one wanted the universal machine to simulate machine number $n$, one would first input, on the tape, the code number $n$ of the machine to be simulated, and then input the data the machine would be required to process.
In this description, the tape plays the role of the memory of the computer; the code number $n$, the program; and the remaining data on the tape, the data processed by the program. Modern computers are essentially implementations of this concept of a universal computer, in which the hardware is fixed but the computer has the capability of doing any mechanical calculation, limited only by the size of its memory.

In terms of Maharishi Vedic Science, we can equate the data acted upon by the program with the Chhandas value, and equate the program itself, the embodiment of the intelligence underlying the computation, with the Rishi value. In terms of the hardware, these are both stored the same way in memory, expressing the unification of the Rishi and Chhandas values. This gives the program the ability to function in a self-referral way, curving back on itself through loops, by having access to the sequential addresses of its own commands, which reside, side by side with the data, in the memory of the computer. The theme of self-referral is thus found at all levels of computer science, from the architecture of the underlying hardware to the highly abstract self-referral content of the intelligence of the programmer that is expressed in the self-referral structure of high-level programming languages.

§11. The Polymorphic Lambda Calculus

We shall consider in this final section an extension of the $\lambda$-calculus, called the \textit{polymorphic} $\lambda$-calculus, which expresses an added dimension of self-referral through abstraction on types. This will provide a basis for modelling the highly self-referral structure of polymorphic programming languages.

The formulation of the $\lambda$-calculus described in Section 9 is called the \textit{untyped} $\lambda$-calculus. It is an example of an untyped language, in which one does not distinguish the types of terms. This contrasts with the structure of most programming languages, in which there are specified data types, and each variable must have a well-defined type. Typical data types are real type (real numbers), integer type (whole numbers), boolean type (true or false), string type (sequences of characters), and so on. When one introduces a variable in a C or Java program, for example, one must explicitly declare the type of the variable, that is, what type of data that variable is intended to represent.
There is a variant of the \( \lambda \)-calculus that reflects this kind of typed structure. This is the *typed* \( \lambda \)-calculus. In the typed \( \lambda \)-calculus there are different types, and each term has a specified type. For any types \( A \) and \( B \), there is required to be a function type \( A \to B \), designating functions from objects of type \( A \) to objects of type \( B \).

In the typed \( \lambda \)-calculus, the application operation is restricted in the following way. If \( r \) and \( s \) are terms, one can only form the application \( rs \) when \( s \) is of some type \( A \) and \( r \) is of type \( A \to B \); the application \( rs \) then has type \( B \). In other words, if \( s \) designates an object of type \( A \) and \( r \) designates a function of type \( A \to B \), that is, a function from objects of type \( A \) to objects of type \( B \), then the application \( rs \) will designate an object of type \( B \).

In the typed \( \lambda \)-calculus, one can never apply a term \( r \) to itself; the self-referral structure of the untyped \( \lambda \)-calculus is lost. In a recent development, this self-referral value has been regained through a second level of application of the \( \lambda \)-abstraction process, this time to types rather than terms. This is the second-order polymorphic \( \lambda \)-calculus, introduced by Reynolds (1974).

In the polymorphic \( \lambda \)-calculus, one introduces a second class of variables, \( \alpha, \beta, \gamma, \ldots \), to represent types, and then introduces type abstractions, indicated by the symbol \( \forall \). This is similar to ordinary abstraction for terms, indicated by the symbol \( \lambda \). For example, the type \( \forall \alpha (\alpha \to \alpha) \) represents the function from types to types, that takes any type \( A \) into the type \( A \to A \), just like the term \( \lambda x.x \) represents the function that takes any term \( s \) to the term \( ss \).

A type of the form \( \forall \alpha \phi \), where \( \phi \) is some type expression, is called a polymorphic type. For example, \( \forall \alpha (\alpha \to \alpha) \) is a polymorphic type. There is a type application rule that allows polymorphic types to be applied to any type. For example, if \( A \) is any type, then if we apply the polymorphic type \( \forall \alpha (\alpha \to \alpha) \) to \( A \) we obtain \((\forall \alpha (\alpha \to \alpha))A = A \to A \). In particular, a polymorphic type can be applied to itself:

\[
(\forall \alpha (\alpha \to \alpha)) (\forall \alpha (\alpha \to \alpha)) = \forall \alpha (\alpha \to \alpha) \to \forall \alpha (\alpha \to \alpha)
\]

Through the introduction of polymorphic types, the typed \( \lambda \)-calculus thus once more becomes self-referral.

The polymorphic \( \lambda \)-calculus provides a natural tool for the study of the denotational semantics of polymorphic programming languages.
These are languages admitting variables of polymorphic type, for which any possible data type can be substituted. Polymorphic languages have much greater expressive power than the familiar languages with rigid type structure, and there is currently great interest in these languages. However, the self-referral nature of polymorphic types presents a great challenge in modelling the semantics of these languages.

Reynolds (1984) analyzes a natural approach to defining a semantics for the polymorphic $\lambda$-calculus based upon category theory. In this approach, the expressions designating types are interpreted as functors, and expressions designating terms are interpreted as natural transformations between these functors. The semantics of the self-referral type abstraction operator $\forall$ involves a highly impredicative construction, in which one is required to form arbitrary products of objects of the underlying category, indexed by the objects of the category itself. (This type of self-referral construction is somewhat analogous to forming a “set of all sets.”) A well-known result due to P. Freyd implies that, in terms of the usual set-theoretic foundation of modern mathematics, one cannot construct categories displaying the required self-referral structure. Reynolds concludes that “polymorphism is not set-theoretic.”

Pitts (1987) shows that the required category-theoretic constructions can be carried out if one works in a suitable universe of sets governed by intuitionistic logic rather than classical logic. The intuitionistic universe that works is the effective topos, introduced by Hyland, Johnstone and Pitts (1980). The effective topos is a generalized universe of sets that is constructed through a sequence of steps, starting from the partial recursive functions.

The effective topos is constructed in the following way. One first constructs the internal logic of the topos. The truth values for the internal logic correspond to all possible sets of natural numbers. The set of all natural numbers, $\mathbb{N}$, represents the truth value $T$. The empty set represents the truth value $F$. Every other subset of the natural numbers represents some intermediate truth value lying in the “gap” between true and false. This logical structure therefore differs fundamentally from the structure of classical logic, in which there are only two truth values, $T$ and $F$.

The logical operations $\land$ (and), $\lor$ (or), $\neg$ (not), and $\rightarrow$ (implies) for these truth values are then defined in an unusual way using the partial
recursive functions. These definitions are based upon the realizability interpretation of arithmetic (see Troelstra, 1973). Once the logical operations are defined, one then generates a universe of sets governed by this logic. This universe is the effective topos. Within this universe, one can then construct a category that is a model for the polymorphic \( \lambda \)-calculus.

There are two very striking features of this construction. One is the way in which topos theory is required to model the semantics of the polymorphic \( \lambda \)-calculus. In this we find the meeting of two of the most striking expressions of self-referral in modern mathematics: topos theory and the \( \lambda \)-calculus.

We discussed earlier how the sheaf semantics of a topos presents a completely self-referral structure of knowledge in which the stages of knowing (the values of the knower) are identical to the sets (the objects of knowledge). It is striking that this self-referral structure of knowledge is precisely what is required to provide a model for the self-referral structure of the polymorphic \( \lambda \)-calculus.

The second, even more striking fact concerns the nature of the specific topos required to model the polymorphic \( \lambda \)-calculus, namely, the effective topos. We have observed that the effective topos is generated starting from a logical system constructed from the partial recursive functions. Within this topos, one then constructs a category that provides a model for the polymorphic \( \lambda \)-calculus. This then provides a natural model for the denotational semantics of polymorphic programming languages. The operational semantics of these same languages is defined in terms of implementations on computers, which compute, as their output, the partial recursive functions.

In this we find the partial recursive functions lying at the two extremes of the spectrum of knowledge of computer science. At one extreme, we find the partial recursive functions expressing the extreme Rishi value, as the basis of the logic of the abstract mathematical universe in which the semantical models are constructed. At the other extreme, the partial recursive functions express the extreme Chhandas value as the concrete numerical output of the computer. Thus the extreme Rishi value is found to be identical to the extreme Chhandas value in a totally unexpected way! This provides a rather amazing expression, in the field of computer science, of a deep principle of Maharishi Vedic Science:
the ultimate identity of the Rishi and Chhandas values at the unified source of creation.

In this identity of the Rishi and Chhandas values we find actually two expressions of self-referral in computer science. The first expression of self-referral is just the identity of Rishi and Chhandas. A second expression of self-referral is found in the nature of the recursive functions themselves, which are the common value of Rishi and Chhandas. We have seen that the recursive functions are defined by self-referential mathematical formulas, through which a recursive function is defined in terms of itself. The recursive functions are thus a self-referral mathematical reality, and this self-referral reality plays the role of both Rishi and Chhandas in the semantics of the polymorphic λ-calculus, as we have analyzed above.

It is interesting to reflect on the relationship between the untyped λ-calculus and the polymorphic λ-calculus in the context of Vedic Science. Both have a self-referral structure. The untyped λ-calculus expresses self-referral in the context of an untyped language, that is, an undifferentiated level of mathematical reality. The polymorphic λ-calculus expresses self-referral in the context of a typed language, that is, a differentiated level of mathematical reality. Of the two, the polymorphic λ-calculus has the richer structure and expresses a more profound value of self-referral.

These two mathematical expressions of self-referral have a parallel in Vedic Science in two levels of experience of the self-referral nature of consciousness. The first is the experience of the self-referral structure of the undifferentiated field of pure intelligence, in which consciousness knows itself on its own abstract, unmanifest level. The second is the experience of the fully matured state of enlightenment, Brahman Consciousness, in which the infinite diversity of creation is experienced within oneself, and the self-referral value of consciousness is found to be permeating all the diverse expressions of natural law.

The very rich expression of self-referral in the polymorphic λ-calculus can be understood as a natural development in the evolution of programming languages. The direction of evolution of programming languages has been towards greater power, abstraction, and ease of application. That is, the direction of development has been towards greater and greater harmony with the natural abstract mathematical functioning of
human awareness, so that the languages can most effortlessly, naturally, and fully express the abstract content of awareness. For this reason, the languages have become more and more expressive of the structure of the inner language of intelligence.

The inner language of intelligence is the language of nature, the Veda. The Veda presents the self-referral structure of natural law, at its most fundamental and unified level, in its own natural language. We can think of the Veda itself as presenting the programming language of nature. Maharishi often speaks of the human brain as the “cosmic computer,” which, through proper programming, can accomplish anything (1986, p. 32). Maharishi has spoken of the Vedic literature as providing the “cosmic software” for the cosmic computer. See Maharishi (1983) and Lester (1987).

Maharishi has presented a vision of the range of application of the cosmic software available in the Vedic literature as extending far beyond the range of numerical computation, far beyond even the range of the totality of modern mathematics. The ultimate application of the Śūtras available in the Vedic literature is in structuring the state of enlightenment. In the fully developed state of enlightenment, the awareness is described as having an “all knowing” quality, called Jyotish Mati Pragyā; this quality of awareness has the ability to spontaneously compute the past, present, and future of anything in creation. This is the supreme computing ability of nature that belongs to the cosmic computer, the human brain. This supreme computing ability can be actualized in the individual life through proper application of the cosmic software available in the Vedic literature.

This supreme computing ability of the cosmic computer is certainly far beyond the range of computing power available in the computer technology of today. Nevertheless, the natural direction of evolution of programming languages has been to express more and more of the self-referral structure of the “cosmic” programming language, the language of nature, the Veda. With the current development of polymorphic languages, this self-referral value has been found to be expressed to a remarkable degree, as we have analyzed above.

This evolutionary theme in computer science suggests that the detailed syntax and semantics of the language of the Veda might have great relevance to the future development of programming language...
theory. The structure of the Vedic language itself is Vedic Sanskrit. One aspect of the Vedic literature, called Vyākaraṇa, is specifically concerned with the grammar of this language, analyzing in depth both its syntax and semantics.

Briggs (1985) notes the striking parallels between the approach to semantic analysis in the ancient texts of Vyākaraṇa and the contemporary approach in artificial intelligence in terms of semantic nets. He comments:

Understandably, there is a widespread belief that natural languages are unsuitable for the transmission of many ideas that artificial languages can render with great precision and mathematical rigor. But this dichotomy, which has served as a premise underlying much work in the areas of linguistics and artificial intelligence, is a false one. There is at least one language, Sanskrit, which for the duration of almost 1000 years was a living spoken language with a considerable literature of its own. Besides works of literary value, there was a long philosophical tradition that has continued to exist with undiminished vigor until the present century. Among the accomplishments of the grammarians can be reckoned a method for paraphrasing Sanskrit in a manner that is identical not only in essence but in form with current work in Artificial Intelligence. This article demonstrates that a natural language can serve as an artificial language also, and that much work in AI has been reinventing a wheel millennia old.

Briggs concludes, “Their [the Indian grammarians’] analysis of language casts doubt on the humanistic distinction between natural and artificial intelligence, and may throw light on how research in AI may finally solve the natural language understanding and machine translation problems.”

Our discussion of computer science in this article has focused on abstract aspects of programming language theory. Lester (1987) gives a comprehensive examination of computer science in the context of Maharishi Vedic Science. Lester notes in particular the way the architecture of the “cosmic computer,” the human brain, has served as a model for the recent development of neural computing networks, which display striking expressions of self-organization and consequent intelligent behavior, such as learning. The mathematical description of the highly self-referral computing of neural networks is still in its infancy.
In the coming years we can expect to see extraordinary developments in the mathematical expressions of self-referral in connection with computer science, as this discipline embraces more and more of the cosmic computing ability of the unified field of natural law, the self-referral field of pure intelligence.

§12. Conclusion

From the examples discussed in this article, we hope to have conveyed some sense of the liveliness and prominence of the theme of self-referral in the foundations of mathematics. Self-referral has historically played an important role in the foundations of set theory and logic, particularly in connection with the paradoxes of the infinite and Gödel’s theorems. In the past two decades, however, there have been a number of striking foundational developments that have given a new prominence to self-referral in a variety of areas of pure and applied mathematics.

Perhaps the three most significant of these recent developments are:

1. Scott’s discovery of lattice models for the self-referral structure of the untyped $\lambda$-calculus and subsequent developments in the field of denotational semantics.

2. The development of topos theory, based upon the self-referral structure of knowledge formalized in sheaf semantics, which has provided a way to construct models for intuitionistic set theory and has thereby led to the creation of models for a number of highly self-referral mathematical “impossibilities.”

3. Aczel’s development of the theory of non-well-founded sets, in which sets can be elements of themselves, which has been applied to model self-referral situations in semantics, information theory, and the $\lambda$-calculus.

These expressions of self-referral are not restricted to the abstract realm of contemplation of philosophers of mathematics, disconnected from the ordinary activity of the working mathematician, to say nothing of the concrete reality of practical life. Rather, they have natural and important applications in many areas not only of pure mathematics but of applied mathematics as well. All three of these developments have, in particular, natural and significant applications to computer science.
Maharishi Vedic Science provides insight into the genesis of the self-referral developments in modern mathematics, by revealing them to be natural expressions of the self-referral nature of intelligence. This we have elaborated in our discussion of the different mathematical examples of self-referral in this article.

In some cases, Maharishi Vedic Science can be expected to provide needed motivation for the acceptance of new intuitive frameworks for developing mathematics. This applies in particular to non-well-founded sets. But it applies elsewhere as well. For example, it can be applied to the process of doing mathematics internally in unusual toposes, such as occurs in synthetic differential geometry. These universes allow one to model “paradoxical” qualities of the self-referral field of intelligence, as described in Maharishi Vedic Science. Through the knowledge of Maharishi Vedic Science, together with the direct experience of the self-referral field of intelligence, one can feel quite at home in these exotic mathematical universes. This expands the range of mathematical activity to different realities, not just the conventional reality formalized in the Zermelo-Fraenkel axioms of set theory and classical logic. In this way, mathematical activity and mathematical thought can embrace a greater range of possibilities that belong to the field of all possibilities, the self-referral field of pure intelligence.

Mathematics has always reflected its cultural milieu. At the present time, we are experiencing a cultural transformation—the transformation of the age of ignorance to the Age of Enlightenment. In this process of transformation, the self-referral nature of intelligence is becoming very lively in the world. This is occurring on the basis of Maharishi Vedic Science and Technology and is occurring on two levels: intellectual understanding and direct experience. The first is based upon Maharishi’s exposition of the principles of Vedic Science over the past 30 years, which have elaborated the detailed self-referral structure and self-interacting dynamics of the field of pure intelligence, the unified field of natural law. The second is based upon the technology of consciousness, the Transcendental Meditation and TM-Sidhi programs, which Maharishi has introduced throughout the world, providing direct experience of the transcendental field of consciousness, and thereby enlivening the self-referral value of intelligence in the collective consciousness of the world.
It is natural that the enlivenment of the self-referral value of collective consciousness should be reflected in the direction of development of mathematical concepts. This is something that we have been witnessing in the past two decades, in the blossoming of mathematical expressions of self-referral. This development is not only enriching to modern mathematics but is also enriching to Maharishi Vedic Science, by providing mathematical models for the self-referral structure of intelligence and its self-interacting dynamics. With the continued growth of the self-referral value in world consciousness, we can expect to see extraordinary new expressions of the mathematics of self-referral that will very completely integrate modern mathematics with Maharishi Vedic Science, structuring a wholeness of knowledge that is fully satisfying to the intellect in its quest for mathematical truth.

References


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Lambek, J. and Scott, P.J. (1986). Introduction to higher order categorical logic. Cambridge: Cambridge University Press.


Part IV

Mathematics:

Geometry, Symmetry,

and Consciousness
Consciousness:
The Last Frontier of Geometry

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We hear of “the last frontier of biology,” “the last frontier of physics,” and “the last frontier of psychology.” These last frontiers are one and the same, human consciousness. This paper proposes that geometry also has consciousness as its last frontier.

The inspiration for geometry is in the physical world, but like every other discipline of knowledge, geometry is the product of its practitioners—it is the product of the consciousness of the geometer. The discipline of geometry can give deep insight into the nature of consciousness. Even the perfection of the circle suggests characteristics of consciousness and many other geometrical concepts reflect qualities of consciousness. This paper will examine aspects of geometry that can shed light on the nature of the consciousness of the mathematicians who originated them: continuity, higher dimensions, infinity, symmetry, and the homogeneity of space.

From these observations, along with reference to recent scientific research on the nature of consciousness, geometers can learn more about consciousness from within the discipline of geometry and teach geometry to better reflect its source in consciousness.

Introduction

Each discipline of knowledge studies different aspects of life and uses different methodologies and procedures. Many disciplines, however, find that their distinct paths reach a common end: questions about consciousness. Thus, consciousness is regarded as the last frontier in humanity’s quest for knowledge. Here, we will describe the discipline of geometry, that part of mathematics that investigates patterns of space, shape, and form. We will see that geometry can also lead, in a very natural way, to concerns relevant to the study of consciousness and can give unique insights into the ultimate nature of consciousness.

Throughout its history, mathematics has been highly regarded as a discipline that can offer valuable insights outside of its designated domain. Thousands of years ago, Plato [1, page 69] affirmed that “geometry will draw the soul towards truth.” Descartes, several hundred years ago, modeled his comprehensive search for truth on the methodology of geometry. Today, Keith Devlin [2, page 9] recognizes that “as an
entirely human creation, the study of mathematics is ultimately a study of humanity itself."

In this paper, we will first look at how other disciplines have encountered this last frontier of consciousness and then focus on several themes of geometry—continuity, higher dimensions, infinity, symmetry, homogeneity of space, and non-Euclidean geometries—to see what connections they have with consciousness. Finally, we will discuss the significance of this view of geometry.

**Last Frontiers**

Physics, the study of matter, force, and energy, has encountered Heisenberg’s uncertainty principle, which guarantees that merely observing a system changes the system in a way that cannot be minimized or avoided. This means that the role of the observer is an integral part of any system being observed. This has two important implications. The first is that understanding the process of observation, whereby an observer is connected to an object of observation, is necessary for any complete theory of physics. The second is that the observer and the observed, consciousness and matter, mind and body, are not separate but are intimately connected at their very basis. Thus, a unified field theory of physics must ultimately account for the role of consciousness.

Biology, the study of living organisms, has located one fundamental structure at the basis of life: DNA. Encoded in an organism’s DNA is all of the information about the growth, development, functioning, and behavior of the organism. Biologists know quite precisely how certain sequences of DNA codons produce specific proteins, but they know nothing about how the structure of DNA and the biological systems it controls are connected to the phenomenon of human awareness or consciousness. Perhaps a complete and correct understanding of this “blueprint of life” could lead to a deep understanding of the nature of consciousness.

Psychology is the study of the mind and behavior; from the beginning, psychologists have considered consciousness to be their main object of study. After a century of research, however, the true nature of consciousness continues to elude psychologists. It appears that current theories are not able to encompass the full range of consciousness,
suggesting that consciousness is a far more challenging and far more holistic object of study than was originally thought.

Even computer science has realized the need for understanding the “cosmic computer,” the human brain. The development of computational algorithms has been a significant factor in increasing computer speeds and establishes the need for the development of artificial intelligence. Still, attempts to mimic even the simplest cognitive skills have met with limited success. It is clear that understanding the functioning of human intelligence will be useful in the future development of computer science.

The Case for Geometry
Given the limited success of other disciplines in describing the nature of human consciousness, what reason is there to expect that geometry has anything to contribute? After all, geometry makes no claim to study the mind or behavior, as does psychology. Geometry has found no fundamental connection between the observer and the physical world, as physics has. The subject matter of geometry has, in fact, surprisingly little to do with the real world. Nevertheless, there may yet be ways in which geometry is able to give us insight into the structure and functioning of human awareness. In discussing the axioms of geometry, Keith Devlin [2, page 163] says,

… these fundamental geometric notions, and the intuitions that accompany them, are not part of the physical world we live in; they are part of ourselves, of the way we are constructed as cognitive entities. Euclidean geometry may or may not be the way the world is ‘made up,’ whatever that may mean, but it does appear to capture the way human beings perceive the world.

Geometry locates key characteristics and properties that can be useful to interpret, compare, and analyze shapes and forms. Geometry, like all of mathematics, is the result of deliberation about completely abstract structures and relationships. The patterns and theorems of geometry are developed by different individuals, but their validity spans individual, cultural, and even chronological differences. It is fair to expect, therefore, that studying geometry should disclose intimate and profound attributes of consciousness that are common to every individual. We begin our discussion with continuity.
Continuity

Continuity is a central idea in geometry; it ensures that two straight lines intersect at a point and provides a basis for measurement. At the heart of the Greek preference for geometry is the lack of continuity of the rationals, as we would now understand their discovery of the irrationality of $\sqrt{2}$. The fifth of Hilbert’s five groups of axioms is about continuity [3, page 26]. Birkhoff’s first and third axioms affirm the continuity of the straight line and of angular measure [6, Appendix 2].

Until very recently, the most appropriate model for space, time, matter, and energy has been the continuum. Calculus was developed to handle changing relationships between continuous quantities. Beginning with the discovery of molecules and atoms, however, each continuous model used by physics has been shown to be inaccurate at sufficiently small time and distance scales. The most recent theories of physics, which model space and time with space-time foam, show that even space and time cannot, at their finest level, be modeled by a continuum.

It must be the case, then, that continuity is characteristic of how the observer, as a mathematician or physicist, interprets or understands space, time, and matter. Continuity must be more closely associated with the consciousness of the observer than with the object of observation. The continuum is an infinite, expanding structure, flowing evenly. It is the basis for measuring or quantifying. We can conclude that these qualities are necessarily attributes of consciousness.

Higher Dimensions

The three dimensions of space that we live in appear to be inviolable, yet they can be extended mathematically in a very natural way to higher- and even infinite-dimensional abstract spaces. Higher-dimensional objects or spaces are typically used to provide a convenient geometric framework in which to consolidate all possible members of a certain set. For example, the infinite-dimensional phase spaces of physics consist of every possible state a specific particle or system could have; the time evolution of the particle or system is then a trajectory through the phase space. Similarly, the feasibility region of a linear program is the many-dimensional set of all possible points satisfying a given collection of inequalities. These higher-dimensional spaces are used because they
organize information in an effective way that allows for further computation. The physical bounds of three dimensions are not boundaries to our awareness; it is natural and useful for the mind to operate in an all-inclusive way, unrestricted by the boundaries of the space we live in. Our consciousness more naturally functions from a unified level, creating a totality or wholeness out of all possible distinct parts.

Infinity
The infinite is at the heart of mathematics. Considerations of continuity and higher-dimensional spaces lead us inevitably to the broader concept of infinity in geometry. The straight line is infinite in extent and has infinitely many points. There are infinitely many geometric shapes, classified according to the various attributes (number of edges, measure of angles, and so on). Fractals show the intricate and beautiful structure that the infinite can support. What can be said of the human awareness that has created, studied, and classified the infinities of geometry? Certainly the faculty that can comprehend and manipulate those infinities must itself be even more infinite. Consciousness is infinite, it is capable of comprehending infinity, and it must be the holistic value of which all other infinities are only a part.

Symmetry
At the basis of the concept of symmetry is the recognition that two different objects can have essentially the same structure. For example, the symmetries of a square depend on the congruence of each side with the other three sides and each angle with the other three angles. Our innate appreciation of symmetry depends on our ability to simultaneously discriminate (between the different parts of an object or between different objects) and to locate sameness or equivalency (of those parts or objects).

Consciousness can locate differences, but more importantly, it can use differences as the basis for harmonizing or balancing those differences through the location of symmetries. The natural enjoyment of unity in the presence of diversity explains the universal use of symmetry in the decorative arts. Thus, we see that consciousness is at once discriminating and harmonizing, synthesizing parts into ever greater wholes.
Symmetry also depends on the comparing of an object to itself, using an object itself as its own measure. This is reflective of the self-referral quality of consciousness, the ability of consciousness to know, observe, and interact with itself.

**Homogeneity of Space**
The fourth postulate of Euclid says that all right angles are equal; this means that space is everywhere the same in Euclidean geometry. This is certainly not how the eye perceives the space around us, since a right angle looks “right” only when viewed straight on. It is also not the best model of the space we live in, as we see from Einstein’s General Theory of Relativity. This suggests that homogeneity is more naturally a characteristic of the consciousness of the mathematician than of the space we live in.

**Different Geometries**
For hundreds of years, Euclidean geometry was regarded as “true,” the only possible geometry and the actual description of the space we live in. The discovery of non-Euclidean geometries that are provably as consistent as Euclidean geometry inspired a reexamination of the nature of mathematical truth, forcing mathematicians to accept mathematical truth as relative truth only. Consciousness is capable of integrating opposites, harmonizing them into a single viewpoint that transcends the diversity created by the intellect.

**Conclusions**
The discussion here of these few geometric topics affirms that the study of geometry is an important and appropriate study for someone interested in consciousness and the meaning of life. In geometry, one is studying patterns of consciousness and gaining knowledge about how consciousness functions within itself. The unique perspective of geometry, focusing on patterns of shape and form, is complementary to other approaches to the study of consciousness.

The study of geometry suggests that consciousness is an infinite, unbounded, homogeneous continuum that can support great diversity. It is more natural and effective for the mind to construct a whole out of infinitely many possibilities than to deal with finitely many cases. The
intellect is harmonizing and unifying, capable of locating the common features of different structures. Finally, human consciousness, even when restricted by the precision and logical correctness of mathematics, is capable of encompassing different, even contradictory, truths.

These conclusions about connections between geometry and consciousness have implications for the student of geometry. Students of geometry should consider themselves to be also students of consciousness. The development and refinement of the intellect has always been considered to be important for mathematicians, but we see now that an intellectual understanding of the nature of consciousness can support and enrich our understanding of geometric concepts.

The integration of the study of consciousness with the study of geometry is part of the curriculum at Maharishi University of Management. At the beginning of their studies, all students take a course covering the nature of consciousness, intelligence, and knowledge led by Maharishi Mahesh Yogi. This course serves as a foundation for their study of all other disciplines, including geometry. All students also practice the Transcendental Meditation and TM-Sidhi programs as an experiential method for research into the nature of their own consciousness. This foundation has proved to be successful and rewarding in the study of geometry.

Finally, I would like to note that Maharishi Mahesh Yogi, the world’s foremost scientist in the area of consciousness, views mathematics as really nothing other than the study of consciousness:

Mathematical knowledge deals directly with the functioning of the field of intelligence—consciousness. The principles of Mathematics are universally valid principles of knowledge that describe the dynamics of the field of intelligence—the functioning of the mathematician’s own consciousness. [5, page 160]

From this perspective, geometry is necessarily one aspect of the study of consciousness. The real fulfillment of mathematics, however, is in what it can achieve for humanity:

Vedic Mathematics provides the steps of invincibility, enlivens the total potential of Natural Law, establishes mastery over Natural Law and thereby places life on the invincible level of ‘Victory before War’—life without problems—invincible defence against anything that is not useful to life. [4, page 351]
This vision of what mathematics can achieve embodies the goals mathematicians have held throughout history and is an inspiration to all mathematicians.

References


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Symmetry:
A Link between Mathematics and Life

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Symmetry is an important and basic concept in mathematics that should be deeply understood by students. This paper describes how symmetry in mathematics can be related to cultural and artistic themes, in particular a verse from the Bhagavad-Gītā, to enhance students' understanding of the mathematical significance of symmetry.

Introduction

In mathematics, certain basic concepts, such as symmetry and infinity, are so pervasive and adaptable that they can become elusive to the student. Understanding these concepts and the tools for studying them is often a long process that extends over many years in a student's career. Students first see infinity appearing as the potential infinite inherent in the positional number system, then implicit in plane geometry, and eventually underlying all of calculus and analysis. Students begin to use symmetry with commutativity and associativity in arithmetic, making more use of it in Euclidean geometry and plane geometry, and may eventually see it in terms of transformation groups. Nevertheless, it is natural to want to teach these concepts in their full value from the very beginning. This paper will describe how I have been introducing students in a general education geometry course to the concept of symmetry in a way that I feel gives them a comprehensive understanding of the mathematical approach to symmetry.

Why Teach Symmetry?

Symmetry is found everywhere in nature and is also one of the most prevalent themes in art, architecture, and design—in cultures all over the world and throughout human history. Symmetry is certainly one of the most powerful and pervasive concepts in mathematics. In the Elements, Euclid exploited symmetry from the very first proposition to make his proofs clear and straightforward. Recognizing the symmetry that exists among the roots of an equation, Galois was able to solve a centuries-old problem. The tool that he developed to understand symmetry, namely group theory, has been used by mathematicians ever since to define, study, and even create symmetry.

Students are fascinated by concrete examples of symmetry in nature and in art. The study of symmetry can be as elementary or as advanced
as one wishes; for example, one can simply locate the symmetries of designs and patterns, or one can use symmetry groups as a comprehensible way to introduce students to the abstract approach of modern mathematics. Furthermore, the ideas used by mathematicians in studying symmetry are not unique to mathematics and can be found in other areas of human thought. By looking at symmetry in a broader context, students can see the interconnectedness of mathematics with other branches of knowledge.

For these reasons, many mathematicians today feel that the mathematical study of symmetry is worthwhile for general education students to explore.

**A Link between Symmetry and Life**

The central idea in the mathematical study of symmetry is a symmetry transformation, which we can view as an isomorphism that has some invariants. For example, a symmetry transformation of a design in the plane is an isometry that leaves a certain set of points fixed as a set. I would like students to realize that this concept of symmetry transformation, as abstract as it may appear, can be connected to ideas that may seem more central to a view of life as a whole; for this, I introduce a verse from the Bhagavad-Gītā.

In the Bhagavad-Gītā, Lord Kṛṣhṇa lays out the complete knowledge of life to his pupil Arjuna, just as a great battle is about to begin. This work has long been appreciated for the great wisdom that is expounded in just a few short chapters. A verse that seems to me to capture the essence of the mathematical study of symmetry is part of Kṛṣhṇa’s explanation of the field of action (Chapter 4, verse 18, translated by Maharishi Mahesh Yogi):

\[
\text{He who in action sees inaction}
\]

\[
\text{and in inaction sees action is}
\]

\[
\text{wise among men. He is united, he}
\]

\[
\text{has accomplished all action.}
\]

How is this related to symmetry? A geometric figure that we wish to study is usually given as a set of points existing in some ambient space. For example, a tiling pattern may be given as a collection of line
segments in the plane. A symmetry transformation can be regarded as “action” and invariants can be regarded as “inaction.” We begin with a non-dynamic situation (the set of points of the tiling pattern sitting in the plane) and then find some dynamism (the symmetry transformation). Thus, in inaction, we see action. But a symmetry transformation is not just any action; it must leave the pattern (as a set of points) invariant. Thus, what is important to us is that in this action (the transformation), we are able to see inaction (the invariance of the set of points making up the pattern).

This is the seed of all that I want students to know about symmetry: action and inaction, a transformation and its invariants, what changes and what stays the same.

With this, students gain a unifying perspective on the concept of symmetry that can help them understand it initially and that can later help them simplify and unify all the occurrences of this concept as they are met and eventually understand symmetry groups, invariants, and so on. This theme can also help students connect all instances of symmetry that they have already seen to this one unifying perspective. For example, in the commutative and associative properties of arithmetic, the positions of the numbers or parentheses change, but the answer does not change. For a tiling, the Euclidean plane can be rotated, reflected or translated in certain ways, but the pattern remains the same. A knot can be moved and redrawn, but its Conway polynomial is invariant, and so on.

This verse from the Bhagavad- ītā not only captures the essence of symmetry, but also helps students understand the importance of invariants wherever they might see them. In his commentary on this verse, Maharishi Mahesh Yogi (1969, p. 278) explains that the phrase “in action sees inaction” means that one sees the nonchanging unmanifest absolute silent level of pure consciousness underlying the normal activity of thinking, perceiving, and acting. This silent level of life is the source of the active levels of life; it is subtler and more abstract than the active levels, but more powerful and more important. Elsewhere, Maharishi (1969, p. 470) explains this using an analogy of the ocean. The ocean is silent at its depths and the dynamism of the waves is just the natural expression of the silent levels; the silent, nonchanging level is more fundamental. Thus, it is the invariants of a transformation that
will be useful to us, even though at first they may seem difficult to grasp because of their subtlety or abstraction. With this perspective, whenever we see a transformation, our first question is, “What are the invariants? What doesn’t change?”

For students at Maharishi University of Management, this understanding takes on a very personal meaning in terms of their practice of the Transcendental Meditation technique, which allows the active thinking mind to settle down to the silent, nonactive state of consciousness at its source. In their own experience, they see that their consciousness has two aspects, active and silent, and that the silent level is more fundamental and more powerful than the dynamic level. In a very concrete way, they are able to connect the ideas of symmetry transformation and invariants to their own personal experience.

**Teaching Symmetry**

Students come to mathematics with rather limited ideas of symmetry; frequently the word symmetry is interpreted to mean “bilateral symmetry” and nothing more. Nevertheless, they will have seen symmetry in many forms already: nature, manufactured objects, art and architecture, and even in mathematics (commutativity, circles and squares, odd and even functions, and so on). It is good for students to have an understanding of symmetry that includes all the examples that they have seen and that lays a foundation for further study. I want to introduce them to the idea of symmetry transformation, even though they may not know what a function is, so that they will remember it, feel that it is important, and be able to make some use of it. Students should realize that symmetry locates some underlying property that may be more abstract and less obvious but is more unifying and more discriminating. I also want students to have some insight into why symmetry is attractive and aesthetically appealing to us.

Using the verse from the Bhagavad-Gītā as a guide, symmetry can be learned in a unifying way that students seem to enjoy.

We begin with a discussion of what symmetry is, recording some of the students’ points on the board. Then we examine some finite designs from the artwork of different cultures and revise our notions of symmetry based on the fact that these designs should come under our definition of symmetry. To motivate this discussion, I bring up the idea that
mathematics needs a precise definition that can allow us to definitively say whether something is symmetric or not and that a good definition will also help us to study objects in terms of the property.

Here, the idea of symmetry transformation is introduced. We look at some of the designs and find rotations and reflections and see that a rotation followed by a rotation is another rotation, a reflection followed by a reflection is a rotation, and the composition of a reflection and a rotation, in either order, is a reflection. Further investigation reveals the fundamental properties of finite designs: (1) a pattern can have only rotations, but not only reflections, and (2) if the identity is treated as a rotation and there are reflections, then there are just as many rotations as reflections.

At this point, I introduce the Bhagavad-Gītā verse and we spend quite a bit of time understanding the verse and how it can be interpreted in terms of symmetry transformations. The questions “What changes?” and “What stays the same?” start to become part of the students’ way of thinking.

The first application of this way of thinking comes when we start working out the group table for the symmetry group of an equilateral triangle. After two symmetries are performed, one needs to determine what one symmetry is equivalent to the composition. We look at what stays the same. If one vertex of the triangle is left fixed, the composition is a reflection. If no vertices are left fixed (so that only the center is fixed), then the composition is a rotation. If we want to determine the type of a given transformation, look at what is fixed: if only a point is fixed, it is a rotation about that point; if a line is fixed, it is a reflection across that line; if everything is fixed, it is the identity.

As we move on to frieze ornaments and wallpaper patterns, to identify all possible transformations becomes more challenging. Now, we can think of “inaction” in terms of sameness, lack of change. Locate a motif or small design that is repeated throughout the whole pattern; then see if there is a way to transform that motif or design to as many of its repetitions as possible. If any of these transformations are symmetries of the pattern as a whole, then we have located a symmetry transformation. And the best way to describe the transformation is to say what stays the same: the center of rotation, the axis of reflection,
the direction vector for a translation and the direction vector for a glide reflection (the lines determined by these vectors are fixed).

The Beauty of Symmetry

When students begin to design their own patterns, they start thinking in terms of aesthetics, what patterns they like and want to work on themselves.

Symmetry is beautiful and fascinating. From the charm of a snowflake to the deep spirituality of Leonardo’s Last Supper, symmetry has an essential role in nature and art. Can the understanding of symmetry that we have gained here help us in any way to understand this role? We have seen that a symmetrical pattern gives rise to symmetries or transformations of the pattern which leave it essentially unchanged. From the Bhagavad-Gītā, we see that life has two aspects, active and inactive. According to Maharishi, the silent level of life is pure consciousness, the source of thought, and it is subjectively experienced as bliss; whenever the active level of the mind begins to move in the direction of the silent level of the mind, there is increasing bliss. An artistic pattern or structure of nature expresses the diversity of relative existence, yet in the repetition of aspects of the design or structure, that is, in the symmetry, an underlying sameness or unifying value is indicated. The mind is spontaneously led to experience activity and silence simultaneously. This is in the direction of the nature of the experience described in the verse of the Bhagavad-Gītā that we have examined. Thus, our analysis can help shed light on the charming nature of symmetry.

Conclusion

Mathematics is part of life; mathematicians doing mathematics are subject to the same natural laws that govern all of life. A deep understanding of the whole of life should give us the kind of insight that will help us understand the parts of life, including some very specific aspect of mathematics. This paper presents how one expression of knowledge about the nature of life from the Bhagavad-Gītā can be used to go deeply into the mathematical study of symmetry and, hopefully, acts as a suggestion that this bringing together of mathematics and life as a whole can be done in other ways.

This article, “Symmetry: A Link between Mathematics and Life,” by Catherine A. Gorini, here updated and revised, and reprinted with permission, originally appeared in the *Humanistic Mathematics Network Journal*, 1996.
Electronic Resources and Publications

LINKS

Education

Maharishi University of Management: www.mum.edu
Maharishi School of the Age of Enlightenment:
  www.maharishischooliowa.org
Maharishi’s Consciousness-Based Education: www.CBEprograms.org
International Foundation of Consciousness-Based Education:
  www.CBEfoundation@ifcbe.org
David Lynch Foundation for Consciousness-Based Education and
World Peace: www.davidlynchfoundation.org

Transcendental Meditation Program

Maharishi’s Technologies of Consciousness: www.tm.org
Maharishi Channel: www.maharishichannel.in
Maharishi Lectures and Interviews (film clips): www.tm.org/maharishi
Invincible America Assembly: www.invincibleamerica.org
Global Country of World Peace: www.globalcountry.org
Global Good News Site: www.globalgoodnews.com
Fortune Creating Homes: www.FortuneCreatingHomes.com
Sthapatya Veda: www.sthapathyaveda.com

Research

Center for Brain, Consciousness, and Cognition: www.drfredtravis.com
Truth about TM: www.truthabouttm.org

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1-888-LEARN TM (1-888-532-7686)
Maharishi University of Management (1-641-472-7000)
These publications are available from Maharishi University of Management Press: http://mumpress.com and at the MUM Bookstore.

Books by Maharishi Mahesh Yogi

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