Consciousness-Based Education: A Foundation for Teaching and Learning in the Academic Disciplines

A Series of 12 Volumes

Managing Editor, Dara Llewellyn
Executive Editor, Craig Pearson

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CONSCIOUSNESS-BASED EDUCATION AND MATHEMATICS

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Higher education faces a complex set of challenges today. We are seeing resources diminish at the same time we are hearing calls for greater access and affordability. Demands for greater transparency and accountability are being sounded by both the general public and the government. Government is exerting increasing controls in this long-independent area.

These challenges, however, are merely financial and political, and they are hardly limited to colleges and universities. The fundamental challenges are educational and center around the students themselves. Challenges include high levels of stress, pervasive substance abuse (particularly binge drinking), lack of preparedness for college-level work, and mental and emotional disabilities. In most of these areas, the problem is serious and worsening. Though colleges and universities are striving to address these challenges, few would claim we are turning the tide.

An encouraging trend is the increasing focus in higher education nationwide on promoting student learning. Yet these laudable efforts do not take into account the powerful forces working in opposition. It is well known that learning is inhibited by stress, sleep deprivation, alcohol, and poor diet—and these are among the most conspicuous features of the college student experience.

Something new is required. Education needs a reliable means of developing students directly from within. We need a systematic method for cultivating their creative intelligence, their capacity to learn, and their natural humanity. All education aims at these goals, of course—but the approach thus far has been from the outside in, and the results have been haphazard at best.

Consciousness-Based education was established to address this need. It integrates the best practices of education and places beneath them a proper foundation—direct development of the student from inside out.

The outcomes of Consciousness-Based education have been unprecedented and scientifically verified. These outcomes include significant
growth of intelligence, creativity, learning ability, field independence, ego development, and moral maturity, among others. These results are remarkable because many of these values typically plateau in adolescence—but Consciousness-Based education promotes this growth in students of all ages, developing potentials that otherwise would have remained unexpressed.

Beyond this rich cognitive growth, Consciousness-Based education significantly reduces student stress, boosts self-esteem, improves health, reduces substance use, and enhances interpersonal relationships. All of this comes together to create exceptional learning environments. This approach even measurably improves the quality of life in the surrounding society.

Consciousness-Based education was founded by Maharishi Mahesh Yogi, the world authority on the science of consciousness. First pioneered at Maharishi University of Management in Fairfield, Iowa, Consciousness-Based education is being adopted by schools, colleges, and universities around the world. It is easily integrated into any school, without any change in mission or curriculum.

Consciousness-Based education recognizes that student learning depends fundamentally on students’ levels of consciousness or alertness. The more alert and awake the student, the more successful and satisfying the learning.

Consciousness-Based education consists of three components:

- a practical technology for directly developing students’ potential from within,
- a theoretical understanding of consciousness that gives rise to a unifying framework for knowledge, enabling students to easily grasp the fundamental principles of any discipline and to connect these principles to their own personal growth, and
- a set of classroom practices, arising from this understanding, that also help promote effective teaching and learning.

**The Transcendental Meditation program**

At the heart of Consciousness-Based education is the practice of the Transcendental Meditation technique. The technique was brought to light by Maharishi Mahesh Yogi from the Vedic tradition of India, the
world’s most ancient continuous tradition of knowledge. It is practiced for 20 minutes twice daily, once in the morning and once in the afternoon, while sitting comfortably with eyes closed. It is simple, natural, and effortless—so simple, in fact, that ten-year-old children can learn and practice it. It has been learned by more than six million people worldwide, of all ages, religions, and cultures.

The Transcendental Meditation technique differs from other procedures of meditation and relaxation in its effortlessness. It involves no concentration or control of the mind. Neither is it a religion, philosophy, or lifestyle. It involves no new codes of behavior, attitudes, or beliefs, not even the belief it will work.

The Transcendental Meditation program is the most extensively validated program of personal development in the world. It has been the subject of more than 600 scientific research studies, conducted at more than 250 universities and research institutions in more than 30 countries worldwide. These studies have been published in more than 150 scientific and scholarly journals in a broad range of fields, including *Science*, *Scientific American*, *American Journal of Physiology*, *International Journal of Neuroscience*, *Memory and Cognition*, *Social Indicators Research*, *Intelligence*, *Journal of Mind and Behavior*, *Education*, *Journal of Moral Education*, *Journal of Personality and Social Psychology*, *Business and Health*, *British Journal of Educational Psychology*, *Journal of Human Stress*, *Lancet*, *Physiology and Behavior*, and numerous others. No approach to education has as much empirical support as Consciousness-Based education.

This approach, moreover, has been successfully field-tested over the past 35 years in primary, secondary, and post-secondary schools all over the world, in developed and developing nations, in a wide variety of cultural settings—the United States, Latin America, Europe, Africa, India, and China.

The Transcendental Meditation technique enables one to “dive within.” During the practice, the mind settles inward, naturally and spontaneously, to a state of deep inner quiet, beyond thoughts and perceptions. One experiences consciousness in its pure, silent state, uncolored by mental activity. In this state, consciousness is aware of itself alone, awake to its own unbounded nature.
The technique also gives profound rest, which dissolves accumulated stress and restores balanced functioning to mind and body.

This state of inner wakefulness coupled with deep rest represents a fourth major state of consciousness, distinct from the familiar states of waking, dreaming, and sleeping, known as Transcendental Consciousness.

In this restfully alert state, brain functioning becomes highly integrated and coherent. EEG studies show long-range spatial communication among all brain regions. This coherence is in sharp contrast to the more or less uncoordinated patterns typical of brain activity.

With regular practice, this integrated style of functioning carries over into daily activity. Research studies consistently show a high statistical correlation between brainwave coherence and intelligence, creativity, field independence, emotional stability, and other positive values. The greater one’s EEG coherence, in other words, the greater one’s development in these fundamental areas. At Maharishi University of Management, students even have the option of a Brain Integration Progress Report—an empirical measure of growth of EEG coherence between their first and last years at the University.

The brain is the governor of all human activity—and therefore personal growth and success in any field depend on the degree to which brain functioning is integrated. The increasingly integrated brain functioning that spontaneously results from Transcendental Meditation practice accounts for its multiplicity of benefits to mind, body, and behavior.

Every human being has the natural ability to transcend, to experience the boundless inner reality of life. Every human brain has the natural ability to function coherently. It requires only a simple technique.

**Theoretical component—
a unified framework for teaching and learning**

Scholars have long called for a way to unify the diverse branches of knowledge. Current global trends are making this need ever more apparent. The pace of progress is accelerating, the knowledge explosion continues unabated, and knowledge is becoming ever more specialized.

Academic disciplines offer a useful way of compartmentalizing knowledge for purposes of teaching, learning, research, and pub-
lication. But each academic discipline explores only one facet of our increasingly complex and interrelated world. The real world, however, is not compartmentalized—an elephant is not a trunk, a tusk, and a tail. Academic disciplines, consequently, are criticized as inadequate, in themselves, for understanding and addressing today’s challenging social problems.

Today, more than ever, we need a means of looking at issues comprehensively, holistically. We need a way of discovering and understanding the natural relationships among all the complex elements that compose the world, even among the complex elements that compose our own disciplines.

Various attempts to address this need have been made under the rubric of interdisciplinary studies—programs or processes that aim to synthesize the perspectives and promote connections among multiple disciplines. Some of these efforts have been criticized as superficial joinings of disciplinary knowledge. But the chief criticism of interdisciplinary studies—levelled even by its proponents—is that looking at an issue from multiple perspectives does not, in itself, enable one to find the common ground among contrasting viewpoints, to resolve conflicts, and to arrive at a coherent understanding.

The diverse academic disciplines can be properly unified at only one level—at their source. All academic disciplines are expressions of human consciousness—and if the fundamental principles of consciousness could be identified and understood, then one would gain a grasp of all human knowledge in a single stroke.

This brings us to the theoretical component of Consciousness-Based education. Consciousness-Based education does precisely this—and not as an abstract, theoretical construct but as the result of students’ direct experience of their own silent, pure consciousness. In this sense, practice of the Transcendental Meditation technique forms the laboratory component of Consciousness-Based education, where the theoretical predictions of Consciousness-Based education can be verified through direct personal experience.

This theoretical component offers a rich and deep yet easy-to-grasp intellectual understanding of consciousness—its nature and range, how it may be cultivated, its potentials when fully developed. This theoretical component also identifies how the fundamental dynamics of
consciousness are found at work in every physical system and in every academic discipline at every level.

With this knowledge as a foundation, teachers and students in all disciplines enjoy a shared and comprehensive understanding of human development and a set of deep principles common to all academic disciplines—a unified framework for knowledge. With this unified framework as a foundation, students can move from subject to subject, discipline to discipline, and readily understand the fundamental principles of the discipline and recognize the principles the discipline shares with the other disciplines they have studied. This approach makes knowledge easy to grasp and personally relevant to the student.

**Pure consciousness and the unified field**

Consciousness has traditionally been understood as the continuous flux of thoughts and perceptions that engages the mind. Thoughts and perceptions, in turn, are widely understood to be merely the by-product of the brain’s electrochemical functioning.

Maharishi has put forward a radically new understanding of human consciousness. In Consciousness-Based education, the foundation and source of all mental activity is understood to be pure consciousness, the most silent, creative, and blissful level of the mind—the field of one’s total inner intelligence, one’s innermost Self. (This unbounded value of the Self is written with an uppercase “S” to distinguish it from the ordinary, localized self we typically experience.) Direct experience of this inner field of consciousness awakens it, enlivens its intrinsic properties of creativity and intelligence. Regular experience of pure consciousness through the Transcendental Meditation technique leads to rapid growth of one’s potential, to the development of higher states of human consciousness—to *enlightenment*.

But consciousness is more, even, than this.

Throughout the 20th century, leading physicists conjectured upon the relation between mind and matter, between consciousness and the physical world; many expressed the conviction that mind is, somehow, the essential ingredient of the universe. But Maharishi goes further. He has asserted that mind and matter have a common source, and that this source is pure consciousness. Consciousness in its pure, silent state is identical with the most fundamental level of nature’s functioning,
the unified field of natural law that has been identified and described by quantum theoretical physicists over the past several decades. Everyone has the potential to experience this field in the simplest form of his or her own awareness. Considerable theoretical evidence, and even empirical evidence, has been put forward in support of this position.

Maharishi has developed these ideas in two bodies of knowledge, the first known as the Science of Creative Intelligence, the second as Maharishi Vedic Science and Technology. The Science of Creative Intelligence examines the nature and range of consciousness and presents a model of human development that includes seven states of consciousness altogether, including four higher states beyond the familiar states of waking, dreaming, and sleeping. These higher states, which develop naturally and spontaneously with Transcendental Meditation practice, bring expanded values of experience of one’s self and the surrounding world. Each represents a progressive stage of enlightenment. Maharishi Vedic Science and Technology examines the dynamics of pure consciousness in fine detail. It reveals the fundamental principles of consciousness that may then be identified in every field of knowledge and every natural system.

Most important for teaching and learning, these sciences reveal how every branch of knowledge emerges from the field of pure consciousness and how this field is actually the Self of every student.

**Strategies for promoting teaching and learning**

Consciousness-Based education also includes a battery of educational strategies that promote effective teaching and learning. Foremost among these is the precept that parts are always connected to wholes and that learning is most effective when learners are able to connect parts to wholes. In Consciousness-Based education, the parts of knowledge are always connected to the wholeness of knowledge, and the wholeness of knowledge is connected to the Self of the student.

One means of doing this is through *Unified Field Charts*. These wall charts, developed by the faculty at Maharishi University of Management and used in every class, do three things: (1) They show all the branches of the discipline at a glance. (2) They show how the discipline emerges from the field of pure consciousness, the unified field of natural law at the basis of the universe. (3) They show that this field is the
Self of the student, which the student experiences during practice of the Transcendental Meditation technique.

In this way students can always see the relation between what they are studying and the discipline as a whole, and they can see the discipline as an expression of their own pure consciousness. Again, this is more than an intellectual formulation—it is the growing reality of students’ experience as they develop higher states of consciousness.

Another strategy is *Main Point Charts*. Developed by the faculty for each lesson and posted on the classroom walls, these charts summarize in a few sentences the main points of the lesson and their relationship to the underlying principles of consciousness. In this way students always have the lesson as a whole in front of them, available at a glance.

**The next paradigm shift**

If higher education is fundamentally about student learning and growth, then Consciousness-Based education represents a major paradigm shift in the history of education. To understand this change, it is useful to reflect on the encouraging paradigm shift that has already been taking place in education over the past several decades.

This shift involves a move from what many call an *instruction paradigm* to a *learning paradigm*. In the instruction paradigm, the mission of colleges and universities is to provide instruction; this is accomplished through a transfer of knowledge from teacher to student. In the learning paradigm, the mission is to produce student learning; this mission is achieved by guiding students in the discovery and construction of knowledge.

This shift is a vitally important advance in education, leading to more successful outcomes and more rewarding experiences for students and teachers alike. But a further paradigm shift remains, and we can understand it by examining a fundamental feature of human experience.

Maharishi observes that every human experience consists of three fundamental components: a knower, a known, and a process of knowing linking knower and known. We may also use the terms experiencer, object of experience, and process of experiencing, or observer, observed, and process of observation.
This three-fold structure of experience is nowhere more evident than in schools: The knowers are the students, the known is the knowledge to be learned, and the process of knowing is what the full range of teaching and learning strategies seek to promote.

Understanding this three-fold structure helps us understand the paradigm shifts that are taking place.

The instruction paradigm places emphasis on the known. It focuses on the information students are to absorb and the skills they are to learn. In this paradigm, the instructor’s role is to identify what students need to know and deliver it to them.

The learning paradigm emphasizes the process of knowing. It recognizes that students must be actively involved in the learning process, that knowledge is something individuals create and construct for themselves, that students have differing learning styles and differing interests that must be taken into account. In this paradigm, the instructor’s role is to create learning environments and experiences that promote the process of learning.

The Consciousness-Based paradigm embraces the known and the process of knowing but places primary emphasis on the knower—on developing the knower’s potential for learning from within. The following diagram shows the respective emphases of each approach:
But the learning paradigm does not so much abandon the instruction paradigm as enlarge it, so that it includes the process of knowing as well as the known. And the Consciousness-Based approach completes the enlargement to include the knower:

Consciousness-Based education, in summary, is a theory and practice grounded in a systematic science and technology of consciousness, making available the complete experience, systematic development, and comprehensive understanding of the full range of human consciousness. More than 30 years’ experience and extensive scientific research confirm the success of this approach and its applicability to any educational institution.
About this book series

This series of twelve volumes is the result of a unique faculty-wide project that began with the founding of Maharishi University of Management in 1971 and continues to this day. Each volume in the series examines a particular academic discipline in the light of our Consciousness-Based approach to education.

Each volume includes:

- an introductory paper introducing the Consciousness-Based understanding of the discipline,
- a Unified Field Chart, if available for publication, for the discipline—a chart that conceptually maps all the branches of the discipline and illustrates how the discipline emerges from the field of pure consciousness and how that field is the Self of every individual. Thus, these charts connect the “parts” of knowledge to the “wholeness” of knowledge and the wholeness of knowledge to the Self of the student,
- subsequent papers that show how this understanding may be applied in various branches of the discipline,
- some examples of student work exploring how the Consciousness-Based approach enhances learning in the discipline, and
- an appendix describing Maharishi Vedic Science and Technologies of Consciousness in detail.
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We welcome inquiries and further contributions to this series.

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Mathematics begins with wonder. Throughout the ages, mankind has innocently asked, and attempted to answer, endless questions about numbers, lines, geometric figures, and other kinds of mathematical objects that seem somehow to be “given” to us for exploration with our mind’s eye. The heart of mathematics is the process of asking such questions, discovering answers, representing both questions and answers in the rigorous language of mathematics, and demonstrating the correctness of these answers through deductive reasoning based on a fundamental set of axioms. One of the remarkable consequences of this play of consciousness is that the patterns of orderliness discovered in this purely abstract way can be used to model the physical world in great detail, even to the extent of allowing scientists to predict nature’s behavior and to develop extraordinary technologies that transform the way life on earth is lived.

The activity of mathematics is by nature a self-referral activity: the mathematician calls upon faculties of his own consciousness to discover answers to questions that arise about mathematical forms and relationships, which are nothing other than forms and relationships of that very same consciousness. Therefore, it is perhaps no surprise that the structures and relationships that have emerged within mathematics, as well as the mathematical method itself, reflect the principles and dynamics of the structure of consciousness, as described by Maharishi Vedic Science.

Discovery, elaboration, and application of these connections, on the basis of the direct experience of pure consciousness through regular practice of the Transcendental Meditation and TM-Sidhi programs, is the heart of Consciousness-Based Education in mathematics. Seeing how mathematical structures and relationships reflect the nature and characteristics of one’s own consciousness, both student and researcher of mathematics find the technical topics of mathematics to be easier to grasp and fundamentally more relevant. Then, the heightened clarity of awareness that develops from practice of the Transcendental Medita-
Consiousness-based education and TM-Sidhi programs permits the student to enjoy an enhanced creativity with concepts that tend to be abstract and complex at times. The goal of this Consciousness-Based approach is that the very process of learning and doing mathematics stirs and awakens the “Cosmic Mathematician” deep within each student, resulting not only in great mathematical accomplishment, but also in the ability to engage the orderly mathematical intelligence of nature itself to create a fulfilling and mistake-free life.

Some examples of mathematical notions for which deep parallels to principles of Maharishi Vedic Science have been discovered—parallels that will be elaborated in this volume—include the following:

- the *structure of the real line*, in which each point is related to the line as a whole through the principle of nested intervals, an analogue to the *Aksbha* principle of Maharishi Vedic Science by which points arise in the collapse of the unbounded continuum, yet retain the memory ($Smriti$) of their source;
- mathematical *quantification of change* in analysis, whereby one mathematically computes the unmanifest impulse of change at a point of a function;
- the concept of *symmetry*, by which one locates nonchange in the midst of change; this fundamental and pervasive idea rests at the core of geometry and was a key factor in the development of the modern-day theory of groups;
- central to the mathematical method itself, the *process of abstraction*, whereby deeper levels of mathematical knowledge are seen to unify diversity; this move toward abstraction from concrete particulars appears throughout mathematics, from applied areas of mathematics, in which, for example, a single differential equation can capture, in a single symbolic expression, the entire range of behavior of a law of nature, to the full abstraction of all mathematical theories, culminating in category theory, which provides a unification of all of mathematics in terms of fundamental *universal constructions*.

One area of mathematics that exhibits Vedic principles especially clearly is the area of mathematical foundations—particularly set theory,
which is the standard foundational theory for all of modern mathematics. Set theory consists of a small collection of axioms formulated in the formal language of sets, from which all the theorems about sets, and, consequently, all of mathematics, may be derived. The universe of sets that is built up sequentially through the use of these axioms contains in principle all mathematical objects, represented as sets. In this way, set theory naturally plays the role of the unified field of mathematics, providing a unified basis for the diverse expressions that collectively constitute the field of mathematics.

A testament to the depth of the knowledge provided by mathematical foundations is that it is able to precisely describe its own boundaries, its own limitations. An important example is found in a famous theorem due to Kurt Gödel, the *Incompleteness Theorem*, which states that no foundational theory, including set theory, can establish by way of logical deduction from its axioms every statement that is *true* in the mathematical universe. His result points to a fundamental limitation in the very method of mathematics, which relies on logical derivations from axioms to discover mathematical truths. Another example is concerned with the mathematical formulation of the concept of the Infinite. While it is known that there is an endless variety of different sizes of infinity in mathematics, a phenomenon that modern set theory has been unable to account for is the appearance within mathematics of notions of infinity so large that (by another theorem of Gödel) they cannot be proven to exist; these infinities are known as *large cardinals*. A desirable solution to this well-known problem would be to expand the axioms of set theory in some natural way in order to account for these enormous infinities, but mathematical intuition so far has not been clear enough to see how to do this.

Maharishi Vedic Science, especially Maharishi Vedic Mathematics—the mathematics of self-referral consciousness (Maharishi Mahesh Yogi, 1994:339)—offers an approach to knowledge that sheds light on these and other limitative results in foundations. For example, the mismatch between truth and provability, demonstrated by the Incompleteness Theorem, points to a fundamental principle of Maharishi Vedic Mathematics: complete knowledge is not available at the level of the intellect alone. For example, the Bhagavad-Gita, a portion of Vedic literature, declares (Maharishi Mahesh Yogi, 1969:3.42)
referring to the nature of the Self, the field of pure consciousness. (This unbounded value of the Self is written with an uppercase “S” to distinguish it from the ordinary, localized self we typically experience.) By contrast, according to Maharishi Vedic Mathematics, for a fully awake consciousness, all knowledge is available instantly, in a single stroke. Comparing modern mathematics to his Vedic Mathematics, Maharishi explains, “Modern Mathematics is the field of steps, whereas Vedic Mathematics is the field of pure intelligence that gets what it wants instantly without steps” (Maharishi Mahesh Yogi, 1994:389). The logical, step-based approach to arrive at mathematical truths in modern mathematics is therefore enhanced by the no-step approach to gaining knowledge through Maharishi Vedic Mathematics.

Likewise, by offering knowledge of the nature of wholeness—Brahman—Maharishi Vedic Mathematics offers a deep understanding of the true nature of the Infinite, an intuition that has already proven useful in moving toward a resolution of the problem of large cardinals. Maharishi Vedic Mathematics reveals the ultimate ground of all numbers, finite or infinite, to be what Maharishi has termed the Absolute Number; indeed, he explains that this Absolute Number is itself the self-referral dynamics of wholeness, the basis for unfoldment of all numbers and mathematical structures in such a way that each individual expression remains connected to its source in wholeness (Maharishi Mahesh Yogi, 1994:380-381, 615-621). He further explains: “This [the fact that everything in the objective world is the expression of wholeness] presents to us the need for an Absolute Number in the field of Mathematics, a number that can help us to account for the infinite number of wholenesses within the universe—a number that will help us to account for the theme of creation and evolution in terms of wholeness” (Maharishi Mahesh Yogi, 1994:611). Two of the articles in this volume explore a possible solution to the problem of large cardinals that makes use of the principles and dynamics of Maharishi’s Absolute Number to supply the heretofore missing intuition.

In these and other ways, Maharishi Vedic Mathematics offers a key to bringing to fulfillment the structuring impulses underlying modern mathematics. By overcoming the inherent limitation of knowledge on the basis of deductive logic alone, Maharishi Vedic Mathematics,
through its Consciousness-Based approach to knowledge, opens the door to complete knowledge, and not just on the level of concepts, but as a living reality, leading to a life and a society that embody the full benefit of nature’s mathematics: a mistake-free life lived in waves of success and fulfillment.

This two-part set on mathematics, as part of the Consciousness-Based education series of volumes, organizes in a single publication some of the most significant articles written by the Maharishi University of Management Department of Mathematics over the past 30 years. These articles probe into some of the deepest areas of modern mathematics to show how the basis of mathematics is itself structured in the qualities, principles, and dynamics of pure intelligence. They also explore Maharishi Vedic Mathematics, showing how this aspect of Maharishi Vedic Science brings fulfillment to the trends and aspirations of modern mathematics. Other articles bring to light Consciousness-Based methods of teaching mathematics, to enliven the Cosmic Mathematician deep within every student. And still another section of articles makes use of Maharishi Vedic Science as a set of principles that are used to refine and bring to fulfillment in a technical way the field of mathematics itself.

**Part 1: Pure Mathematics in the Light of Maharishi Vedic Science and Maharishi Vedic Mathematics** locates principles of Maharishi Vedic Science in the various fields of pure mathematics, including their foundation in set theory, and elaborates on the role of Maharishi Vedic Mathematics in bringing fulfillment to the field of mathematics.

**Part 2: Applications of Maharishi Vedic Science to Mathematics Education and Mathematics Research** includes articles that discuss applications of Maharishi Consciousness-Based approach to the field of mathematics education, and other articles in which Maharishi Vedic Science is used as a tool for deeper research into mathematics itself.

**Subject Headings for Part 1**

Section I: *Maharishi Vedic Science and Foundations of Mathematics* begins with Michael Weinless’s article “The Samhita of Sets,” which explores the deep connections between modern mathematics, especially modern set theory, and Maharishi Vedic Science. The author discusses the sense in
which the “absolute infinite” notions of the universe $V$ of sets and the class $\Omega$ of all ordinals are analogous to the transcendental wholeness of Maharishi Vedic Science, and ways in which the three-in-one structure of pure consciousness finds expression in these expansive mathematical models. Venturing into a study of alternative foundations, the article finds that \textit{topos theory}, playing the role of \textit{Vedanta}, successfully integrates intuitionistic mathematics into the classical set-theoretic foundation. The article also explains how recent results on the Axiom of Determinacy illuminate the connection between Vedanta and Jyotish.

Weinless’s article is followed by the “Unified Field Chart for Mathematics,” developed by the Maharishi University of Management Department of Mathematics. This chart locates the source of mathematics within the unified field of natural law, which is seen to be at once the fundamental field of existence, as described by modern quantum field theory, and the source of thought, experienced directly during the practice of the Transcendental Meditation program. The emergence of mathematical knowledge from this field starts from the abstract level of foundations and unfolds through pure and applied fields, into mathematical applications for the development of technology and for the well-being of society. The chart provides a snapshot view of the whole field of mathematical knowledge from its source to its expressions in its most applied values.

As a natural commentary on the Unified Field Chart, the next article in the volume is an excerpt from the \textit{Heaven on Earth} book, originally published in 1989. Highlighting themes of unfoldment from Maharishi Vedic Science, the article surveys the entire range of mathematics from its source in pure intelligence, through foundational theories and pure mathematics, to all the applied areas of mathematics.

The “Richo Akshare Chart for Mathematics,” which follows next, elaborates, from the perspective of eight different mathematical disciplines, the role of the \textit{Richo Akshare} verse of Rk Veda (1.164.39) in the fabric of mathematics. Using the lines of the verse as a template for discussion, each discipline locates the significance of the collapse of infinity to a point, and the emergence of the fundamental impulses of intelligence from this collapse. In addition, each discipline locates within its own structure a deep and vital aspect of mathematical intelligence, without which a proper understanding of the discipline would
be impossible, but knowing which, the depths of the discipline become easily accessible.

Further elaborating this chart with respect to set theory, Michael Weinless’s article “Mathematical Foundations in the Light of the Richo Akshare Verse,” revisits some of the themes presented in the first article of the volume, “The Samhita of Sets,” now appreciated from the point of view of this pivotal verse from Rk Veda.

The final article in Part I, Paul Corazza’s “The Wholeness Axiom,” takes up the challenge raised in Weinless’s “Samhita of Sets,” to use Maharishi Vedic Science as a means to provide the necessary intuition to solve the “problem of large cardinals.” To that end, the article introduces and motivates a new axiom, to be added to the standard set theory axioms, which asserts, in the language of set theory, that “wholeness moves within itself, knows itself, remains unchanged by its own transformation, and remains ever present at each point in creation.” From this axiom, virtually all large cardinals are shown to be derivable.

Section II:
Modern Mathematics in the Light of Maharishi Vedic Mathematics
addresses the vision of possibilities in raising modern mathematics to its supreme level through the knowledge and technologies of Maharishi Vedic Mathematics. John Price’s article, “Maharishi’s Absolute Number: The Mathematical Theory and Technology of Everything,” demonstrates how, by providing direct access to the dynamics of pure intelligence, beyond the senses and even beyond the intellect, Maharishi Vedic Mathematics and Maharishi’s Absolute Number provide a natural completion of the knowledge of modern mathematics and go on to offer profound solutions to problems ordinarily considered to be outside the scope of mathematics, such as providing invincible defense for any nation. Catherine Gorini’s article, “Maharishi Vedic Mathematics: The Fulfillment of Modern Mathematics,” takes up a similar theme, arguing that modern mathematics’ innate drive toward unification through abstraction and toward a complete understanding of natural law through techniques of mathematical modeling find fulfillment in Maharishi Vedic Mathematics, where nature’s intelligence and the intelligence of the mathematician come together in the dynamic source of all knowledge, the
unified field of natural law, that field of intelligence that transcends the intellect and governs the entire manifest universe.

Section III:
Self-Referral Dynamics and Mathematics is concerned with those mathematical theories and structures that reflect the self-referral dynamics of pure intelligence, as described by Maharishi Vedic Science. Michael Weinless’s article, “Self Referral in the Foundations of Mathematics,” surveys self-referral expressions of knowledge in the foundations of mathematics by examining themes from the theory of non-wellfounded sets, topos theory, impredicative definitions in analysis, the \( \lambda \)-calculus, and denotational semantics.

Section IV:
Geometry, Symmetry, and Consciousness brings to light the natural way in which fundamental notions in geometry can be seen to display the characteristics of pure consciousness. In Catherine Gorini’s article, “Consciousness: The Last Frontier of Geometry,” she argues that the deeper themes explored in the field of geometry, such as continuity, higher dimensions, infinity, symmetry, and the homogeneity of space, reflect qualities of the very creator of those concepts, the intelligence of the geometer himself. She concludes that the “last frontier” of the field of geometry can be nothing other than the field of consciousness. In “Symmetry: A Link Between Mathematics and Life,” the last article of Part I, Gorini shows how the important geometric notion of symmetry can be effectively communicated to students of mathematics by referring to parallel themes found in artistic and cultural contexts, and in particular in the context of the Vedic literature.

Subject Headings for Part 2

Section V:
Consciousness-Based Mathematics Education begins the second volume with articles that discuss the educational value of the Consciousness-Based approach to teaching mathematics. Articles address not only teaching techniques and styles, but also in some cases present insights that have proven effective for instructional purposes in the classroom. One article is Catherine Gorini’s “How Maharishi Vedic
Science Answers the Questions of the Unreasonable Effectiveness of Mathematics in the Sciences,” which addresses the long-held question in the scientific community: Why is mathematics, a subjective creation of the human intellect, so effective in the sciences, which study the objective, physical world? The answer she proposes is based on the observation by Maharishi Vedic Science that the subjective and objective fields of knowledge have their common basis in consciousness.

Anne Dow, “A Unified Approach to Developing Intuition in Mathematics,” raises the issue, well-known to mathematics educators, that the crucial step in understanding and doing mathematics, involving a mixture of intuition and analysis, is actually the aspect of mathematics that is hardest to impart to students. Using Maharishi’s model of the thinking process, she proposes that the experience of transcending thought, through the Transcendental Meditation program, directly addresses this need.

“Preparing the Student to Succeed at Calculus,” also by Anne Dow, points out that modern approaches in teaching calculus have successfully addressed only two of the three fundamental components of gaining knowledge: the content of the subject matter (the known) and the process of knowing. These approaches have failed to address the need to expand the container of knowledge, the knower—the third component of gaining knowledge. This becomes necessary, she argues, when the student is asked to grasp deeper ideas in the subject, such as the concept of a “limit” in calculus. She proposes that regular practice of the Maharishi Transcendental Meditation technique addresses this need, and cites her own teaching experience at Maharishi University of Management (formerly Maharishi International University), together with published scientific research, to support this claim.

Catherine Gorini’s article “Using the Study of Consciousness to Teach Calculus” also points to the importance of addressing the need to expand the consciousness of the knower, the student of calculus, through the practice of the Transcendental Meditation program, in order to optimize his or her educational experience. She also demonstrates the use of principles from Maharishi Vedic Science to illuminate challenging concepts in the subject (the continuum, the limit, the derivative, the integral); these principles, she argues, provide much easier and deeper access to these mathematical notions because of the
student’s growing experience of transcending and of deeper layers of consciousness.

**Section VI: Maharishi Vedic Science as a Research Tool for Modern Mathematics** contains articles in which Maharishi Vedic Science is used to advance or more deeply elucidate the foundations of modern mathematics. Paul Corazza’s, “Vedic Wholeness and the Mathematical Universe: Maharishi Vedic Science as a Tool for Research in the Foundations of Mathematics” offers a first look at the Wholeness Axiom (a subject that is developed further and from a different perspective in the author’s other article on this topic in Part I), using as a starting point the following question: Which qualities and dynamical principles of wholeness, described by Maharishi Vedic Science, are clearly present and lively in the universe of sets, as described by modern set theory, and which ones seem to be absent? The article proposes that a proper answer to this question leads to the introduction of new dynamics in the mathematical universe, expressible in the form of a new set-theoretic axiom: the Wholeness Axiom. The article shows that the axiom succeeds not only to “awaken” in the set-theoretic universe qualities and dynamics of wholeness that had apparently been absent before, but also results in an elegant solution to the problem of large cardinals.

Exploring an alternative foundation based on arrows instead of sets, Michael Weinless’s final article in Part 2, “Categories and Toposes: Dynamism at the Foundation of Mathematics,” gives a short course on a highly successful foundation based on category theory, called topos theory. A topos is a category exhibiting such strong closure properties that it is almost a model of set theory, a universe of sets, but not quite. Because its closure properties have a geometric feel, toposes provide natural models for deep results in areas of geometry such as sheaf theory and algebraic geometry. The generality of the concept of a topos makes it possible to define toposes that are genuinely models of set theory, modeling various set theoretic axioms (for example, topos theory provides an alternative view of set-theoretic forcing for producing models). At the same time, because the internal logic of a topos is intuitionistic, topos theory provides the richest known source of intuitionistic theories. The versatility of topos theory leads the author to suggest that toposes play the role of Vedanta in mathematical foundations, synthesizing
Diverse, even incompatible, foundational theories. The article gives a systematic development of this emerging field and uses Maharishi Vedic Science to shed light on its underlying principles.

Section VII: The Appendices include Dr. Kenneth Chandler’s “Modern Science and Vedic Science: An Introduction,” which served as the introduction to the inaugural issue of the journal Modern Science and Vedic Science and which presents an overview of Maharishi Vedic Science and the new technology of consciousness developed by Maharishi Mahesh Yogi. The second appendix in this section provides a list of relevant links and resources for this volume.

References
Section V

Consciousness-Based Mathematics Education
How *Maharishi Vedic Science* Answers the Question
of the Unreasonable Effectiveness
of Mathematics in the Sciences

Catherine A. Gorini, Ph.D.
ABOUT THE AUTHOR

Catherine A. Gorini received her A.B. in mathematics from Cornell University, M.S. and Ph.D. in mathematics from the University of Virginia, and D.W.P. from Maharishi European University. She is Dean of Faculty and Professor of Mathematics at Maharishi University of Management. She is the editor of *Geometry at Work* and author of *Facts On File Geometry Handbook*. Her numerous awards for teaching include the Award for Distinguished College or University Teaching of Mathematics from the Mathematical Association of America.
Mathematicians and scientists have for a long time tried to understand why mathematics, a subjective creation of the human intellect, is so effective in the sciences, which study the objective, physical world. Satisfactory reasons have not been found because there has not been a comprehensive understanding of the relationship between the subjective and the objective aspects of life. In this paper we will see that Maharishi Vedic Science, by explaining the link between the subjective realm where mathematics is located and the objective world that science examines, can resolve this problem in a natural way.

Introduction

Mathematics is fundamental to all areas of science and technology. The language of mathematics has been used since antiquity to express our knowledge of the physical world, to derive new knowledge from old, and to predict the behavior of physical systems. For example, Newton’s Second Law of Motion says that a force exerted on an object is the product of its mass times the resulting acceleration, $F = ma$. Newton used this law, together with his newly developed calculus and the law of gravitation, to derive the elliptical shape of the planetary orbits discovered by Kepler. Today, mathematical analysis similar to Newton’s has placed a man on the moon.

With such dramatic successes, it is not surprising that many people, particularly those who have been at the forefront of developing new applications of mathematics, have wondered why mathematics has proven to be so practical and why the laws of nature are so effectively expressed by mathematical formulas. Mathematics is theoretical and completely abstract, created in moments of inspiration and afterward verified by the intellect. Science, on the other hand, seeks to accurately and objectively describe and predict how the physical world around us behaves.

Nevertheless, these two approaches to knowledge have been intimately linked since we began observing and thinking about the world. The basis for understanding the role of mathematics in science must depend on an understanding of how the subjective world of the mind and intellect, the source of mathematics, is connected to the world of matter, forces, and energy studied by science. This connection can be understood through Maharishi Vedic Science, which gives a compre-
hensive explanation of the nature of consciousness and its manifestations in the physical world and how the subjective world of consciousness and the mind is connected to the objective physical world around us. Because Maharishi Vedic Science is so comprehensive, an analysis of the nature of mathematics according to its principles can provide the link between the subjective and the objective aspects of knowledge necessary to properly explain the role mathematics plays in the sciences.

According to Maharishi Vedic Science, the mind and the physical world are not two separate entities, but two different aspects of one reality. The mind is subtler, more abstract, and more intimate to us than the physical world, but both exist simultaneously and inseparably. As we will discuss in later sections, Maharishi sees both the mind and the physical world as having their source in the self-interacting dynamics of pure consciousness, which he identifies as the total potential of natural law (Maharishi Mahesh Yogi, 1986). Both mathematics and science are studying those aspects of natural law which are quantifiable and exact, although using different methodologies. Thus, the effectiveness of mathematics in the sciences should be no surprise but is, in fact, natural and expected.

Moreover, this understanding of Maharishi Vedic Science shows us that to make mathematics even more powerful, effective, and complete, mathematicians must go even deeper into their own subjective nature and connect themselves to their source in consciousness. The same can be said for scientists, who can make science more productive by linking the objective, natural world that they study to the same source in consciousness.

In this paper, we will first look at mathematics and the question of its effectiveness in the sciences as it has been posed by the physicist Eugene Wigner and the mathematician Richard Hamming. This will be followed by a discussion of points from Maharishi Vedic Science relevant to this question, a resolution of the question based on these ideas, and a look at the implications of this resolution.

**The Question: The Role of Mathematics in the Sciences**

Throughout time, mathematics has been always been associated with its applications and from these applications, mathematicians have derived new impetus and new directions. For example, the Sulba Sutras, one of
the earliest records of mathematics from the Vedic civilization, includes geometric constructions that were used to describe the procedure for the construction of ceremonial platforms (see Henderson, 2000, and Price, 2000). The Rhind Papyrus of the Egyptians gives computational techniques alongside sample problems for applying the techniques to everyday situations such as computing the size of a barn used to store grain. Babylonian clay tablets give mathematical tables for astronomical predictions as well as for business transactions (van der Waerden, 1971).

With the Greeks, however, the discipline of pure mathematics was separated from its applications. As seen in Euclid’s *Elements*, mathematicians had become concerned not with applied problems, but rather with the logical foundations (or postulates) of geometry and the rigorous, systematic derivation of new results from the postulates and previously established results. Mathematical proof became the central feature of the research, communication, and exposition of mathematics.

As mathematics progressed from the classical study of geometry and calculus to the more abstract areas of group theory, non-Euclidean geometry, and topology, its ancient connection to applications weakened still further. In the nineteenth and twentieth centuries, mathematics became replete with concepts that, on the surface, appear to be unrelated to science and the physical world. For example, in certain abstract algebraic systems, the equation $2 + 4 = 1$ can be correct. In hyperbolic geometry, one can draw many different lines through a point parallel to another line, something strictly forbidden in Euclidean geometry.

Topologists and analysts regularly study infinite-dimensional spaces, even though the space around us is only three-dimensional. As mathematicians pursued these and other more abstract ideas for their own intrinsic interest and without regard for possible applications, a large body of seemingly “useless” mathematics was developed. This mathematics nevertheless proved to be beautiful and profound. It provided new insights into applied mathematics and became the core of mathematical research. Some purely theoretical mathematicians, notably G. H. Hardy, even expressed disdain for concerns with applications and pride that their work could have no applications. For Hardy (1940), the value of mathematics is purely subjective, purely in the realm of ideas:
A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. . . . A mathematician . . . has no material to work with but ideas, and so his patterns are likely to last longer, since ideas wear less with time than words. . . . The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. (pp. 84–85)

Hardy sees mathematics as essentially disconnected from the world of applications. In discussing the mathematical significance of the proofs of the infinitude of the number of primes and the irrationality of $\sqrt{2}$, he says,

There is no doubt at all, then, of the ‘seriousness’ of either theorem. It is therefore the better worth remarking that neither theorem has the slightest ‘practical’ importance. In practical applications we are concerned only with comparatively small numbers; only stellar astronomy and atomic physics deal with ‘large’ numbers, and they have very little more practical importance, as yet, than the most abstract pure mathematics. I do not know what is the highest degree of accuracy which is ever useful to an engineer—we shall be very generous if we say ten significant figures. Then

$$3.14159265$$

(the value of $\pi$ to eight places of decimals) is the ratio

$$\frac{314159265}{100000000}$$

of two numbers of ten digits. The number of primes less than $1,000,000,000$ is $50,847,478$: that is enough for an engineer, and he can be perfectly happy without the rest. (pp. 101–102)

He goes on to claim that what he considers “real mathematics,” the purest, most abstract mathematics, is without applications:

There is one comforting conclusion which is easy for a real mathematician. Real mathematics has no effects on war. No one has yet discovered
any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years. (p. 140) . . . I have never done anything ‘useful’. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. (p. 150)

These deep-seated ideas notwithstanding, history had a surprising twist in store for mathematicians. At the beginning of the twentieth century, developments in quantum physics and relativity theory required the most abstract theories of algebra, analysis, and geometry. Furthermore, computer technology has required precisely the mathematics that Hardy felt to be impractical. In fact, one multi-million dollar company, RSA Cryptosystems, specializes in finding for its customers prime numbers 100 to 200 digits long, primes which far exceed the numbers considered by Hardy to be “enough.” This mathematics has even proven to be crucial to the military; for instance, extremely large prime numbers are used daily in securing military communications.

As the abstract mathematics that had seemed so irrelevant to the pragmatic world began to have exciting and unexpected applications, it was inevitable that scientists would search for an explanation. One such individual was Eugene Wigner. Noted for his deep insights into mathematical physics, he gave fresh insight into the usefulness of mathematics in his now classic paper, “The Unreasonable Effectiveness of Mathematics in the Natural Sciences,” first published in 1960 (Wigner, 1967).

Wigner begins his paper with the belief, common to all those familiar with mathematics, that mathematical concepts have applicability far beyond the context in which they were originally developed. Based on his experience, he says “it is important to point out that the mathematical formulation of the physicist’s often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena” (p. 230). He uses the law of gravitation, originally used to model freely falling bodies on the surface of the earth, as an example. This fundamental law was extended on the basis of what Wigner terms “very scanty observations” (p. 231) to describe the motion of the planets and “has proved accurate beyond all reasonable expectations.” Another oft-cited example is Maxwell’s equations, derived to
model familiar electrical phenomena; additional roots of the equations describe radio waves, which were later found to exist. Wigner sums up his argument by saying that “the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it” (p. 233). He concludes his paper with the same question he began with:

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning. (p. 237)

Wigner has drawn many others into this discussion on the applicability of mathematics. R. W. Hamming (1980) repeats Wigner’s observation about its usefulness: “constantly what we predict from the manipulation of mathematical symbols is realized in the real world. . . . The enormous usefulness of the same pieces of mathematics in widely different situations has no rational explanation (as yet)” (p.82). Hamming carefully examines his own experiences of using mathematics, his understanding of the origins and history of mathematics, the nature of mathematics, mathematical discovery and proof, the foundational crisis of mathematics, and the nature of science and scientific laws, and then finally proposes some explanations. Nevertheless, he is unsatisfied with his reasoning and must, like Wigner, leave the question of the role of mathematics unanswered:

From all this I am forced to conclude both that mathematics is unreasonably effective and that all of the explanations I have given when added together simply are not enough to explain what I set out to account for. I think that we—meaning you, mainly—must continue to try to explain why the logical side of science—meaning mathematics, mainly—is the proper tool for exploring the universe as we perceive it at present. I suspect that my explanations are hardly as good as those of the early Greeks, who said for the material side of the question that the nature of the universe is earth, fire, water, and air. The logical side of the nature of the universe requires further exploration. (p. 90)

Thus, we are left with the question of why mathematics, which is developed and verified by mathematicians according to human logic
and reasoning, is so perfect a tool for investigating the physical world around us.

*Maharishi Vedic Science*

This question of the effectiveness of mathematics can be answered by considering the Vedic knowledge brought to light by Maharishi Mahesh Yogi in his Vedic Science. Everywhere we look in nature, whether as a scientist or not, we see orderliness and growth. Natural laws, still not yet understood by scientists, govern the universe of billions and billions of stars moving throughout space in perfect harmony. The delicate balance of the environment on earth is the result of thousands of species living together in an intricately organized way. Maharishi (1994) points out that observations such as these lead us to recognize that intelligence is inseparable from life.

We see things around us exist. We also see that things around us change and evolve. We also see that there is order in evolution—an apple seed will only grow into an apple tree, etc. Thus it is obvious that existence is endowed with the quality of intelligence—existence breathes life by virtue of intelligence. (pp. 57–58)

Maharishi goes on to locate consciousness at the basis of life, as fundamental as existence and intelligence, “Consciousness is the existence of everything, and consciousness is the intelligence of everything” (p. 58). Science and mathematics are intimately linked to questions of existence and intelligence, so knowledge of the field of consciousness is important for the question of the role of mathematics in science. To give experiential knowledge of the total range of consciousness, Maharishi has made available the Transcendental Meditation technique, a simple, natural, effortless technique:

During this technique, the individual’s awareness settles down and experiences a unique state of restful alertness: as the body becomes deeply relaxed, the mind transcends all mental activity to experience the simplest form of human awareness—Transcendental Consciousness—where consciousness is open to itself. This is the self-referral state of consciousness. (p. 260)
In the pure self-referral state of transcendental consciousness, consciousness is conscious of itself, and the subject of knowledge is the same as the object of knowledge. Since consciousness is the link between itself as subject and as object, it is also the process of knowing. Maharishi (1986) describes the importance of this fundamental relationship, “This state of pure knowledge, where knower, known, and knowledge are in the self-referral state, is that all powerful, immortal, infinite dynamism at the unmanifest basis of creation” (p. 27). In particular, this dynamism of consciousness is the source of subjective experience: “When consciousness is flowing out into the field of thoughts and activity, it identifies itself with many things, and this is how experience takes place” (p. 25). Furthermore, since knowledge has organizing power, Maharishi (1980) concludes that the field of pure consciousness is also a field of absolute organizing power and from there the laws of nature emerge:

Knowledge has organizing power and therefore in the absolute structure of knowledge, in the state of the absolute observer-observed relationship, we have absolute organizing power. Once we have the field of absolute organizing power in this state of pure transcendental awareness, the seat of absolute knowledge, we have the source of all the streams of organizing power in nature. All the laws governing different fields of excitation in nature, all the innumerable laws known to all the sciences have their common source in this field of absolute organizing power. (pp. 74–75)

In this way, we see that the self-interacting dynamics of pure consciousness is at once the source of subjective experience and of the laws of nature governing all aspects of the world around us. The principles of intelligence and orderliness inherent within consciousness therefore govern all the expressions of consciousness—and, as Maharishi explains 1994, that is all that there is.

All speech, action, and behaviour are fluctuations of consciousness. All life emerges from and is sustained in consciousness. The whole universe is the expression of consciousness. The reality of the universe is one unbounded ocean of consciousness in motion. (pp. 67–68)

Since all the fundamental frequencies of creation are lively in the Veda, Maharishi refers to the Veda as the Constitution of the Uni-
verse, “The structure of this level of self-referral pure intelligence is the structure of Veda, which is the very well structured Constitution of the Universe” (pp. 208–209). Thus, the laws that govern all manifest and unmanifest aspects of creation are structured within the consciousness of each individual.

With this explanation of the fundamental role of consciousness and the intimate connection of consciousness and the physical universe, we are ready to answer the question about the connection of mathematics, a subjective creation of the human mind, with the structure of the objective physical universe around us.

**Resolution of the Question**

We now consider how the description of consciousness as the source of life in Maharishi Vedic Science resolves the question that Wigner and Hamming have set before us. The question is, simply put, why is mathematics, which is developed as a subjective discipline, so effective in its applied forms in the natural sciences, which describe nature in a purely objective manner. First, we clarify what is meant by mathematics so that we can more easily put it into the framework of Maharishi Vedic Science.

Mathematics is the search for and study of abstract and precise patterns of orderliness in number, shape, and form. The objects studied by mathematics—numbers, shapes, sets, patterns, relationships, and so on—do not have any real physical existence. Rather, as pointed out by Hardy (1940), they exist as ideas in the awareness of the mathematician, and they are, therefore, part of the subjective realm of life. Accordingly, new mathematical ideas are discovered on the subjective level by intuition, insight, and creativity, and mathematics is considered to be an art by those who practice it. The results of mathematics are expressed in very precise language as formulas and theorems and are verified and proved according to strict standards of logic, so mathematics has the reliability and objectivity associated with science, but it is nevertheless a subjective study.

Mathematics investigates the structure of the laws governing the subjective values and functioning of intelligence and consciousness; it quantifies subjective and abstract patterns in a precise way; and it offers an exact and systematic description of purely subjective phenomena.
Science, on the other hand, investigates the underlying structure of objective phenomena. Wigner and Hamming made the seemingly obvious assumption that mathematics and science were therefore studying two completely separate worlds. However, in Maharishi Vedic Science, we understand that these two worlds are both the expressions of the same underlying field of consciousness and are both governed by the same natural laws.

Thus, mathematicians and scientists are both studying the same laws of nature. Furthermore, they are both looking for those properties of natural law that are general enough to capture the underlying structure of many different situations, as for example in the way the law of gravity applies to objects on earth, planets orbiting the sun, and galaxies in the heavens, or in the way the quadratic formula can solve all possible quadratic equations. Mathematicians and scientists are both looking for exact, concise, and systematic representations of their discoveries. Both demand that knowledge be non-variable and verifiable.

There are differences between mathematics and science, however, and these differences have given rise to the question of the effectiveness of mathematics in science. Mathematicians, by going deep into the structure of their own intellect, are studying how the laws of nature govern subjective aspects of creation, and they verify their discoveries by the intellect. Scientists, by looking out at the world around them, are studying how the laws of nature govern objective aspects of creation and they verify their discoveries by experimentation. The understanding given by Maharishi Vedic Science allows us to reconcile these differences. Although from two different vantage points, mathematicians and scientists are both looking at the same phenomena, the same “unbounded ocean of consciousness in motion,” so the patterns and structures which the mathematician sees on an abstract level are exactly those that the scientist studies on the physical level. There must be not only parallels in what they find, there must be perfect coincidence—and this is exactly what so puzzled Wigner and Hamming. Maharishi (1996) explains this as follows:

This universality of applications can be traced back to the fact that all aspects of Nature and areas of life are governed by the same principles of order and intelligence that have been discovered subjectively by mathematicians by referring back to the principles of intelligence in their
own consciousness. Great scientists like Einstein have marveled in the past about this profound relation between the subjective and objective aspects in creation, a relation which has its foundation in the identity of the Unified Field of Natural Law and the field of pure self-referral consciousness displaying the universal principles of intelligence and order. (pp. 304–305)

Working on the level of the intellect where understanding about natural law can be expressed in concise and exact mathematical formulations, the mathematician is able to provide powerful and comprehensive tools for the scientist. Abstract mathematical formulations are able to capture in a simple way the understanding of the scientist, and scientific laws are generally expressed as mathematical equations. Since the principles of order and intelligence expressed in the mathematical model of a physical system are the same as the principles governing the behavior of the system, we see that the computational consequences of a mathematical model of a physical system can exactly describe or predict the evolving conditions of that system. The great speed and efficiency with which the mind can derive predictions from a mathematical model give science great power. For example, in a few minutes, one can set up and solve the equations describing a trajectory that can take a comet months or years to traverse.

Finally, then, in Maharishi Vedic Science, we are able to find a resolution to the question of the role of mathematics in the sciences. The same laws of nature, with their source in consciousness, are responsible for both the subjective and objective aspects of creation. The mathematician intellectually studies the subjective side of creation; the intimacy of the intellect with the subjective side of creation gives mathematics its profundity, elegance, and naturalness. The scientist intellectually studies the objective side of creation. The subjective language and tools of the mathematician provide the precise and appropriate intellectual structures for the scientist to comprehend the physical world.

**Conclusion**

This explanation of the role of mathematics based on the principles of Maharishi Vedic Science allows us to come to a number of conclusions and to suggest some new directions. Firstly, because mathematicians are studying the same principles of order and intelligence that are
studied by science, but in a subjective and abstract way, mathematics is the natural language for scientists to record their understanding of the physical world and the methodology of mathematics provides the natural means for predicting the behavior of the physical world. On the other hand, new discoveries and problems arising in the sciences are naturally a resource for the mathematician looking for new ideas and directions.

Next, we see the value for mathematicians to pursue pure mathematics without consideration of its applications. There has been concern in the discipline that by following their individual aesthetics and judgments, mathematicians might go off in directions that are unproductive. But we see here that it is precisely by following their own tastes and preferences that mathematicians are able to uncover deeper and deeper principles governing the structure of subtler and subtler values of natural law. According to Maharishi (1996), “These principles describe the dynamics of Cosmic Intelligence—the Unified Field of Natural Law—as it functions within itself, and are directly cognized on the level of the consciousness of the mathematician “ (p. 302). Since these principles are also responsible for the physical world, they must have some reflection in the physical world, and whether they have been located now or not, eventually they will be. As Lobachevsky (see 1984), a founder of non-Euclidean geometry, said, “There is no branch of mathematics, however abstract, that will not eventually be applied to the phenomena of the real world.”

Finally, this understanding of the role mathematics plays in the sciences shows us that in order to have a complete science, we must have complete mathematical knowledge, and in order to have complete mathematical knowledge, we must have complete knowledge of all levels of life. This means that mathematicians must have complete knowledge of the structure of pure knowledge and complete knowledge of the structure and functioning of consciousness. To be a good mathematician, one must develop one’s consciousness fully—from the finest level to the grossest level. Maharishi Mahesh Yogi has provided theoretical knowledge and practical techniques, including the Transcendental Meditation and TM-Sidhi programs, for this purpose. In his Vedic Mathematics, Maharishi has gone on to show how this knowledge of consciousness can be applied to fulfill the goals of modern mathemat-
ics. Maharishi Vedic Mathematics is the mathematics of consciousness itself.

Vedic Mathematics is the mathematics of the absolute, eternal, unbounded, which deals with the absolute reality, self-referral singularity—the total potential of infinite diversity at the unmanifest basis of creation, the transcendental level of consciousness. (Maharishi Mahesh Yogi, 1996, pp. 366–367)

With the comprehensive knowledge of Maharishi Vedic Mathematics (see also Price, 1997), mathematics will be able to rise to its full potential and guide life in a more holistic, mistake-free, and evolutionary way.

References


A Unified Approach to Developing Intuition in Mathematics

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ABOUT THE AUTHOR

M. Anne Dow, Ph.D., was born in England, grew up in the USA, and spent time in Canada and Australia before joining the Department of Mathematics at Maharishi University of Management in 1984. She has a BA in Honors Mathematics from the University of British Columbia, Canada, an MA in Mathematics from the University of Western Ontario, Canada, and a PhD in Mathematics from the University of Queensland, Australia, where she was on the faculty of the Department of Mathematics for 11 years. She also has a Doctorate of World Peace from Maharishi European Research University. Research interests are partial differential equations, mathematics education, and the relationship between principles of Maharishi Vedic Science and mathematics. She began the practice of Maharishi’s Transcendental Meditation in 1973 and became a teacher of Transcendental Meditation in 1978. She has taught all levels of university mathematics, and, while at the University of Queensland, developed complete written materials for over 12 undergraduate math courses, which she taught by correspondence for the Division of External Studies. At Maharishi University of Management she now chairs the Department of Mathematics and enjoys teaching mathematics using the Consciousness-Based approach.
ABSTRACT

This paper proposes that creative insight in mathematics and the understanding of mathematics have in common a simple, highly integrated, creative process involving both intellectual analysis and intuitive insight. Further, many of the problems currently facing mathematics education can be traced to the inability of educators to directly address the crucial step in this process, the spontaneous step of illumination, in which the mind integrates and expands new information and existing mental structures into a whole new mental structure and one directly grasps a solution or understands a proof. This process is analyzed in the light of the model of the mind described by Maharishi Mahesh Yogi’s Vedic Science, and it is suggested that the experience of transcending during the practice of the Transcendental Meditation technique trains the mind in this crucial step, thus completing the processes by which we learn and do mathematics and making the learning and doing of mathematics a joyful, fulfilling experience.

Introduction

It is well known that mathematics education is in trouble. At the school level, the Second Mathematics Assessment of the National Assessment of Educational Progress [4, p. 134] found that for the 9-year-olds mathematics was the best liked of 5 academic subjects, for the 13-year-olds it was the second best liked, and for the 17-year-olds it was the least liked subject. At university level many are unable to pass even first year calculus, and, of those who do, few would admit to enjoying the subject. Many reasons have been cited: lack of teachers interested in mathematics, lack of understanding of cognitive processes involved in learning and doing mathematics, teaching concepts before the student’s cognitive development is ready for them, and so on. Even if one completes a degree in mathematics, one has, according to Littlewood [29], the “agony of research” to look forward to: a life spent mostly in frustration punctuated by rare inspirations. Contrast this scenario with the following experience.

In 1985 I had the good fortune to visit the Department of Mathematics at Maharishi University of Management (previously Maharishi International University), a nonsectarian private university in Fairfield, Iowa, with a strongly conservative curriculum, employing a system of education founded on the principles of Maharishi Vedic Science.
I taught courses in calculus, linear algebra, analysis, and differential equations. It turned out to be the most rewarding and enjoyable teaching experience I have ever had. I found my students happy, bright, awake, interested in what I was saying, fearless, and enjoying mathematics. Furthermore, they seemed to learn more easily, and to display a better intuitive grasp of new concepts and more creative approaches to problem solving than students I had taught elsewhere.

My experience at Maharishi University of Management changed my thinking dramatically about what it is possible for mathematics education to accomplish. In this paper I would like to share with you my views on the reasons for the enormous discrepancy between my experience at Maharishi University of Management and the general experience elsewhere, and to suggest that the methods used at Maharishi University of Management be adopted elsewhere.

My basic thesis is that mathematical knowledge progresses and is understood and learned through a common, simple, highly integrated creative process involving both intuitive insight and intellectual analysis. Most parts of this process have been successfully analyzed and taught, but a crucial step in the part of this process that pertains to gaining intuitive insight has so far evaded our efforts to teach it. This is the spontaneous step of illumination, in which the mind is able to cognize deep connections between its own structures and new information, putting them together into a new whole. At Maharishi University of Management, everyone practices the Maharishi Transcendental Meditation technique, in which the active thinking mind spontaneously settles down to experience Transcendental Consciousness, a state of restful alertness in which awareness is only aware of itself. Maharishi describes Transcendental Consciousness as the source of all mental activity or thought, a reservoir of latent creativity and intelligence. It is my view that regular experience of this process of transcending trains the mind in this crucial intuitive step, thus completing the basic creative process through which we learn and do mathematics. This step is also the step of the creative process that makes mathematics a joyful and fulfilling experience, which would explain my experience that students at Maharishi University of Management not only succeed at mathematics but also enjoy it.
In this paper, I will discuss the common features of the creative process and the process of understanding, bringing out the important role of intuition and of schemas (or intuitive biases) and pointing out the weak link. I will explain how experience of Transcendental Consciousness strengthens not only this crucial link, but also other aspects of the creative process that often go wrong, thus completing the creative process and making it always accessible to students and mathematicians.

In addition to the practice of the technique, a second aspect of the approach at Maharishi University of Management is the drawing of parallels between principles of mathematics presented in class and more universal principles common to all disciplines and to the growth of human consciousness. These principles, which are drawn from the body of knowledge known as Maharishi Vedic Science, appear to provide a valuable source of insight into mathematical processes, functioning as a body of established schemas, or intuitions.

The paper is presented in the following sections:

1. The role of intuition in modern mathematics.
2. Solving current problems in mathematics education.

Appendix. A model for cognitive development.

1. The Role of Intuition in Modern Mathematics

The importance of intuition, both in research and in education, is probably appreciated more in mathematics than in any other field. In the preface to Geometry and the Imagination, Hilbert and Cohn-Vossen describe the two complementary tendencies in mathematics:

In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations [24, p. iii].
Poincaré refers to the role of intuition in integrating parts into a whole:

In the edifices built up by our masters of what use is it to admire the work of the mason if we cannot comprehend the plan of the architect? Now pure logic cannot give us this appreciation of the total effect; this we must ask of intuition [41, p. 436].

**The Two-Fold Process of Acquiring Mathematical Knowledge**

All levels of mathematical knowledge, both informal and formal, both concrete and abstract, progress by means of a deeply integrated process of reflective, analytic thinking and intuitive thinking. On the one hand, analytic thinking emphasizes precision, objectivity, logic, symbols, operations, rigor, details. It aims to arrange accumulated knowledge in a rigorous, logically sound, deductive sequence of steps to arrive at safe conclusions. It tends to be highly reliable. One thinks one has some idea how to teach it. It seems to fit into current models of human cognitive development, particularly Piaget’s endstage of adolescent development, formal operations. On the other hand, intuitive thinking is subjective, immediate, global, nonverbal, synthetic. “Such a cognition is felt by the subject to be self-evident, self-consistent, and hardly questionable” (Fischbein & Gazit [18, p. 2]). Compared to analytic knowledge, intuitive knowledge is generally held to be unreliable. Skill in it can only be sought, not taught. And modern psychology offers no single comprehensive model of cognitive development that encompasses it.

Let us examine some of the ways in which these two processes, subjective and objective, are interwoven. On the one hand, after an initial stage of exploration involving both intuitive and analytic steps, the solution to a mathematical problem usually occurs to one as a sudden intuitive grasp. Because of its global subjective nature, an intuitively grasped cognition then needs to be crystallized and brought out by means of reflective thinking into the realm of mathematical discourse. Usually one does this by placing it in the context of first informal, then formal theory (see, e.g., Wittman [54] for a useful discussion of the role of informal mathematics). This procedure also serves to verify the correctness of intuitions by testing them against the rules of deductive logic, our main criterion for the objective correctness of mathematical knowledge.
On the other hand, intuition is used even in this process of bringing intuitive knowledge into the formal structure of mathematics. For example, when constructing a rigorous proof, we use intuitive interpretations, geometric insight, diagrams, analogies, particular concrete examples from our experience. Afterwards, intuition may serve as a check on the correctness of the formal proof that we have constructed. For example, Wittman points out that it is intuition that makes us suspect there is an error in the “proof” of the assertion that all triangles are isosceles. On the basis of this intuition, one then discovers that the diagram used is faulty. A recent conference of the Humanistic Mathematics Network [53] stressed the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be “merely technical.” In creating new mathematics, the investigator “as in every other science, does not work in this rigorous deductive fashion. On the contrary, he makes essential use of his imagination and proceeds inductively aided by heuristic expedients” (Felix Klein, quoted in [28, p. 274]).

As mathematics educators, most of us appreciate the need to use an integrated approach in the classroom. Petitto [39] studied strategies used by 9th grade students to solve algebraic equations. She found two groups: one that leaned towards an intuitive global approach, trying to capture numerical relationships among numbers in an equation without transforming the equation itself, and one that relied on memorized or routine step-by-step procedures for transforming the equation and producing the answer. Those that moved easily between the two proved most successful. Wittman [54] proposes that the main task of mathematics teaching at the school level is the development of informal mathematics. He advocates practice in intuitive, operative studies of a rich variety of examples and models, in discovering patterns and constructions through one’s own activities, and then reflecting on these intuitive activities by finding general formulations, finding proof ideas for the patterns and constructions, and so on. This he regards as prerequisite to axiomatics, to formal mathematics. Fischbein’s comprehensive studies of the role of intuition in learning (below) also bring out the crucial integration of intellectual and intuitive understanding.
In the field of science in general, Hanson [23] has argued that all knowledge, whether gained by subjective or objective means, is based in intuition.

Let us now turn to the concept of intuition itself.

The Nature of Intuition

Intuition appears in many different guises. In mathematics it seems to function in two main processes. The first commonly comes up in problem solving or mathematical research. It is the classic intuitive leap or sudden insight described by Poincaré [42], Littlewood [29], and others. The process is usually held to involve four steps. Following Littlewood’s description, the first is a period of preparation in which one strips the problem of accidentals and brings it clearly into view, surveys all the relevant knowledge, ponders possible analogues. Newton, he says, suggested keeping it constantly before the mind during intervals of other work.

The second is a period of incubation, which could last a moment or several years, and during which one may not be consciously thinking about the problem at all. The third, called illumination, often happens in a fraction of a second, when the creative idea is suddenly grasped in its wholeness, before any explicit details and justification are given. The fourth then involves working the idea out, the reflective activity of verifying its truth and integrating it into formal mathematics. This experience of creative insight may take place briefly when solving a problem in class, or may take place over a period of years when a practicing mathematician discovers the solution to a research problem.

Secondly, we apply the term intuitive to an explanation or interpretation directly accepted as natural, self-evident, intrinsically meaningful. For example, Fischbein [15] points out that most people find the equality of opposite angles formed by two intersecting lines highly intuitively obvious, but find the fact that the sum of the angles of a triangle is equal to two right angles not intuitively obvious. Fischbein [15], [17], [18] has written extensively on the need to provide an intuitively acceptable explanation of a proof or to provide a “basis for belief” based on active personal involvement along with or prior to a formal step-by-step logical argument. Without this more holistic direct feeling for the correctness of a proof, he finds, students tend not to be able
to remember the proof nor to effectively use it later. A related idea is Skemp’s “learning with understanding.” In mathematics, Skemp [48] says, agreement depends on pure reason, not on authority. Therefore, one must learn mathematics with understanding, not by rote. One must establish mental patterns, or schemas, rather than memorize facts. Only then will the knowledge be useful later on.

The concept of schemas is fundamental to Piaget’s model of cognitive development [40] and also to contemporary cognitive psychology. One of the most powerful tools of the human mind is that of conceptualization, the ability to abstract a concept from a list of examples displaying that concept, to treat that concept as an entity in itself, perform operations on it, compare it with other concepts, and so on. This ability is fundamental to mathematics. A conceptual mental structure, a body of concepts interrelated in some way, is often called a schema. For example, a student builds up a collection of concepts involving numbers and some relationships between them. His conception of how they all fit together and are used is a familiar mental schema approximating what we call arithmetic.

Later on when some problem requires addition, he spontaneously evokes this schema and is able to find the sum. Later in his mathematical career he may see this same problem in the context of his understanding of group theory, another schema. Schemas may be very local or very comprehensive, very simple or very complex. They may involve geometry or algebra or everyday life. A single example may invoke many different schemas from which one must choose, either automatically or consciously. Once in place in the mind they seem to be invoked automatically when the need arises. (See Skemp’s book [48] for a clear introduction to mathematical schemas.)

According to this model, what makes a mathematical explanation feel natural and intuitively acceptable, or a problem easy to solve, is that it fits into, easily extends, or easily integrates our previously established mental schemas. These mental schemas may already be in place, coming from one’s stage of cognitive development or from previous experience, mathematical or otherwise, or they may be put in place by a teacher who gives an intuitive justification of a proof before giving the formal proof, or who provides examples and informal experience before
introducing an abstract concept or definition. The more schemas one has available, the better the chance of coping with new knowledge.

The notion of having schemas in place is also useful in understanding the nature of mathematical research. This concept of schematic structures appears to correspond to Dieudonné’s conception of intuitions [8]. He held that intuition regarding mathematical objects is acquired by growing experience and that there are different intuitions in different fields, geometric intuition, combinatorial intuition, and so on. Progress in modern mathematics was achieved essentially by transfer of intuition from one mathematical field to another. For example, linear algebra came about as a transfer of geometric intuition to algebra. According to Dieudonné, the more schematically and abstractly mathematical objects are comprehended, the more deeply relationships can be understood and transferred intuitively. Hilbert was a living example of this principle. He mastered one field of mathematics after the other, leaving one field for the next just as he had reached his peak of success. In this way he was able to make profound use of transfer of intuition from one field to another.

At first glance these two processes, creative insight and the combining of intuitive acceptance with formal mathematics to produce integrated understanding, may appear unrelated. But they have key aspects in common. Both involve a preparatory stage that serves to enliven the mathematical content and the intention to solve the problem or understand the explanation. Both evoke established schemas, or mental structures. Both processes involve a crucial step of integration and expansion of knowledge and schemas into a new more comprehensive whole.

This move from parts to whole involves letting the attention disengage itself from its identification with the content and activity of the preparatory stage, thus leaving the mind lively in the problem but free to settle down. In this lively, free state the mind, if it is going to, will spontaneously settle down and directly grasp the whole. This whole may be experienced as the solution to a problem, or as a sharp concept that was being sought, or as an understanding of a proof or explanation. As a result of this step, the new integrated whole becomes available to intellectual analysis and verification, and ultimately may
become a schema for future use in further cycles of the creative process and understanding.

The step in which the mind settles down and an integrated whole dawns in the awareness is invariably a blissful experience, often called the “aha” experience. Because of its direct, spontaneous nature it is called intuition. In fact, because of its immediacy, Gödel [51, pp. 84–86], Littlewood [29, p. 113], Polanyi [43, p. 118], Fischbein [15], and others have related this sudden direct grasp to sense perception. It is the union of the intellectual understanding of a mathematical truth (for example, a proof or concept) with a more immediate, global feeling for that truth that produces the joy of mathematics and makes doing mathematics fulfilling.

It is this union that Fischbein says is so necessary for real understanding. In his paper on intuition and proof, Fischbein explains, “a mathematical truth can become really effective for productive mathematical activity if, together with a formal understanding of the respective truth, we can produce that kind of synthetic, sympathetic, direct acceptability of its validity. The same holds for a concept, a statement, a proof, and for the basic principles of generalizing, deducing, and proving in mathematics” [15, p. 18]. Sometimes this is easy. For example, most people find that the equality of opposite angles where two straight lines intersect is obvious. Sometimes there is no simple way, but Fischbein holds that such a fusion makes the knowledge satisfying, acceptable, remembered. One needs a feeling that the proof is right, not just a memory of the steps of the proof.

Poincaré [42] connected the creative process with the faculty of feeling when he said that mathematical creation requires strength of memory (but not prodigious), attention, and a delicate feeling, an intuition of mathematical order, that makes us divine hidden harmonies and relations.

In the light of these deep parallels between the creative process and the process of integrated understanding, it is my view that these two apparently unrelated processes are really expressions of a common underlying creative mechanics. Its key is the ability to allow the mind to leave the superficial active level of thinking about the problem or explanation and spontaneously settle to a level where intellect and feeling are
united, where new information and established schemas are integrated and expanded, where intellectual knowledge becomes immediate.

It is my view that this type of process permeates every level of doing and learning mathematics. The process, however, has eluded our efforts to teach it. We have addressed much effort to the preparatory stage, teaching heuristics, trying to set up appropriate schemas in the education system, trying to relate new knowledge to old as we teach, and so on. Many of us have emphasized informal mathematics, in which we structure into our classes opportunities for the creative, integrative process to take place. And all of this has helped to some degree, but we have been unable to directly address the integrative step itself, in which the mind spontaneously settles down and puts everything together, uniting intellect and feeling to create true understanding. It is this crucial step that regular practice of the Transcendental Meditation technique addresses directly. The technique provides the basis for completing the creative integrative process required for mathematics to progress.

Let us now look at the way in which the Transcendental Meditation technique fills this gap and helps to solve the problems of mathematics education.

2. Solving Current Problems in Mathematics Education

The experience of mathematicians and mathematics educators, that formal intellectual understanding and intuitive grasp must be united in order to produce the creative process and real understanding, can be viewed in the light of the model of the mind presented by Maharishi Vedic Science. According to Maharishi [32], the mind is structured hierarchically in increasingly abstract, powerful, and comprehensive levels: senses; active thinking mind (the associative faculty, including apprehending and comparing); intellect (the abstract discriminative faculty including analysis and synthesis); subtle feeling (responsible for intuition), which is usually said to be the most refined and subtle level of intellect; ego (the most subtle integrative function of individuality); and Self (a completely abstract level also called pure awareness or pure consciousness).\(^1\) Note that subtle feeling, or intuition, is considered in

\(^1\) This unbounded value of the Self is written with an uppercase “S” to distinguish it from the ordinary, localized self we typically experience.
this model to be the most refined and powerful level of intellect, and that when the mind functions from this level, intellect and feeling are united as an integrated means of gaining knowledge.

Maharishi further describes the Transcendental Meditation technique as a process whereby the attention is spontaneously and naturally freed from mental and physical objects of perception and allowed to settle to more and more subtle and comprehensive levels until it transcends thought altogether and one arrives at one’s own simplest, most integrated and comprehensive state of awareness, a state of restful alertness in which awareness is aware only of itself. This is the level of mind called Self or pure consciousness, and the experience is called Transcendental Consciousness. This experience is said to enliven the more subtle and powerful levels of the mind, thus enhancing creativity and allowing one to use one’s full mental potential in all areas of life. (See Section 3 for research supporting this.)

Maharishi speaks of the role of education in developing in students the ability to spontaneously think deeply on quieter levels of the mind, where knowledge is stored (Lecture, May 5, 1971). He explains that as one’s attention goes to the quieter levels of the mind, the quiet, tender, comprehensive level of the mind is enlivened, where deep connections are made. The purpose of education, he says, is to enliven those quiet levels.

Further, Maharishi explains that regular experience of Transcendental Consciousness during the Transcendental Meditation technique results in growth to higher states of consciousness, each of which presents even more powerful direct means of gaining knowledge. These higher states would be recognized from a Piagetian viewpoint as stages of cognitive development beyond formal operations. Evidence that practice of the Transcendental Meditation technique is helpful in stabilizing stages of cognitive development up to and including formal operations is cited in Section 3. See also [2] for a survey of research suggesting that it promotes development beyond formal operations. A discussion of the full range of human development to higher states of consciousness is beyond the scope of this paper. (See the Appendix and [1] and [2] for a discussion of the way in which the model of cognitive development presented by Maharishi Vedic Science provides a single
comprehensive model of cognitive development and for a comparison of this model to current psychological theories of cognitive development up to and beyond formal operations.

By promoting growth of successively more comprehensive and powerful stages of cognitive development, regular experience of Transcendental Consciousness is clearly a most valuable asset in the study and doing of mathematics. I will not extend this paper into this vast area, however. Let us look instead at how, by itself, experience of Transcendental Consciousness is the basis for solving current problems in mathematics education.

**Completing the creative process**

Experiencing Transcendental Consciousness regularly enables the key aspect in creative insight and in mathematical understanding to happen: the crucial step of deep integration and expansion of knowledge and schemas into a new more comprehensive whole, described earlier.

It does this by promoting the habit and ability of allowing the attention to settle to deeper, more comprehensive levels of the mind. The spontaneous, natural process of allowing the attention to free itself from mental or physical objects of perception and to effortlessly settle down to the most comprehensive and integrated level of all, Transcendental Consciousness, becomes a familiar process. Regular practice twice daily of the Transcendental Meditation technique allows this to happen naturally and regularly. As this process becomes more familiar, the experience of practitioners of Transcendental Meditation is that it becomes readily available during daily activity.

Thus if we are working on a problem in class, or working on a research problem, or just listening to a proof of a theorem, the mind will take the opportunity to settle down and experience deeper levels, so that the step of illumination, when we grasp the whole of whatever it was we were thinking about, will take place. Without this habit and without a state of relaxed alertness, when trying to solve a problem or understand a proof, we tend to try to keep thinking about it, which prevents the mind from settling to deeper levels. Even trying to think about the possible wholeness does not help, because that is still holding the mind on the active level of involvement with thoughts. The experience of wholeness occurs spontaneously when the attention is freed
from thoughts and is able to go within to a deeper level of the mind. This is what regular experience of transcending seems to promote.

As research mathematicians, Littlewood and Poincaré knew the value of letting a problem go. They both structured into their lives time for allowing the attention to disengage from the level of actively thinking about a problem. Littlewood went for walks in the country. Poincaré went on a geology excursion. And occasionally, with good luck, the sudden flash of deep inspiration occurred.

In my opinion we can now take the element of luck out of the process. We can directly train students or ourselves in the ability to take that intuitive step of illumination or understanding, simply by teaching students the practice of the Transcendental Meditation technique. This would make our efforts to teach heuristics and to structure schemas at the right time effective and fruitful. With the completion of the creative process, students should be able to solve problems significantly better than before and be able to learn, remember, and use new knowledge. Quite an extensive body of research suggests that this should be the case (see Section 3). Further, research appears to support the view that the Transcendental Meditation technique provides this ability to use the quieter, more comprehensive levels of the mind spontaneously at any time. In tests of field independence, students have been found to grow in the ability to focus sharply and maintain broad awareness (see, e.g., [38], [10]) possibly indicating growth to the stage of development where the mind permanently functions from its deepest, most comprehensive level.

Alleviation of math anxiety and other affective factors

Other factors affecting the creative process are also affected by experience of Transcendental Consciousness. Both the ability to focus and the ability to settle to a deeper, more comprehensive level of the mind require a state of relaxed alertness. Stress and anxiety interfere with this. It is well known that higher mental activities are the first to be affected by stress and anxiety (see, e.g., Skemp, [48, p. 118]). Negative views about one’s ability and about the difficulty of doing mathematics also affect the creative process. They create anxiety and also may prevent one from even beginning the preparatory stage. Practice of the Transcendental Meditation technique has been found to provide deep
rest to the nervous system and to relieve anxiety and stress. (For a comprehensive survey of physiological research on the technique, see [12].) It also increases self-confidence and self-actualization (see Section 3). This in itself should decrease math anxiety and improve confidence in doing mathematics. With growing relief from anxiety and stress experienced during the practice of the Transcendental Meditation technique, the state of relaxed alertness experienced during becomes more and more a part of everyday life, always there even in dynamic activity, hence always available to the creative process. With the complete creative process available, a student will find himself actually able to do mathematics. Success is also good for overcoming math anxiety.

**Overcoming the feeling that mathematics is remote**

It is widely held that one reason potentially good students do not pursue mathematics as a career is that they find it remote from personal experience. Intuition, as we have pointed out, is a crucial step in creating and understanding mathematics. In order to bring intuitive knowledge into the mainstream of mathematical discourse, however, we embed it in the framework of formal theory, using deductive logic as our criterion of right knowledge. This approach has made mathematics a body of reliable, true, useful, and elegant knowledge. But at the same time its very success depends on removing any element of ourselves from the knowledge.

Further, from the viewpoint of cognitive development, a strict diet of logical formal theory in the classroom may lock one into the stage of formal operations, which Cook-Greuter [6, p. 96] has called a stage of maximum distance between thinker and object of thought. Formal operations is the stage of the reflective intellect. It allows the thinker to deal only indirectly with objects of thought, through mental representations of them. Even the self becomes one more object to reflect on indirectly through representation. And as Maharishi has said: “When the knower does not know himself, then all knowledge is baseless. And baseless knowledge can only be non-fulfilling” (Lecture, August 1, 1986). Since formal operations begins to stabilize around age 12, perhaps there is some relationship between this side effect and the growing distaste for mathematics that students exhibit from ages 13 to 17, which I referred to in the introduction.
One way we as teachers try to overcome the feeling of remoteness in our students is by relating the mathematical theory to applications, which, it is hoped, mean something to the student. This is useful, but does not address the above, much deeper problem. What is required is to overcome the intrinsic separation of student from mathematics imposed by representation itself. This is what is achieved in the step of illumination in the creative process and in the process of understanding, in which the whole is grasped directly before any representation or intellectual analysis can be made.

Ensuring correct schemas are in place beforehand helps facilitate this integrative step. Also, teaching informal mathematics sets up many opportunities for these processes to happen. But in both cases these processes should be far more likely to happen, and deeper levels of integration should occur, when the ability to transcend is also available.

Students who have the direct experience of this profound process of integration of intellect and feeling regarding a mathematical truth several times in each class are not likely to feel mathematics is remote from them. The processes take place within them, and the mathematical knowledge produced by them is part of them, a mental structure they have uncovered within themselves. It is very personal knowledge.

At Maharishi University of Management students find mathematics personally relevant also because it is pointed out to them that the principles by which their own consciousness unfolds are the same as the principles by which mathematics unfolds. The body of knowledge concerned with the unfolding of consciousness is called Maharishi Vedic Science.

This aspect of Maharishi University of Management is explained in the next subsection. The point here is that, as Maharishi remarks, when one knows that mathematics is really nothing but the study of one’s own intelligence, which has its source in one’s own simplest state of awareness, Transcendental Consciousness, which one experiences every day, then the study of mathematics becomes completely intimate. Transcending through practice of the Transcendental Meditation technique and using the principles of Maharishi Vedic Science together make studying mathematics intimate and hence deeply fulfilling.

Further means by which the student is connected to mathematics at Maharishi University of Management are classroom charts that place
the topics covered in the lesson in the context of the whole lesson, place the lesson in the context of mathematics as a whole, place mathematics in the context of the wholeness of all knowledge, and connect the wholeness of all knowledge to its basis in the student’s own consciousness. Although students focus sharply on the mathematical content, they never lose sight of the most global perspective on the topic being taught, and never lose sight of themselves, as the source of the knowledge.

**Making the most of schemas**

New information or problems that do not fit easily into established schemas, or that do not easily extend existing schemas often cause conflicts that students are unable to resolve satisfactorily. Either students give up trying to understand mathematics, or they blindly apply wrong schemas to the new information. Both avenues lead to frustration.

Fischbein seeks the source of difficulties students encounter in the basic intuitive patterns they use when solving mathematical questions, that is, in their established schemas. His extensive and illuminating research into intuitive biases of primary and secondary students led him to the conclusion that “the problem of identifying the natural intuitive biases of the learner is important because they affect—sometimes in a very strong and stable manner—his concepts, his interpretations, his capacity to understand, to solve and to memorize in a certain area. We are naturally inclined to retain interpretations which suit these natural, intuitive biases, and to forget or to distort those which do not fit them” [20, p. 491]. Thus intuition, which gives us the creative process itself, may degenerate into blind rigidity when one’s schemas are not correct.

Feller [14] and, more comprehensively, Fischbein [17] found children’s natural intuitions about probability usually contradict the way probability actually works. Also, Fischbein and others [19] found in studies of children’s intuition of infinity that conservation mechanisms from lower stages of cognitive development may be wrongly applied to problems concerning infinity.

Students also have inadequate schemas regarding formal mathematics itself. Fischbein [15] found intuitions regarding the nature of formal mathematics are often absent. High school students were not necessarily convinced of the truth of a statement simply by understanding a
formal logical proof. Also to this point, Schoenfeld [46] gave students a geometry problem and found that most used pictures to try and solve it. None of them used theorems from Euclidean geometry. They did not realize that theorems and constructions of formal geometry could be used in solving a problem.

Another reason schemas may be absent at the right time in the course of education is that a student may not have reached the level of cognitive development in which such schemas are established. For example, Driscoll [13, pp. 19–24] claims many topics taught in school disadvantage students who have not yet reached the Piagetian stage of formal operations: formal proof, the concept that an equation remains the same when the variable is changed, fractions, conservation of length, volume, area, and word problems. He felt that some, but not all, of this could be overcome by appropriate teaching methods. In this regard Carpenter [4, p. 144] points out that mathematics instruction has been found to aid development and stabilization of new cognitive stages if it is done carefully, but it is rarely done carefully.

Skemp [48] advocates structuring the mathematics curriculum in such a way that when a topic is taught, the mental schemas that are needed for its understanding are already in place. He also points out that in the course of simplifying material so that it can be understood by elementary students, teachers must be very careful to give students schemas that are easily extendable at a later stage rather than schemas that have to be unlearned. For example, children develop strong intuitions that addition and multiplication make something bigger and that division and subtraction make something smaller. This leads to trouble in first meeting negative numbers and fractions less than 1.

We can see that the problem of inadequate or wrong schemas is an important one. Identifying wrong schemas or intuitive biases and changing them is necessary. This is not always easy. Fischbein points out that to change a wrong schema, one has to entirely re-elaborate the process. Further verbal instruction does not help. It is my view that the Transcendental Meditation technique should help here as well. As I suggested earlier, by culturing the ability to allow the mind to settle spontaneously to deeper, more integrating levels, practice of the technique promotes integrated intellectual and intuitive understanding.
of mathematical arguments. Thus, understanding a re-elaboration at a deep level should be smooth and easy.

Further, according to the model of cognitive development presented by Maharishi Vedic Science, more comprehensive perspective develops as one is able to function from deeper, more comprehensive levels of the mind. This makes the mind more flexible, less rigid. Broader perspective should help students to let wrong intuitions go when they become aware of them, and even help them become aware of contradictions in their thinking themselves. Research supports this predicted flexibility of mind (see Section 3).

Let us now turn to establishing schemas.

Schemas (bases of belief, intuitive patterns, established mental structures) are an important ingredient in the creative process and the process of understanding, as I have brought out. Bad ones interfere with the process, good ones are important for success. Where do they come from?

They seem to be the result of past occurrences of the same creative integrative process we have been discussing. Indeed, if this process is our basic mechanism of assimilating knowledge and developing cognitively, one imagines a long sequence of recurrences of the process throughout life going back to assimilation of sensory-motor experience in childhood. At each step one directly experiences the new information integrating with the old mental structures to produce new structures.

Establishing a basis for belief, according to Fischbein [15], requires that one “live the process” (p. 14). One needs to be involved directly, personally, behaviorally, either mentally or physically. For example, Fischbein felt the reason students were not convinced of the truth of a theorem after accepting a formal proof of it was that “the concept of formal proof is completely outside the mainstream of behavior” (p. 17). In this scientific age, conviction that a conclusion is true is instead “derived from a multitude of practical findings that support the respective conclusion” (p. 11).

A most interesting example is the following. One of Fischbein’s fascinating projects has been the study of the intuitive patterns of primary and secondary students regarding mathematical concepts of infinity [15], [16], [19], [20]. Aristotle considered infinity only as a potential-
ity. For example, a line can be extended indefinitely or a segment can be divided indefinitely. Not until Cantor introduced infinite cardinal numbers did actual infinity become part of mathematics. In the studies cited above, students in grades 8 and 9 appeared to be comfortable with the notion of an indefinitely repeated process, potential infinity, but highly uncomfortable with actual infinity. Fischbein’s explanation of this was that “actual infinity has no behavioral meaning and therefore is not congruent with an intuitive, insightful interpretation.” That is, it is beyond any direct experience we could have, either mental or physical. Further, studies of students in grades 5 to 9 suggested that development of concepts of infinity were tied to cognitive development. Up to age 12, roughly the beginning of the formal operational stage, intuitive interpretations of potential infinity were still in the process of formation. After age 12 intuition of infinity appeared to be stable, but was still based on finitist schemas. It appears that actual infinity is not available as a schema within the formal operational framework.

This raises some interesting questions. Is such a schema available to a person who has attained post-formal operational stages of development, for example, to Cantor? Are there mathematical truths for which one could never have a schema, that is, for which one could never have a direct intuitive feel?

I suggest that experience of Transcendental Consciousness and development of stable higher states of consciousness vastly extend the range of direct experience from which we can draw, and that schemas based on this experience are fundamentally related to mathematics. Maharishi Vedic Science is the science of consciousness itself. It studies principles governing all orderly change and growth. These principles are directly experienced in transcending during practice of the Transcendental Meditation technique and the TM-Sidhi programs or in our everyday life. According to Maharishi, the principles described by Vedic Science, governing the way consciousness unfolds, are the deep principles governing any discipline, including mathematics. Thus study of Maharishi Vedic Science, together with direct experience of transcending, should establish in one a body of very deep schemas upon which intuitions of a mathematical nature can be based.

A comprehensive discussion of the principles of Maharishi Vedic Science and how they relate to mathematics is beyond the scope of
this paper. For an introduction to Maharishi Vedic Science, see Maharishi Mahesh Yogi [31] and Chandler [5], and for a comprehensive discussion of deep relationships between Vedic Science and foundational areas of mathematics, see Weinless [52]. Briefly, here is an application to the intuition of actual infinity from Weinless’s paper.

The experience of Transcendental Consciousness itself provides a behavioral basis for actual infinity. A completely non-representational state, awareness, by virtue of being aware, is aware of itself. This is not awareness thinking about a representative of itself because the active thinking mind has settled down to its ground state. It is direct experience of wakefulness, with no boundaries, unbounded awareness. Maharishi describes this state as the Absolute. This corresponds to Cantor’s description of the absolute infinite, which includes and surpasses all the different levels of the actual infinite in set theory.

According to Weinless, Maharishi Vedic Science provides deep principles of the structure and functioning of intelligence that can unify our understanding of mathematics, and which, together with the experience of Transcendental Consciousness, can provide an intuitive basis for the deepest principles of mathematics.

3. Research into the Effects of the Transcendental Meditation and TM-Sidhi Programs on Factors Affecting Success in Mathematics

There is an extensive literature on research into the effects of the Transcendental Meditation and TM-Sidhi programs on factors affecting education. For the sake of brevity, we shall summarize some of these, and refer the interested reader to the papers [12], [2], [37], [30], which provide in-depth surveys of the research behind these statements and many other studies. Although these factors affect education in general, many of them have been shown to affect mathematics education specifically. For example, field independence [26], [45], [50], intelligence [25], and self-concept [44] have been found to correlate positively with mathematics achievement, whereas field dependence [22] correlates with math anxiety, and building a positive and realistic self-concept can prevent math anxiety [7].

Cognitive, affective, and physiological characteristics that contribute to effective learning have been found to improve in students practicing
the technique. Improvements have been found in alertness, memory, fluid intelligence [3], [10], [47], field independence [10], creativity [47], [49], reasoning ability [11], and academic achievement [12]. There have also been many studies indicating improved self-concept; increased self-actualization; reduced depression, neuroticism, and anxiety; reduced aggression and increased tolerance; greater emotional stability; and greater physiological resistance to stress.

In Section 2, I pointed out that solving mathematical problems requires cognitive flexibility, which includes the ability to recognize when a previously learned schema no longer works and must be abandoned in order to find new ones. Research has verified that such cognitive flexibility increases through the Transcendental Meditation and TM-Sidhi programs. Dillbeck [9] found that after learning the technique, students become more efficient in applying schemas to solve problems, and at the same time they become more flexible in discarding old schemas in situations in which they no longer applied. Another study [11] found that the TM-Sidhi program increased cognitive flexibility and that cognitive flexibility was correlated with frontal EEG coherence. In this study of concept learning, the correct concept was changed in mid-experiment. After practicing the TM-Sidhi program for four months, the students became more efficient and flexible in shifting to a new concept.

I also pointed out earlier that mathematical insight involves an incubation period. This period evidently includes spontaneous organization of information in memory. A study of memory [33] has demonstrated improvement in just such an ability in students after they learned the Transcendental Meditation technique. Subjects were presented with a list of 40 words to memorize. These words were in random order, but contained items that could be organized into taxonomic groups of metals, professions, fruits, and animals. After a recall interval of a few minutes, subjects were asked to recall as many words as they could, and these words were scored for clustering into the taxonomic groups. The subjects did not know that they would be scored for clustering, and they were given a filler task of arithmetic problems to do during the recall interval so that they could not consciously rehearse. Subjects underwent this test before they learned to meditate and again after forty days of practice. They were compared to controls tested at the same times
who rested twice a day over the forty days instead of meditating. At
pretest, the group and controls were similar. At posttest the group had
improved significantly in conceptual organization of the words. Interest-
estingly, they also improved more than controls on speed in solving
the filler arithmetic problems. This study demonstrates that both spon-
taneous organization of information in memory and speed in mental
processing are improved through practice.

Implementing the Transcendental Meditation program and
Maharishi Vedic Science in existing classes has been successful as well.
Findings are consistent with those above. For example, Shecter [47]
found that when compared to non-meditating control groups, students
in classes practicing the Transcendental Meditation technique showed
increases over a 14-week period in fluid intelligence, creativity, energy
level, innovation, self-esteem and tolerance, and decreases in anxiety
and conformity. This program has been introduced in all levels of
education in over 20 countries, including Australia, Brazil, Denmark,
Dominican Republic, Great Britain, The Netherlands, India, Kenya,
Korea, Norway, Philippines, Puerto Rico, Taiwan, Thailand, and the
US. Studies in India and in England, for example, found improved
reading comprehension, memory and concentration, and improved
grade-point average.

A number of studies have monitored the educational experience at
Maharishi University of Management and at its affiliate, Maharishi
School of the Age of Enlightenment (MSAE) (kindergarten through
grade 12). Cross-sectional studies indicate that Maharishi University of
Management students score higher than controls and norms on scales
of self actualization. Longitudinal studies of Maharishi University of
Management undergraduates over four years found increases in fluid
intelligence, in contrast to no change in normative trends [3], [10],
increased field independence [10], and increased social maturity and
psychological health as indicated by personality tests [3].

Longitudinal studies over shorter periods indicate that students at
Maharishi University of Management who learn the more advanced
TM-Sidhi program, compared with matched Maharishi University of
Management students who practice only the program, show signifi-
cantly increased abstract learning ability (concept learning), increased
flexibility of the central nervous system (faster recovery of the paired
Hoffman reflex), and increased orderliness of brain functioning (as indicated by EEG coherence in frontal brain areas)(see, e.g., [11]). Indeed, frontal EEG coherence has been specifically correlated with mathematics achievement [34]. These cognitive and neurophysiological developments occur together as an integrated whole [11] and higher levels of these traits predict higher academic performance.

It appears that their educational system enhances cognitive development (and hence mathematical ability) in MSAE students. In two different studies, children at MSAE performed better than non-meditating children elsewhere on standard Piagetian tasks in conservation. Secondary students at MSAE performed better on a test of creativity than control subjects taken at random from a normative data bank of the test. Length of time practicing the Transcendental Meditation technique was significantly correlated with level of performance. Other studies found greater field independence on cognitive-perceptual tasks in meditating children ages 7–11 than matched controls. (See [12] for references.)

Despite the liberal admissions policy, academic achievement has been extremely high at MSAE. The majority of classes score in the top five percent nationally on the Iowa Test of Basic Skills (administered to grades K–8) and the Iowa Tests of Educational Development (administered to grades 9–12) and many classes score in the top one percent. Individual students show an increase in percentile level on the tests of language, reading, and mathematics over the course of one school year [35], [36]. New students entering MSAE increased 31 percentile points in mathematics in one year. Teachers and visitors to the school remark on the children’s greater awareness, greater ability to focus, increased creativity, broader perspective, flexibility, and enjoyment of their studies, in comparison to students at other schools. All of these are needed to succeed at mathematics.

**Conclusion**

Concerning the National Council of Teachers of Mathematics’ current “decade of problem solving”, Galovich said, “We seem to face the unhappy situation of wanting to do something to help our students become better problem solvers, taking what seem to be the obvious steps to bring about this goal, and discovering that our efforts are only
marginally successful [21, p. 68].” The same is true of understanding mathematics, and doing mathematics generally. Bruce Vogeli of Columbia University Teachers College was quoted in Time magazine as calling innumeracy the “major untouched educational issue of the decade” [27, p. 66]. Why is the problem untouched in spite of all our efforts to improve mathematics teaching?

In this paper I suggest that the key to improving the doing and learning of mathematics is the addition of the regular experience of transcending, that is, of the Transcendental Meditation program, to the present curriculum. I argue that this experience directly structures in students the ability to take the one step in the creative process and the process of assimilating new knowledge that none of our other approaches has been able to address directly: the intuitive step, whereby new information and existing mental structures are spontaneously synthesized and extended into whole new mental structures, the step of illumination. It is this step that enables the student to grow, to expand his mathematical awareness into a greater whole, and to experience the direct relationship between himself and mathematics. It is this step that makes mathematics a joy.

Transcending is also the experiential basis of fundamental principles of the unfoldment of consciousness embodied in Maharishi Vedic Science. If we introduce lessons on Maharishi Vedic Science along with practice of the Transcendental Meditation technique, then we are also able to address the problem of providing deep schemas for mathematical concepts, such as actual infinity, for which students, at least those who have not progressed beyond the cognitive stage of formal operations, have no experiential bases. This body of deep unifying principles governing the mechanics of consciousness may also provide a profound intuitive basis for research areas of mathematics.

It is a popular belief that there can be no one solution to all the seemingly diverse problems of mathematical education. Yet the concept of one crucial idea that makes a whole body of mathematics work—for example, the limit process in analysis—is the stuff mathematics is made of. Also central to mathematics is the idea that a single weak link in a proof can invalidate the entire proof. Why not one simple process, transcending, that will enable the entire structure of mathematics education, as we know it, to work? We need to understand and teach heu-
istics, to establish correct schemas and bases of belief, to identify and correct intuitive biases, to teach informal mathematics as well as formal mathematics. Many of us do. But we also need to teach the ability to transcend. Only then will all our other efforts bear fruit.

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A Model for Cognitive Development

In this appendix, we examine the model for cognitive development provided by Maharishi Vedic Science in order to understand the role of transcending in more detail.

One reason it has been so difficult for education to deal with intuition is that modern psychology has not yet found a single comprehensive model of cognitive development that includes cognitive abilities beyond Piaget’s endstage of formal operations. Formal operations is the stage attained in middle or late adolescence in which one becomes able to think about thinking. One can make mental representations of concepts and operate on them; higher levels of abstraction and formal logic become possible [58]. This is the level of the adult scientist, and leads to the objective, scientific approach to knowledge. Recently researchers have proposed more abstract, postformal stages associated more with subtle feeling and intuition (see, e.g., [55] for a comprehensive survey). These are often associated with very creative people (see, e.g., [56]). Apparently, these stages are hardly ever reached by most of the adult population, another reason why intuition is not well understood.

According to Alexander [2], Maharishi Vedic Science provides an integrated model of the mind and human cognitive development that includes Piaget’s model and postformal models, but goes far beyond, positing three higher stages of consciousness that are truly postrepresentational. Let us briefly look at this model. (For a complete description and comparison with current theories of cognitive development, see [1], [2].)

According to Vedic Science [32], the mind itself is structured hierarchically in increasingly abstract, functionally integrated levels: action and senses; desire (which directs attention to the objects of the senses) and representation; active thinking mind (the associative faculty, including apprehending and comparing); intellect (the abstract discriminative faculty); feelings and intuition, which are usually said to be the most subtle level of intellect; ego (the most subtle integrative function of individuality); and Self (a completely abstract level also called pure awareness or pure consciousness). All mental levels operate to some extent throughout development, but a cognitive stage corresponds to the ability of the awareness primarily to identify with and function
through a particular level of the mind. For example, identification of the awareness with the level of intellect may correspond to Piaget’s formal operational stage, identification with the feeling and ego levels to postformal stages. Development of cognitive stages then correspond to the awareness sequentially becoming able primarily to identify with and function through successively deeper levels. At each stage, abilities and concepts of previous stages are integrated into the new stage but used from the viewpoint of the new stage.

This structure contains within it Piaget’s structure and postformal structures. The mechanism for development has some similarities but is different. According to Piaget’s model, new mental structures are created by a process of equilibration [59]. New experience or information is actively taken into existing mental structures and schemas. A conflict results because the fit is not perfect. This causes the structures themselves to adjust, and/or the information to be distorted or rejected, until a state of equilibrium is reached between the environmental demands and the cognitive system. This mechanism operates both within stages of development to assimilate new knowledge and create new schemas, and also cumulatively, together with maturing of the physical brain, to gradually structure the next higher stage of development.

In the Vedic Science model, mental structures are inherent, not created. They only need to be enlivened, and this is accomplished by the attention transcending to progressively deeper inherent levels of the mind. As the attention regularly transcends to a deeper level, the awareness becomes more familiar with that level, gradually establishes itself there, and eventually becomes able to function from that level, with all the previous levels integrated within it.

The main principle underlying the mechanics of this model is that shifts are sequentially made possible (1) through maturing of the physical structure of the brain during childhood, (2) by freeing the attention from identification with the previous state or level, and (3) by removing accumulated physical stress that inhibits normal functioning of the nervous system. (1) is just a matter of time. (2) and (3) require something more. An example of (2) from modern psychology is that of a child who begins learning language. Bruner [57] suggests that this process may serve to free the attention from identification with objects, thus enabling the child to progress from the sensory-motor stage to
representational stages. According to Vedic Science both (2) and (3) are facilitated at all levels by the practice of Transcendental Meditation. Through a simple mental device, the Transcendental Meditation technique allows the attention to be disengaged from both objects of perception and thoughts, allowing it to settle spontaneously and naturally into its simplest form of awareness, a state of restful alertness in which the mind is completely silent but awake. This is the level of mind called Self or pure consciousness, and the experience is called Transcendental Consciousness. Also, this experience has a profound effect on the body, giving it deep rest and allowing deeply rooted stresses and abnormalities to be removed. (For a comprehensive survey of physiological research see [12].) According to Maharishi, as the awareness settles to this state, attention shifts progressively through all the deeper levels of the mind.

Thus in one stroke, the Transcendental Meditation technique both provides the inward shift of attention and also removes the obstacles in the way of the shift. There is evidence that the Transcendental Meditation technique is helpful in stabilizing Piagetian stages of cognitive development (see Section 4), and that promotes development beyond formal operations (see [2] for a survey of research on this aspect). According to Maharishi Vedic Science, this experience of transcending is the basis for developing higher states of consciousness, in which the deepest, most powerful level of the mind then organizes the structure and functioning of all mental activity. The point here is that this model is comprehensive enough to contain current psychological theory and also explain how experience of transcending in the Transcendental Meditation technique develops the more subtle levels of mind, particularly the level of subtle feeling, integrating its functioning with the intellect, so that intuitive grasp, where intellect and feeling come together, can happen easily whenever one is presented with a mathematical problem or explanation. Further, daily alternation of intellectual activity, which involves representation, with transcending, which frees the mind of representation, should provide a firm foundation for the intuitive step of the creative process we have described, the step that requires integration of reflective intellect and feeling at a deep level.
Additional References Cited in the Appendix

Preparing the Student to Succeed at Calculus

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Learning calculus involves three aspects: the knower, or student; the known, or content of calculus; and the process of knowing, which links them. Neglect of any of these three aspects leads to incomplete knowledge. Today’s calculus reform deals admirably with the last two, but still leaves the development of the student’s full mental potential largely to chance. Comprehension of the limit process at the heart of calculus requires use of deeper levels of the mind than previous courses, and also requires an intuitive basis. A major aim at Maharishi University of Management (previously, Maharishi International University) is to develop the inner resources of students at the same time as they are acquiring knowledge and skills. The practice of Maharishi Mahesh Yogi’s Transcendental Meditation program at Maharishi University of Management gives calculus students (and teachers) at least two distinct advantages: a relaxed, alert mind with which to think deeply and also an experiential framework for feeling at home with the limit process.

Introduction

There are three aspects to any knowledge: the knower, the known, and the process of knowing, which links the knower with the known. All three must be addressed in order to successfully impart knowledge. In teaching calculus the knower is the student, the known is the content of calculus, and the process of knowing is the way we teach and the way the student comprehends calculus. Neglect of any of the three leads to incomplete, unfulfilling knowledge. In my view, calculus reform is addressing the last two admirably, but is still unable to directly address the development of a student’s inner mental resources. To understand the infinitary concepts of calculus, students need more than ever to develop very deep levels of thinking.

To fully comprehend and be able to use calculus, students must come to grips, really for the first time, with the concept of completing an infinite process. In order to grasp calculus conceptually, and not just use it as a series of formulas and procedures to be applied on recognition of a particular situation, students need to be able to understand and feel at home with infinitary methods. Some preliminary infinite concepts they may have previously been exposed to, such as geometric progressions and their sums. But a full appreciation of what is involved even in summing series is not really required until calculus. A course in precalculus
provides understanding of many concepts on which calculus is based, but does not prepare the student for the depth of thinking required to comprehend and use limits.

A second problem that students encounter when first introduced to limits is that the concept of completed infinity, essential to the manipulation and application of limits required in calculus, usually has no basis in a student’s past experience. There is no ‘hook’ on which to hang the idea of completed infinity. Potential infinity, that a process can be repeated indefinitely, does not go far enough in this direction.

Both these problems have to do with the ability of students to grasp deep concepts. The first asks how to create a receptive student, one who is alert, bright, and capable of thinking clearly at deeper, more abstract levels of the mind than ever before. The second asks how to provide a ‘basis for belief’ for infinitary concepts, so that the student finds the idea of a limit intuitively acceptable, familiar, and feels at home with it. This paper describes how we address these two problems at Maharishi University of Management.

The two problems fit in with a major aim at Maharishi University of Management, namely, to develop the inner resources of students at the same time as they are acquiring knowledge and skills. The most important way in which we address these problems at Maharishi University of Management is by daily practice of the Transcendental Meditation program. Everyone at Maharishi University of Management, students, staff and faculty, practice this simple, natural technique that allows the mind to settle down and gives the body and mind deep rest. Another way in which we develop the students’ ability to learn and understand is by relating principles uncovered in every discipline to deep principles of growth and the expression of intelligence in nature in general, described by Maharishi Vedic Science. This is done very briefly during class and gives students the understanding that the concepts of calculus are not remote and unique to mathematicians but are based on principles common to all areas of life, and to the way in which their own intelligence unfolds.

In addition to solving the above two problems, this approach overcomes many of the problems students have with calculus: math anxiety, stress, lack of previously developed schemas, fragmentation, the feeling
that calculus is remote from them personally, lack of motivation, and lack of creativity.

Let us now turn to the way in which this approach addresses the two basic problems we have just described.

**Developing a Receptive Student**

The power of mathematics lies in its ability to crystallize abstract concepts from a complex problem and deal with them on deep abstract levels of the mind. The concept of infinity is very abstract. In order for a student to deal with it, he or she must be able to think clearly at deep levels of the mind. It is well known that abstract levels are the first to go when one is anxious or stressed. Therefore the mind needs to be relaxed but very highly alert.

At Maharishi University of Management, the regular practice of the Transcendental Meditation program provides a very deep rest twice a day. This prepares students to be clear, awake, and dynamic in class. The deep rest also releases stress and reduces anxiety, so students are relaxed and focused in class. Further, the Transcendental Meditation program expands the conscious capacity of the mind by allowing the mind to experience its deepest, most silent level, thereby making that level familiar and spontaneously available later, in class, when a settled mind is required.

What is the Transcendental Meditation program? Maharishi describes the technique as a simple, natural mental process during which the active thinking mind spontaneously settles down to experience Transcendental Consciousness, a state of restful alertness in which thinking is completely transcended and the mind experiences its own unbounded nature. At the same time the body experiences deep rest while remaining fully awake. This is the simplest state of awareness—awareness is only aware of itself. In this state knower, process of knowing, and known are completely unified. The knower is one’s own consciousness, the process of knowing is one’s own consciousness, and the object being known is one’s own consciousness. This unified state is called pure consciousness or the Self, and the experience of it is called Transcendental Consciousness. (This unbounded value of the Self is written with an uppercase “S” to distinguish it from the ordinary, localized self we typically experience.) According to Maharishi, this unified
state of knowledge is the basis for all knowledge and in fact for the entire creation.

What does this do for the student? Experience of this state, Maharishi says, enlivens the more subtle and powerful levels of the mind, thus enhancing creativity and allowing one to use one’s full mental potential in all areas of life. Every experience of this simple state of restful alertness provides the body and mind with deep rest, and regular practice leads to this restful alertness being carried over into one’s daily activity. Over 500 studies at over 200 research institutions around the world have demonstrated that regular practice of the Transcendental Meditation program

1. settles the nervous system and increases coherence in the brain;
2. increases intelligence, creativity, learning ability, and memory;
3. reduces tension and anxiety; and
4. increases self confidence and psychological maturity.

Here is just a small sample of this research.

A statistical meta-analysis ([8]) of 31 physiological studies found that respiration rate and plasma lactate decreased and basal skin resistance increased much more during Transcendental Meditation practice than during eyes-closed rest. Further, outside the practice, the Transcendental Meditation subjects maintained significantly lower levels of respiration rate, plasma lactate, spontaneous skin conductance, and heart rate than did controls. This suggests that over time the deep relaxation of the Transcendental Meditation program begins to be carried over into activity outside meditation.

Numerous studies have shown that regular practice decreases stress and anxiety and enhances physical and mental health. An exhaustive statistical meta-analysis of 146 independent outcomes from Transcendental Meditation practice and other forms of meditation and relaxation found that the effect of the Transcendental Meditation program on trait anxiety, an indicator of chronic stress, was significantly greater than that of other techniques ([11]).

During practice of the Transcendental Meditation program the deep relaxation is accompanied by heightened mental alertness. For example, alpha and theta EEG power and coherence increase (see, for example, [21]). Heightened EEG coherence during Transcendental Meditation
practice has been found to be significantly correlated with enhanced H-reflex recovery, concept learning, creativity, fluid intelligence, moral reasoning and decreased neuroticism (see, for example, [27]). That reaction time is faster for meditators than for controls (see, for example, [5]) suggests that this alertness is also carried over into activity outside meditation.

A wide range of studies have found that practice of the Transcendental Meditation program has a positive influence on cognitive and personality development. For example, developmental changes through Transcendental Meditation practice include enhanced flexibility of perception and verbal problem solving ([7]), increased creativity ([34]), increased fluid and analytical intelligence (remarkably, Maharishi University of Management students increased in standard IQ 5 points in 2 years and 9 points in 4 years, while students at other universities showed no change) (see, for example, [5]), improvements in overall academic achievement ([25]), and greater field independence [28]. Cranston’s study ([5]) also indicated development of broad comprehension (simultaneous processing of the elements of a complex stimulus field) and the ability to focus sharply (quick and accurate responses).

High scores on the Personal Orientation Inventory (POI), a standard test of self-actualization, have been correlated with creativity. ‘Self-actualization’ was proposed by Maslow to describe full development of the unique potential of the individual self. He and others claim that only a very small proportion of adults achieve self-actualization. A recent exhaustive meta-analysis of over 40 studies on the POI found that the effect produced by practice of the Transcendental Meditation program on overall self-actualization was very much larger than that produced by other forms of meditation and relaxation ([2]).

Many of the above factors have been shown to affect mathematical education specifically. For example, field independence ([20], [32], [35]), intelligence ([17]), and self-concept ([31]) have been found to correlate positively with mathematical achievement, whereas field dependence ([16]) correlates with math anxiety, and building a positive and realistic self-concept can prevent math anxiety ([6]).

The above is only a small sample of research available on the effects of the Transcendental Meditation program applicable to mathematics education. The interested reader is referred to the following com-
prehensive overviews for the principles, practice and research on this approach to education: primary and secondary education[24], tertiary education[19], and education generally[9]. The five-volume collection [26], [3], and [36] contains a broad spectrum of the published research on the Transcendental Meditation program. For an in-depth treatment of the effects of the Transcendental Meditation program in ‘unfreezing’ human development, that is, in developing higher cognitive stages beyond formal operations, see [1].

All of this research supports the traditional goal of meditation. Maharishi has pointed out ([23], pp. 135–138) that the goal of meditation is expressed in the Bhagavad-Gita as “Established in yoga [that is, in restful alertness], perform action.” In other words, the purpose of meditation is to establish a coherent state of mind and body as preparation for more dynamic effective activity.

Maharishi describes the role of education in developing in students the ability to spontaneously think deeply on quieter levels of the mind, where knowledge is stored (Lecture, May 5, 1971). He explains that as one’s attention goes to the quieter levels of the mind, the quiet, comprehensive level of the mind is enlivened, where deep connections are made. The purpose of education, he says, is to enliven those quiet levels.

In this connection, a previous paper ([10]) describes how regular experience of transcending during Transcendental Meditation trains the mind to settle spontaneously to that quiet powerful level of the mind in which the spontaneous step of illumination can take place. This is the step of the four-step creative process described by Poincaré ([30]), Littlewood ([22]), and others, that occurs (if one is lucky) after the steps of preparation and incubation. It is the step in which the mind spontaneously integrates and expands new information and existing mental structures into a whole new mental structure and one directly grasps a solution to a problem or understands a proof or concept.

Let us now turn from development of the inner resources of students as a preparation for calculus to the development of schemas that can serve as ‘hooks’ for the infinitary processes of calculus, so that these deep concepts can be easily assimilated.
Making the Concept of Infinity Intuitively Acceptable

Most calculus teachers appreciate the need to make the concepts of calculus intuitively acceptable to their students. Is this possible for a concept such as infinity? Perhaps it is possible for the Archimedean concept of potential infinity, the idea that a process can be repeated indefinitely. However, it appears that it has not been possible for Cantor’s concept of actual or completed infinity. Let us look at the importance of intuitive acceptance more closely. (For a lively, insightful, comprehensive discussion of the role of intuition in the highly integrated process of intellectual analysis and intuitive insight by which we learn and do mathematics, I highly recommend the work of Fischbein ([12], [13], [14], [15]). See also [10].

Fischbein ([12], p. 10) has applied the term ‘intuitive’ to an explanation or interpretation directly accepted by us as natural, self-evident, intrinsically meaningful. He points out, for example, that most people find the equality of opposite angles formed by two intersecting lines highly intuitively obvious, but find the fact that the sum of the angles of a triangle is equal to two right angles is not. Fischbein has researched extensively the need to provide students with an intuitively acceptable explanation of a proof or to provide a ‘basis of belief’, based on active personal involvement, along with or prior to a formal step-by-step logical argument. In [12] he explains,

A mathematical truth can become really effective for productive mathematical activity if, together with a formal understanding of the respective truth, we can produce that kind of synthetic, sympathetic, direct acceptability of its validity. The same holds for a concept, a statement, a proof, and for the basic principles of generalizing, deducing, and proving in mathematics. (p.18)

Without a holistic, direct feeling for the correctness of a proof, he finds, students tend not to be able to remember it nor to use it effectively later on.

A related idea is Skemp’s ‘learning with understanding’([33]). In mathematics, Skemp says, agreement depends on pure reason, not on authority. Therefore, one must learn mathematics with understanding, not by rote. One must establish mental patterns, or schemas, rather than memorize facts. Only then will the knowledge be useful later on.

The idea of a ‘hook’ on which to hang a new concept was well illustrated by Millie Johnson in her delightful contribution to the recent
Conference on the Teaching of Calculus ([18]). Everything that could possibly come to hand, from beans to baby shampoo bottles, was used as raw materials for hands-on experiments that would give her students a feel for concepts and solutions to calculus problems.

Thus in order for the infinitary concept of limit at the core of calculus to be understood and useful later on, a ‘hook’ or ‘basis for belief’ based on active personal involvement must be found. Does one exist?

Fischbein ([15]) asked students in grades 8 and 9 the following question:

“Given a segment $AB = 1m$. Let us suppose another segment $BC = \frac{1}{2} m$ is added.

Let us continue in the same way, adding segments of $\frac{1}{4} m$, $\frac{1}{8} m$, etc.

(a) Will this process of adding segments come to an end?

(b) What will be the sum of the segments $AB + BC + CD + \ldots$?”

The students were quite comfortable with (a), confirming Poincaré’s views ([29]) on potential infinity:

Why then does this judgment force itself upon us with an irresistible evidence? It is because it is only the affirmation of the power of the mind which knows itself capable of conceiving the indefinite repetition of the same act when once this act is possible. The mind has a direct intuition of this power, and experience can only give occasion for using it and thereby becoming conscious of it. (pp.23–24)

The same students were very uncomfortable, however, with (b). Fischbein’s explanation of this is that “Infinity exceeds every real human (mental or practical) experience” ([15], p. 497), and,

For our usual logic and for an intuitive understanding, infinity is only a potentiality. An actual infinity (as it appears in the Cantorian approach) has no intuitive representation. The concept of actual infinity is intuitively contradictory. (p. 506)

Understanding limits requires students (and teachers!) to feel at home with actual infinity, with actually taking the limit, with manipulating several limits at once by adding them, multiplying them, composing them as in the chain rule for differentiation. Definitions are stated and proofs are given in terms of potential infinity, but to fully comprehend
and motivate them, a feeling for completed infinity is required. A striking example is in applications of integration in which one wants to find some quantity, such as a volume. The standard procedure is to imagine how to break that volume into small pieces or slices, to approximate the volumes of those small pieces by volumes of related small pieces whose volume is easy to find by finite methods, then to add up the approximations, and take the limit of the sum as the size of the pieces approaches zero (that is, calculate an integral). The result is hopefully the volume sought. In this process it is essential that the student be able to conceive of the quantity being sought as a completed infinity.

How can one give students an intuitive feel for infinity? I claim that students who practice the Transcendental Meditation program have an experiential basis for both potential and completed infinity. The spontaneous transcending process that takes place several times during each sitting of meditation provides one. As the mind settles, one spontaneously experiences subtler and subtler, fainter and fainter levels of thought until thought vanishes altogether and the knower is left alone with himself. The transcending process is analogous to the passage to a limit. And the state of transcendental consciousness, in which all thought has been left behind, is analogous to the limit itself, to completed infinity. The process of transcending does come to its goal, a completely non-representational state. Awareness, by virtue of being aware, is aware of itself alone. This is not awareness thinking about itself, it is direct experience of wakefulness itself, unbounded awareness. In that state one is beyond, deep, infinite. One doesn’t observe this state; one is this state.

In a calculus class at Maharishi University of Management, we simply point out the similarity between the transcending process they experience every day in meditation and the limit process of calculus. It takes only a minute of class time, and it makes all the difference in how our students feel about limits. Limits are no longer some strange, remote concept thought up by mathematicians; suddenly limits are a concept completely intimate to themselves, an aspect of their own inner experience. They realize “Hey, this course is about me.” This naturally makes them very excited about calculus. Both potential and completed infinity appear to be perfectly acceptable to students who practice the Transcendental Meditation program.
Many concepts in calculus are related to more universal principles common to all the disciplines and to the growth and expression of human consciousness. In every class we take a moment to draw a parallel between the major principle covered in the lecture and one of these principles. These principles are drawn from the body of theoretical knowledge known as Maharishi Vedic Science, which describes the unfoldment of intelligence from its source in that state of pure consciousness experienced during the Transcendental Meditation program. (See [4] for an introduction to Vedic Science.) These simple fundamental principles governing the expression of intelligence in creation provide a valuable source of insight into mathematical concepts and processes, functioning as a body of established schemas, or intuitions. (See [37] for a discussion relating some of these principles to foundational areas of mathematics.)

It is important to note that Vedic Science works as a common body of principles because every principle is based in the student’s own experience, not simply on intellectual understanding of the principle. The principles of Vedic Science are actually established schemas, available as ‘hooks’ for the deep concepts of calculus.

Students are further connected to the content of calculus at Maharishi University of Management by classroom charts that place topics covered in the lesson into the context of the whole lesson, place the lesson into the context of mathematics as a whole, place mathematics into the context of the wholeness of all knowledge, and connect the wholeness of all knowledge to its basis in the student’s own consciousness. Although students focus sharply on calculus, they never lose sight of the most global perspective on the topic being taught, and never lose sight of themselves, as the source of the knowledge.

Lest the reader fear that the content of calculus may be diluted or lost in this approach, I hasten to point out that the mathematics is kept separate from Vedic Science. Statements about mathematics are just about mathematics and principles from Vedic Science use only the language of Vedic Science. Except for one or two minutes, the class consists of the mathematical treatment of calculus topics. Only one or two minutes are devoted to giving the one or two main mathematical points of the lesson and to pointing out parallel principles from Vedic
Science. Practice of the Transcendental Meditation program technique itself is completely separate from class time.

Conclusion

In this paper I have argued that Maharishi University of Management addresses directly two important aspects of the teaching of calculus that educators have had difficulty addressing directly before: providing the student with a clear alert mind capable of deep abstract thinking, and providing the student with an experiential intuitive basis for grasping the deep concepts involved in the limit process, particularly the concept of actual, or completed, infinity.

Both are provided at Maharishi University of Management by the practice of the Transcendental Meditation technique. This practice makes the process of education complete by developing the knower along with the process of knowing and the object of knowing, that is, by developing the inner resources of the student along with developing the calculus syllabus and developing the most effective approaches to teaching calculus. This makes the knowledge gained in our calculus courses useful and fulfilling. In the words of Maharishi Mahesh Yogi, the founder of Maharishi University of Management, “When the knower does not know himself, then all knowledge is baseless. And baseless knowledge can only be non-fulfilling.” (Lecture, August 1, 1986)

On the basis of my own experience with calculus students at Maharishi University of Management, I highly recommend that calculus teachers take Maharishi University of Management’s approach seriously by introducing their students to the practice of the Transcendental Meditation technique. If the new approaches to calculus suggested by today’s calculus reform are supported by this practice, they should be successful in producing students who fully comprehend the content, scope, and power of calculus and who will be able to use it easily later on. Furthermore, teachers will find that their students actually enjoy calculus—and that they themselves enjoy teaching it.

Only by including development of the inner resources of our students can we as teachers of calculus do justice to our students and to the marvellous power of calculus given to us by the great mathematicians of the past few centuries.
References


Using the Study of Consciousness
to Teach Calculus

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ABSTRACT
This article suggests a new approach to the problem of teaching students the core abstract insights of calculus, based on the method of teaching at Maharishi University of Management. This new approach makes use of two observations: First, abstract ideas are themselves expressions of consciousness; second, as has been verified by numerous scientific studies, the quality of consciousness itself may be refined through the use of the Transcendental Meditation program. These observations suggest that if mathematical concepts, such as limits, the continuum, the derivative, and the integral, can be appreciated as structures of consciousness, then, as consciousness itself becomes more refined, these concepts can be grasped more easily and profoundly. This is the approach that has been successfully implemented at Maharishi University of Management.

There have been many definitions of mathematics, none of which is totally adequate and certainly none of which would be agreed upon by a majority of mathematicians. One wry mathematician, no doubt trying to avoid the issue, has said, “Mathematics is what mathematicians do.” This remark points out that aspect of mathematics which is often overlooked or taken for granted, and which I want to emphasize here, the human element or consciousness of the mathematician.

Like mathematics, consciousness is a word that eludes definition. For the purposes of this paper, we will assume that it includes the full potential open to the mind: waking, dreaming, and sleeping, as well as experiences of “higher” states of consciousness, states of increased physiological and psychological balance and integration which are as yet hypothesized. I won’t be concerned with the idea that mathematical concepts may have a real existence apart from human consciousness; there are too many excellent arguments on either side. I will, however, explore the idea that mathematics, as we know it and do it, is based completely in our own consciousness, a view that has been supported by mathematicians since the origin of mathematics itself. In the words of several mathematicians:

This branch of study [the study of calculation] really seems to be indispensable for us, since it plainly compels the soul to employ pure thought with a view to truth itself. —Plato, The Republic, 7.5266, 1930
A mathematician . . . has no material to work with but ideas. —G.H. Hardy, 1940

Mathematics is thought moving in the sphere of complete abstraction from any particular instance of what it is talking about. —Alfred North Whitehead, 1925

Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. —Courant & Robbins, 1996

**What is Mathematics?**

These mathematicians all indicate that mathematics is intimately associated with the thinking process. Any concept of mathematics is a structure of consciousness itself, so the student’s comprehension of a concept involves not only intellectual skills but an appreciation of how the concept fits in with the student’s own consciousness. In calculus, the key concepts are the continuum, the limit, the derivative, and the integral. How can the student come to structure in his or her own consciousness these ideas, ideas that are the result of careful refinement over hundreds of years, from the clear intuition and profound insight of Newton and Leibniz to the rigorous axiomatization of the continuum done in this century?

At Maharishi University of Management, the approach to understanding these fundamental concepts is by relating them to the student’s experiences and understanding of his developing consciousness. Consciousness is viewed as a real, concrete value of human existence that can be studied in an objective, scientific way. The Transcendental Meditation and TM-Sidhi program is practiced by each student at Maharishi University of Management as a way to experience and develop a greater range of states of consciousness. The theoretical framework giving an intellectual basis to this practice is studied in the Science of Creative Intelligence, founded by Maharishi Mahesh Yogi. Thus, students not only have experiences of consciousness available to them, but they also have a scientific approach to the study of these experiences.
To use this understanding in the classroom, each mathematical concept is viewed as a structure of consciousness. As such, each concept must reflect to some degree the properties of consciousness. (See, for example, Hamming, 1980.) If those properties of consciousness underlying an important mathematical idea can be located and related to the students in terms of direct experiences of their own consciousness, there is no doubt that they will find this idea as relevant and exciting as when it was first discovered. To illustrate this, I would like to describe how the concepts of continuum, limit, derivative, and integral can be related to properties of consciousness.

**The Continuum** Basic to calculus is the completeness of the continuum, the property that brings together the continuity of the straight line of geometry and the discreteness of the numbers of algebra. Students are usually quite at home with the rationals and can easily see how between any two rationals, there must be another rational. The proof that $\sqrt{2}$ is irrational is short and simple, so it is clear that the rationals cannot name all the points on a straight line. At this point, however, the completeness of the real numbers depends on some construction, such as the Dedekind cut, which is not appropriate in an introductory calculus class.

We can convey the idea more intuitively by comparing the straight line to the continuum of consciousness. We experience thoughts, one after another, in sequence much like the integers, and these thoughts are structured in a continuity of awareness, much like the straight line. We can isolate a particular thought or perception in our awareness, but there is some backdrop to experiences, something which we could call the self. The self maintains itself continuously while thoughts or perceptions come and go. Seeing this comparison between the continuum of the real numbers and the continuity of his or her inner self, his or her own consciousness, the student can grasp intuitively the mathematical nature of the continuum, uniting the discrete and the continuous. In this, a student’s experience with the Transcendental Meditation program and TM-Sidhi programs is valuable. The Transcendental Meditation technique allows one to experience the underlying continuum of consciousness alone, in its pure state, without the presence of a thought. This gives a greater appreciation of
the basis of pure consciousness or pure awareness that underlies all our activity.

An analogy helps here. You can describe to your friend, seeing a film for the first time, how the pictures are formed by colored light falling on a white screen, and your friend will probably understand the process even though he may not have a good idea of what the screen looks like. But once the film is over and the lights are turned on, the screen will be clearly visible, and your friend will know exactly what you were talking about. Pure consciousness is like the cinema screen, with thoughts and perceptions constantly appearing; it is not until these thoughts and perceptions can cease their activity, while awareness is yet maintained, that the true nature of pure consciousness can be perceived.

This state of pure consciousness, the state of awareness without the activity of thinking, has been proposed by physiologists to be a fourth state of consciousness, different from waking, dreaming or sleeping. Subjectively, this state is experienced as unbounded awareness or as non-changing, abstract bliss, complete wholeness, one’s deepest inner nature; this experience is recognized as more fundamental than the experience of waking-state consciousness. By comparing the real number line to this most fundamental level of pure consciousness, the student gains an intuitive grasp of the nature of the continuum and understands that the structure of the real numbers is more profound and of a different quality than the structure of the integers or the rationals.

**Limits** The idea of what a limit really is often gets lost by students once they become so adept at algebraic manipulations that computing a limit as the variable \( x \) approaches some fixed value \( a \) is nothing more than “plugging in” \( a \) for \( x \). The idea of completing an infinite process and the distinction between \( x \) as it approaches \( a \) and \( x \) when it is equal to \( a \) can be made very clear to students by making a parallel with the experience during the Transcendental Meditation technique which I will now describe.

To begin meditation, one introduces a specific thought (mantra) and then, in a completely natural and effortless way, begins to experience this thought at quieter, less active levels of the mind. The mind remains fully awake and alert as the thinking activity of the mind becomes less and less, until finally all activity of the mind has ceased. This process is
referred to as transcending, going beyond; the state where the mind is quiet yet still awake is called transcendental consciousness. Both subjectively and as measured by physiologists, transcendental consciousness is distinctly different from waking-state consciousness. Transcendental consciousness has been described as unbounded, transcendental to the limitations of the waking state.

We can now compare taking a limit to the process of transcending. In taking a limit like \( \lim_{x \to a} f(x) \) we must let \( x \) get closer and closer to \( a \) without ever reaching \( a \). Letting \( x \) get closer and closer to \( a \) is an infinite process; the limit represents the completion of an infinite process. For a Transcendental Meditation meditator, this idea of the completion of an infinite process is a very real experience. In meditation, as the mantra becomes finer and mental activity becomes less, the mind begins to perceive more and more of the pure consciousness that exists at its base. Pure consciousness grows more and more in the mind until it is experienced as infinite and unbounded; the idea of taking a limit and getting a result is thus a real and concrete experience.

In the process of taking a limit, we may never let \( x = a \), so expressions like \( \frac{1}{x - a} \) are well-defined and the rules of algebra can be used to simplify such expression. However, once we have taken the limit, the expression \( \frac{1}{x - a} \) becomes undefined (or labeled “\( \infty \)”). This transition, from \( x \not= a \) to \( x = a \) is like the transition between states of consciousness. Each state of consciousness has unique properties distinguishing it from the others, both in terms of subjective experience and objective measurements (EEG, metabolism, etc.). In dealing with a limit as long as we are in one state (\( x \not= a \) or \( x = a \)) one set of algebraic rules apply; as soon as we change states, another set of rules applies. Transition from one state of consciousness to the next is brought about by the functioning of the nervous system in the first state; transition from \( f(x) \) when \( x \not= a \), to the actual limit itself, \( \lim_{x \to a} f(x) \) is brought about by considering the behavior of \( f(x) \) when \( x \not= a \). Even though this analogy works when we consider any two states of consciousness, it is particularly appropriate to use the transition from waking state consciousness to transcendental consciousness during the Transcendental Meditation technique because awareness is maintained throughout the process.
The Fundamental Theorem of Calculus. The Fundamental Theorem of Calculus describes the way in which taking the derivative (finding the instantaneous rate of change of a function or, equivalently, finding the slope of the tangent to the graph of the function) is inverse to integrating (finding the area under a function). Most students are frequently puzzled by this theorem and do not have any conceptual framework to put it into, perhaps because they do not have a conceptual framework for the idea of limit. Using the above discussion of limit, it will be very simple to see how the processes of differentiation and integration are related.

Differentiation and integration, as limiting processes, can be compared to transitions from one state of consciousness to another. The experience during the Transcendental Meditation technique of going from waking state to transcendental consciousness is used at Maharishi University of Management because it is so clearly experienced by the students, but certainly other examples could be used. Going from waking state to transcendental consciousness is called the inward stroke of meditation; a state of consciousness is reached where there is only the potential for activity. A derivative gives “instantaneous velocity.” There obviously can be no motion at a point, but the lively potential for activity at a point is measured by the derivative. Thus, the inward stroke of meditation is parallel to differentiation. Going from transcendental consciousness back to waking state is called the outward stroke of meditation; the potential for activity that was not active in transcendental consciousness is brought out into the activity of the waking state. As one meditates day after day, it is found that this latent potential of the mind is brought more and more into activity until the hypothesized state of full enlightenment is achieved. When a function $f(x)$ is integrated, it is regarded as a derivative, an expression giving potential for activity or motion at every point; the integral $\int f(x)\,dx$ represents the activity produced by this lively potential. Thus it can be seen that the outward stroke of meditation is parallel to integration.

In the student’s own experience, the inward and outward strokes of meditation are opposite or inverse to each other. The inward stroke is a settling down of the active mind to its pure potentiality; the outward stroke takes the mind from this silent level back into thinking, perceiving and acting. Both the inward stroke and the outward stroke
are essential to the Transcendental Meditation technique. The inward stroke comes first and is necessary for the outward stroke. In calculus, which is the study of relationships and change, both differentiation and integration are essential. Furthermore, practice and skill in differentiation is necessary for skill in integration. Looked at in this way, the Fundamental Theorem of Calculus is parallel to everyday experiences of consciousness and instead of being puzzling, is familiar and reasonable.

**Conclusion** When the fundamental concepts of calculus can be presented in terms of consciousness, a student’s most intimate experience, they take on a much deeper relevance than can be achieved even by the most skillful use of applications. Studying consciousness becomes a way to more fully understand calculus. And as the student finds calculus a means to quantify and describe precisely the experiences of consciousness, growth of consciousness naturally occurs. Students can now regard the study of calculus as a relevant learning experience.

**References**


Section VI

Maharishi Vedic Science
as a Research Tool
for Modern Mathematics
Vedic Wholeness and the Mathematical Universe:

*Maharishi Vedic Science* as a Tool for Research

in the Foundations of Mathematics

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Paul Corazza, Ph.D., received his Bachelor of Arts degree in Western Philosophy from Maharishi University of Management (formerly Maharishi International University) in 1978 and his M.S. and Ph.D. degrees in Mathematics from Auburn University in 1981 and 1988, respectively. He was awarded a Van Vleck Assistant Professorship at University of Wisconsin for the years 1987–1990. He worked in the Mathematics Department at Maharishi University of Management in the years 1990–95. Following a career as a software engineer, he rejoined faculty at Maharishi University of Management in 2004 and currently serves a joint appointment in the Departments of Mathematics and Computer Science. Dr. Corazza has published more than a dozen papers in Set Theory, focused primarily on the quest for providing an axiomatic foundation for large cardinals based on a paradigm derived from Maharishi Vedic Science.
ABSTRACT

In this paper we make use of Maharishi Vedic Science as a tool to consolidate mathematical intuition about the structure of the mathematical universe and the nature of mathematical infinity. We consider the inability of the standard axioms of set theory to account for the presence of large cardinals in mathematics as a serious failing and suggest that the shortcoming at the root of this failure is the omission of any axiomatic principle describing the nature of the wholeness of the universe $V$. We then formulate such an axiomatic principle, called the Wholeness Axiom, which is based on insights into the nature of wholeness derived from Maharishi Vedic Science and from the dynamics suggested by the strongest large cardinal axioms, well-known to set theorists. We illustrate how the universe $V$ exhibits new dynamics in the presence of the Wholeness Axiom, more in accord with the dynamics of wholeness described in Maharishi Vedic Science. We then show that virtually all known large cardinal axioms are naturally accounted for by this new axiom. We conclude that Maharishi Vedic Science, used in conjunction with the frontiers of modern mathematics, can provide the profound intuition needed to build a truly successful foundation for all of mathematics.

§1. Introduction

If the expansion of Rishi, Devata, and Chhandas into the infinite universe does not remain in contact with the source, then the goal of expansion will not be achieved. —Maharishi (1991a)

And do you not also give the name dialectician to the man who is able to exact an account of the essence of each thing? And will you not say that the one who is unable to do this...does not possess full reason and intelligence about the matter? —Plato (The Republic SN 534)

For nearly 100 years, mathematicians interested in the foundations of mathematics have sought a simple set of axioms from which the rest of mathematics could be derived. Georg Cantor, the founder of modern set theory, was among the first to notice that the fundamental concepts used in mathematics—numbers, points, lines, circles, ordered pairs, functions—could be formulated as sets. His insight led to the conclusion that a theory of sets could provide a foundation for mathematics.
Unfortunately, in Cantor’s time, the notion of sets was not well understood; the common idea that a set is simply any collection of objects led to logical contradictions. No direct definition of set seemed to avoid basic paradoxes. As an alternative, mathematicians at the turn of the century devised a set of axioms that would describe properties that sets ought to have; these axioms would then provide a basis for proving theorems about sets, and, hence, about all objects of study in mathematics.

The set of axioms which has become most widely accepted as the foundation for set theory is known as Zermelo-Fraenkel Set Theory with the Axiom of Choice, or ZFC for short. In addition to setting forth basic properties of sets, these axioms have, buried within them, “instructions” for building a universe of sets, a universe in which all mathematical objects could, in principle, be located. In order to indicate that the construction of sets begins with the merest point value, the empty set, and expands outward to generate all possible sets, the universe of sets is denoted by the letter $V$.

As a unifying foundation, ZFC, together with its universe $V$, has been highly successful. Yet, in the past few decades, several advances in mathematics have challenged its adequacy as a foundation. One of the most serious concerns has been the discovery of extremely large infinite sets, called large cardinals, whose existence cannot be proven from ZFC, yet whose central presence in a significant portion of mainstream mathematics makes it unreasonable to simply deny their existence. It was the hope of many set theorists that an “intuitively evident” principle would emerge that would provide sufficient motivation for including (or excluding) large cardinal axioms among basic axioms of set theory. Efforts to formulate such motivation have been only partially successful; the problem has been that there is no fundamental intuition concerning the nature of enormous mathematical infinities that is generally agreed upon by experts in Foundations—even less so among mathematicians generally.

Traditionally, mathematicians have derived their mathematical intuition on the basis of long years of experience with the objects of study in their respective fields. Certainly the axioms of ZFC arose from an intuitive familiarity with sets; the axioms had to be formulated so as to preserve this familiarity while eliminating undesirable paradoxes.
But how does one decide, on an intuitive basis, whether certain types of enormous infinities exist or belong in the universe? An evaluation of the consequences of assuming—or not assuming—that various large cardinals exist has not helped to answer the question.\footnote{As we shall see, none of the other attempts to find an answer to this question have been successful either.}

The general feeling in the set theory community concerning the universe of sets is that it is supposed to represent, in an imprecise sense, the “real” world. Sets in the universe should combine the way we expect “real” sets in the “real” world to combine. This “real world” is a combination of the natural world and the world of mathematics as it has developed through its long history.\footnote{According to Maddy’s account (1988b, p. 758), our basic intuitions concerning mathematical objects like sets begin with our first perceptual encounters with objects in the world and then are shaped by the mathematical concepts and training we encounter later.} Certainly, observing the physical world tells us how to form the union of two disjoint sets and how to extract a subset from a given set. On the other hand, mathematical experience is required to form and study the collection of all subsets of a given set. Likewise, although most people are not accustomed to locating anything infinite in nature, still, mathematical experience guides the mathematician to postulate that indeed there is an infinite set.

When mathematicians try to decide about whether the universe should include large cardinals, however, they are faced with a unique problem: Nature does not provide well-known examples of enormous infinities, and mathematical experience, although it can provide an intuitive feel for mathematical consequences of large cardinal axioms, does not equip the mathematician to decide whether such cardinals should exist. Indeed, P. Maddy (1988a, 1988b) carried out a fascinating survey of philosophical justifications for large cardinals; her work detailed virtually all known intuitive principles that have ever been used to justify the better known large cardinal axioms. Each principle has clear intuitive motivation but succeeds in justifying only a very few of these large cardinal axioms. As Maddy herself aptly remarks, “...the axiomatization of set theory has led to the consideration of axiom candidates that no one finds obvious, not even their staunchest supporters” (1988a, p. 481).

We can imagine a number of different reasons for this wide variation in the mathematical intuitions that guide set theorists in their attempt
to answer the deepest questions about the structure of the universe and the Infinite. One reason could be, as a formalist might argue, that there is no basic underlying reality about which to have clear intuitions in the first place; talk about the “right” foundation for mathematics should not be understood as a commitment to believe that there is some underlying “reality” that is being “described,” but rather as a device to motivate new and interesting—but purely formal—mathematical systems.

On the other hand, if indeed there is an underlying truth which mathematicians with growing clarity are glimpsing as they formulate ever more fundamental axioms for mathematical foundations—and this was certainly the view of Plato, Cantor, and Gödel—then it may well be that foundational experts are glimpsing this basic reality with quite different levels of lucidity. Certainly, Cantor and Gödel are examples of mathematicians whose intuitions went far deeper than those of their contemporaries in foundational matters. Indeed, Plato (1961, *Republic*, SN 518) anticipated such disparities as inevitable; according to his account, these variations are due to varying degrees of skill in the use of those mental faculties which allow clear perception of these underlying mathematical realities. For those whose intuitive faculties are in a sense sleeping, this underlying mathematical reality will be simply a fiction, much as a blue sky must remain a fiction for one deprived of the sense of sight. For those whose faculties are awake, the reality of mathematical objects and those forms which give rise to them will comprise a quite lively reality, which they could hardly consider refuting as a fiction,

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3 Plato held that the objects of mathematical study, however much they may resemble objects in the physical world, properly belong to an independent timeless world beyond the senses, apprehended by a higher faculty of reason (for instance: “... geometry is the knowledge of the eternally existent” (*Republic*, SN 527; also, cf. Plato’s *Meno*). Moreover, he held that, whereas ordinary mathematics begins with certain unquestioned assumptions and derives rigorous conclusions from these, the process by which these assumptions are themselves questioned and transcended activates a new kind of knowing in which the highest level of reality begins to be known (cf. *Republic*, SN 510–511).

4 M. Hallett (1988) remarks: “As Cantor himself says (1883/1980, p. 206, n. 6), what he proposes is a Platonic principle: the ‘creation’ of a consistent coherent concept in the human mind is actually the uncovering or discovering of a permanently and independently existing real abstract idea.”

5 According to Gödel (1948/1983), “Evidently the ‘given’ underlying mathematics is closely related to the abstract elements contained in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality” (p. 60).
even as one having the sense of sight would be unable to deny the existence of a lustrous blue sky.

Whether or not this Platonic view of mathematical reality is correct, it certainly has led to fruitful mathematical consequences. Gödel, for example, claims to have arrived at his famous completeness and incompleteness theorems in logic precisely because of this Platonic world view; he felt that, had he viewed the symbolism of Peano arithmetic as mere formalisms, he would never have made his discoveries (see Wang, 1974). Indeed, according to Moschovakis, it is the experience of nearly all mathematicians—whether they admit it or not!—that a world view which takes mathematical objects to be “real” and which views theorems as discoveries rather than inventions accords more with their experience of creative research than does a more formalistic view. There is also a certain amount of evidence within foundational studies themselves that suggests that mathematicians are “discovering” a mathematical landscape rather than “inventing” it; we have in mind the remarkable confluence of mathematical methods and insights that occurs in the large cardinal hierarchy. It has been observed by many set theorists that the fact that this hierarchy of principles, demarcating the extensions of ZFC in terms of their consistency strength, form a linear order and yet arise from such diverse mathematical considerations suggests that something profound about the hidden fabric of mathematics has been unearthed.

For these reasons, we take the view in this paper that there is indeed an underlying reality that set theorists and experts in foundations are, consciously or not, attempting to describe; and we adopt this point of view for the pragmatic reason that this stance has historically proven to lead to more interesting and deeper mathematics than the opposite view. We propose to lay the groundwork for a program of research that

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6 “The main point in favor of the realistic approach to mathematics is the instinctive certainty of most everybody who has ever tried to solve a problem that he is thinking about ‘real objects,’ whether they are sets, numbers, or whatever; and that these objects have intrinsic properties above and beyond the specific axioms about them on which he is basing his thinking for the moment” (Moschovakis, 1980, p. 605).

7 See Weinless’s (2011) discussion on Vedic objectivism for an approach that uses Maharishi Vedic Science (pp. 157 ff) for his discussion of Vedic objectivism—an approach that uses Maharishi Vedic Science as a basis for a philosophy of mathematics that takes “mathematical reality” to be as “real” as any “physical reality,” since these are all simply expressions of the internal dynamics of the field of pure consciousness.
will (1) clarify the nature of this reality and determine its structure and salient characteristics, and (2) use these insights as the basis for building a new, enriched, consistent foundation for mathematics that accomplishes the purpose of foundations.

How are we to study the “underlying reality” of mathematics? From our observations so far about research in Foundations, it should be clear that the depth and clarity of intuition in this sphere of mathematical endeavor tends to vary widely from one mathematician to the next. For most specialized mathematical endeavors, a disparity in facility at an intuitive level balances out as experts keep abreast of the main new theorems and proofs in their fields. In Foundations, however, there is a need for more uniformity of vision; to formulate the right axioms we need to be seeing the same reality with equal clarity. More theorems using the same old axiom system will not in any significant way equalize our collective vision.

We suggest that the reality that the deepest thinkers in Foundations have been glimpsing on an intuitive basis and have been attempting to express through various axiom systems like ZFC has in fact already been systematically investigated by great seers throughout history. We feel that the deep research of these exceptional individuals has tended to be overlooked by those working in the foundations of all the sciences.

In this paper, we propose to make use of the most ancient of these systematic investigations, the Veda, revived by Maharishi Mahesh Yogi in the form of Vedic Science, as a new, explicit source of intuition for advancing the current research into intuitive principles on which to found set theory and account for large cardinals. This tradition, like many others, asserts that the vast diversity of the universe is the expression of a single wholeness; it offers a detailed account of the structure of this wholeness and its relationship to the manifest universe.

Among the many traditions of knowledge that speak of such a wholeness, we have chosen the Vedic tradition for several reasons. First, this tradition provides the most comprehensive extant treatment of the

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8 The traditions of knowledge of every culture include insights and information about the fundamental intelligence underlying nature. Among these, the tradition most often linked with modern mathematics is ancient Greek philosophy, most notably, Plato’s philosophy. Plato’s philosophy offers a wealth of insight about the ultimate nature of existence as a fundamental wholeness; see his discussions of the *good* and the *one* in the *Republic*, *Parmenides*, *Timaeus*, *Phaedrus*, and *Sophist* (Plato, 1961).
nature of wholeness, not only from the point of view of detailed insights and information, but also because it provides the procedures and technologies needed to awaken and refresh individual awareness\(^9\) so that these truths of wholeness may be perceived directly; secondly, these procedures, unlike those of many other traditions, are readily available to individuals everywhere, so that truths about wholeness need not be taken merely as articles of faith; and finally, there is at least some evidence that all the most deeply cherished traditions of knowledge of mankind may have had their origin in the Vedic tradition.\(^{10}\) Whether or not this is the case, the many connections and similarities to be found in comparing the Vedic tradition with those that have historically followed it suggest, at the very least, that these traditions are giving expression to the same basic reality.

The central truth unfolded through Maharishi Vedic Science is that the natural world, from the microcosm to the macrocosm, is the lively expression of a fundamental infinite wholeness; that this wholeness has its own qualities and dynamics; that it can be experienced directly and effortlessly as the most intimate, quiet level of one’s own awareness; and that direct experience of this wholeness enlivens the entire range of its life-nourishing qualities and dynamics within the individual and the society in which he or she lives, resulting in a more successful, powerful, fulfilling, and stress-free life.

Maharishi Vedic Science has been used successfully by physicists, most notably John Hagelin (1987, 1989), to understand and motivate research into the functioning of nature at its deepest levels. Indeed, recent research in quantum field theory has led to the discovery that all the fundamental force and matter fields of nature are expressions of a single, infinite, self-interacting, highly energetic, self-created, completely unified superstring field. This field, in any of its formulations, has been shown by Hagelin, in collaboration with Maharishi, to exhibit the very qualities and dynamics that characterize the field of pure consciousness as it has been portrayed in the Vedic tradition of knowledge.

\(^9\) The value of, and, indeed, the need for such technologies is recognized in most traditions of knowledge in the world. For the reader most familiar with the Western tradition of knowledge, consider these remarks of Plato: “. . . there is in every soul an organ or instrument of knowledge that is purified and kindled afresh by such studies when it has been destroyed and blinded by our ordinary pursuits, a faculty whose preservation outweighs ten thousand eyes, for by it only is reality beheld” (Republic, SN 527).

\(^{10}\) Cf. Mead (1965, pp. 9–25).
Hagelin (1987) argues strongly in favor of the contention that indeed, the unified field discovered by modern physics is nothing other than the field of pure consciousness described by the ancient texts of the Veda.

Our plan is to use this Vedic vision of wholeness as an intuition to guide the construction of the universe $V$ and to account for large cardinal axioms. We will begin by examining the structure of the universe $V$, as it is currently understood in modern mathematics, in the light of Maharishi Vedic Science. We will see that in some respects, the dynamics of unfoldment of $V$ directly parallel those of pure consciousness in its expression into manifest existence, and in certain other respects the model falls short. Then, using principles from Maharishi Vedic Science, we will formulate a new axiom to be added$^{11}$ to the present ZFC axiom system with the intention of capturing within the resulting universe more of the qualities and dynamics of pure consciousness. This axiom, which we will call the Wholeness Axiom, asserts (in nontechnical terms) that the nature of the universe of sets as a whole is to move within itself and know itself through its own self-interaction. We will see that this new axiom brings the qualities and dynamics of the universe $V$ in much closer alignment with those of the wholeness of pure consciousness. As a satisfying consequence of this new theory, we will be able to give a complete account of the origin of virtually all known large cardinals.

In this paper, we shall not attempt to address the natural question, “What should a foundation of mathematics provide?” We believe that any answer to this question must at least include an account of large cardinal axioms. For the present, we take our success in this latter regard as sufficient evidence that our program is on the right track, and permit this more universal question, concerning the nature of a proper foundation, to motivate further research.

It is important to mention that our solution to the problem of the origin of large cardinals could easily have been discovered without the use of Maharishi Vedic Science; in fact, set theorists are quite familiar with the fact that axioms like the one we propose are strong enough to imply the existence of all known large cardinals.$^{12}$ What has been miss-

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$^{11}$ Our axiom should properly be called a metatheoretic axiom since it cannot be directly formulated in the language of ZFC. Nevertheless, it can be formulated in an expanded language as an axiom schema. See below and Corazza (2000).

$^{12}$ See, for example, Maddy (1988a, 1988b).
ing up until now has been a compelling reason to adopt such an axiom; without basic insights into the nature of the very wholeness that set theorists have been attempting to give expression to all these years, the large cardinal axioms all appear rather arbitrary. Once we have gained a glimpse of the dynamics that ought to underlie any foundational wholeness (since these are the dynamics which underlie nature itself), an axiom such as our Wholeness Axiom begins to stand out as extremely natural.

Our audience for this paper is intended to be wide in scope. For nonmathematical readers who are interested in applications of Maharishi Vedic Science, we have attempted to make the threads of reasoning leading to the Wholeness Axiom and its ramifications direct and relatively free from unnecessary technicalities, while elucidating the connections to Maharishi Vedic Science as accurately as possible. For mathematicians who may be unfamiliar with axiomatic set theory, we quickly review the basic ideas of the subject from ground level and include several highly readable references. And we hope that the experienced set theorist will find the strong connections between the familiar world of large cardinals and elementary embeddings on the one hand, and our intuitive model from Maharishi Vedic Science on the other, a pleasant surprise. Readers who are new to Maharishi Vedic Science may find the article (Corazza, 1993) to be a good starting point.

The paper is organized as follows. We begin with a brief review of modern set theory and the structure of the universe $\mathcal{V}$. As a first suggestion that this structure differs in important ways from the structure of wholeness described by Maharishi Vedic Science, we observe that certain central properties of wholeness do not appear to be present in the structure of $\mathcal{V}$, at least not in the way that we would expect to find them. This divergence in structures becomes more evident when we next consider the dynamics of wholeness described in Maharishi Vedic Science—the lack of any real analogue to the self-referral dynamics of wholeness suggests that some new principle of dynamism ought to be introduced. We then give a brief introduction to the theory of large cardinals and the model theory of ZFC, leading to a natural candidate for an explicit representation of the hidden dynamics of the universe of $\mathcal{V}$ as a whole: a nontrivial elementary embedding from $\mathcal{V}$ to itself. We then discuss K. Kunen’s (1971) surprising result that, under certain natural assumptions, such embeddings don’t exist! Because such an embedding
has seemed particularly natural to large cardinal experts, there have been numerous attempts to bypass Kunen’s theorem; we review some of these efforts. Using Maharishi Vedic Science as motivation, we offer another such attempt, formulated as the Wholeness Axiom, which also bypasses Kunen’s theorem, and which at the same time introduces new dynamics in $V$ that correspond remarkably well to the dynamics of wholeness described by Maharishi. After observing the new character of mathematical proofs that arises from using the Wholeness Axiom, we develop the analogies between the wholeness of $V$ in the presence of the Wholeness Axiom and the wholeness of pure consciousness. We then present proofs that, from the Wholeness Axiom, virtually all large cardinals can be derived. Finally, we observe that our analogy between $V$ and the wholeness described by Maharishi extends even further than previously suggested: Using the technical language of Maharishi Vedic Science, we suggest that the eightfold collapse of infinity to a point within wholeness, in its three phases corresponding to Rishi, Devatā, and Chhandas, are actually mirrored in eight fundamental large cardinal axioms that increasingly approximate the Wholeness Axiom.

§2. The Need for a Theory of Sets

In Cantor’s time it was believed that a set is simply any collection of objects that can be defined by some property. For example, the even numbers $0, 2, 4, \ldots$ form a set, namely, the set of all those natural numbers having the property of being divisible by 2. As another example, the collection $\{2, 5, 7\}$ is also a set; in this case the defining property of its elements is that of being equal to either 2, 5, or 7.

For most purposes, this concept of a set does not cause any problems; but technically, it is seriously flawed because, as Russell (1906) showed, one can invent very innocent-looking properties that determine collections which cannot rightly be considered sets. In particular, Russell showed that if we attempt to form the set of all sets having the property that each is not a member of itself, we arrive at a paradox.\(^{13}\)

\(^{13}\) Russell’s paradox is the following: Let $T(x)$ be the property of sets $x$ that asserts, “$x$ is not an element of $x$.” Let $S$ denote the collection of all sets $x$ that satisfy $T(x)$. By the intuitive notion of ‘set,’ $S$ is a set. Thus, given any set $y$, we should be able to use the property $T$ to decide whether or not $y$ belongs to $S$ (if $T(y)$ is true, then $y \not\in S$; if $T(y)$ is false, then $y \in S$). However, if we attempt to decide whether $S$ itself belongs to $S$, we arrive at the logical absurdity that “$S$ belongs to $S$ if and only if $S$ does not belong to $S$.” If $S$ does belong to $S$, then $T(S)$ is false, whence $S$ does not belong to...
For this and other reasons, at the turn of the century a number of mathematicians focused on the task of developing a theory of sets. The idea was to set forth the most intuitively evident properties that sets ought to have, formulate them in the formal language of first-order logic, and use these formal statements as a set of axioms from which, hopefully, all the properties of sets, and hence of every other mathematical object, could be derived.

To be more concrete, we consider now a couple of the most widely accepted of these basic properties of sets. One basic property that sets ought to have is that two sets should be considered the same if they have the same elements. This property is known as \textit{Extensionality}. Another basic property is that if $A$ and $B$ are sets, there ought to be another set $\{A, B\}$ which contains both $A$ and $B$. This property is known as \textit{Pairing}.

A number of different sets of axioms emerged from this early research. The most widely accepted axiom system, which has served as an excellent foundation for more than 50 years, is known as \textit{Zermelo-Fraenkel Set Theory} with the Axiom of Choice, or \textit{ZFC} for short.

\textbf{§3. Zermelo-Fraenkel Set Theory with the Axiom of Choice}

The axioms of ZFC, given below, detail the essential properties that sets should have and describe implicitly a procedure for building a universe of sets. The axioms do not explicitly tell us what a set is; rather, they list the essential properties sets must have in relationship both to themselves and to other sets. The intent is that if we can conceive of a vast aggregation of collections such that the collections in this aggregation obey the axioms of ZFC, then each collection in the aggregation may be called a set, and the aggregation itself may be called a universe of sets. The universe $V$ mentioned above is such an aggregation, known as the \textit{standard universe} and its members are \textit{standard sets} (or just \textit{sets}).

\textbf{Axioms of Set Theory}

\begin{itemize}
  \item \textit{(Empty Set Axiom)} There is a set with no element.
  \item \textit{(Axiom of Extensionality)} Two sets are equal if and only if they have the same elements.
  \item \textit{(Pairing Axiom)} If $X$ and $Y$ are sets, so is the collection $\{X, Y\}$.
\end{itemize}

$\mathcal{S}$; conversely, if $\mathcal{S}$ does not belong to $\mathcal{S}$, then $\mathcal{T}(\mathcal{S})$ is true, whence $\mathcal{S}$ does belong to $\mathcal{S}$. See Weinless (2011) for further discussion of Russell’s paradox.
• (Union Axiom) If $X$ is a set whose members are also sets, then $\bigcup X$ is also a set.

• (Power Set Axiom) If $X$ is a set, so is $P(X)$, the collection of all subsets of $X$.

• (Axiom of Choice) If $X$ is a set whose members are nonempty pairwise disjoint sets, then there is a set $Y$ which contains exactly one element of each member of $X$.

• (Axiom of Foundation) Every nonempty set $X$ has a member $y$ such that no member of $y$ is in $X$ ($y$ is called an $\epsilon$-minimal element of $X$).

• (Axiom of Separation) For every set $X$ and every property $r$, the collection of all members of $X$ which satisfy the property $R$ is itself a set.

• (Axiom of Replacement) Suppose $X$ is a set and we replace each member $x$ of $X$ with some set $y_x$, according to some well-defined rule. Then the resulting collection $\{y_x : x \in X\}$ is a set.

We take a moment here to discuss the meaning of these axioms (see Weinless, 2011, for a more detailed discussion relating the axioms to Maharishi Vedic Science). The first two axioms guarantee that certain sets actually exist. The empty set, the set with no element, is usually denoted $\emptyset$. The Axiom of Infinity asserts that there is an infinite set. It is not surprising that we require our universe of sets to include an infinite set since such sets are the most commonly used in actual mathematical practice; a familiar example of such a set is the set of natural numbers $\{0, 1, 2, \ldots \}$. The Axiom of Extensionality provides a criterion for testing when two sets are equal. The Axiom of Foundation is a technical axiom which guarantees that any universe of sets that satisfies the axioms must unfold sequentially so that each set emerges only after all its members have emerged. This axiom implies that there is no “circular” set, i.e., no set which is a member of itself.

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14 It can be shown that the Axiom of Infinity is equivalent to the assertion that the collection of all natural numbers is a set.
The Axiom of Foundation and Circular Sets

We consider here how the Axiom of Foundation proscribes sets containing themselves as elements. Suppose there were a set \( x \) which contained itself as an element. We show that the set \( \{ x \} \) would then violate the Axiom of Foundation: Since the only member of \( \{ x \} \) is \( x \), and since there is a member of \( x \) (namely, \( x \) itself) which is also in \( \{ x \} \), the set \( \{ x \} \) has no \( \epsilon \)-minimal element. Similar reasoning can be used to establish the result given in the following exercise.

Exercise  Show that the Axiom of Foundation implies that there do not exist sets \( x \) and \( y \) such that \( x \in y \) and \( y \in x \). (Hint: If such sets \( x \) and \( y \) did exist, show that \( \{ x, y \} \) would violate the Axiom of Foundation.)

The Pairing, Union, and Power Set Axioms say that if certain simple operations are performed on sets, new sets are produced. Pairing, as mentioned earlier, asserts that from any two given sets, a third set can be formed having as its only elements the given two sets. The Union Axiom tells us that given any set whose elements are themselves sets, say, \( X_0, X_1, X_2, \ldots \), a new set can be formed, called the union of \( X_0, X_1, X_2, \ldots \), which consists of all the elements of each of the given sets \( X_0, X_1, X_2, \ldots \). Finally, the Power Set Axiom guarantees that when we form the collection of all subsets of a given set, this new collection is itself a set. The set of all subsets of a given set is called its power set.
Applications of the Union Axiom

As an example of the Union Axiom, consider the sets $X = \{1, 2, 4\}$ and $Y = \{3, 4, 9\}$. The union of $X$ and $Y$, written $X \cup Y$, is the collection $\{1, 2, 3, 4, 9\}$. The Union Axiom asserts that the collection $X \cup Y$ is itself a set. A precise definition can be given as follows: the union of a collection of sets is the collection formed by including as members those (and only those) objects which are members of at least one of the sets in the original collection. As another example, consider the sequence of sets $X_0 = \{0\}$, $X_1 = \{0, 2\}$, $X_2 = \{0, 2, 4\}$, . . . . The union of this infinite collection of sets is the set $\{0, 2, 4, 6, \ldots \}$ of all even numbers. We could write this union in either of the following ways:

$$X_0 \cup X_1 \cup \ldots \cup X_n \cup \ldots = \{0, 2, 4, \ldots \}$$
$$\cup \{X_0, X_1, \ldots X_n, \ldots\} = \{0, 2, 4, \ldots \}.$$  

The second of these notations is used in the statement of the Union Axiom, where $X = \{X_0, X_1, \ldots, X_n, \ldots\}$.

Applications of the Power Set Axiom

As indicated in the Power Set Axiom itself, the power set of a given set $X$ is the collection of all subsets of $X$. As an example, consider the set $X = \{1, 2, 4\}$. The subsets of $X$ can be listed: $\emptyset, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}$. The power set of $X$, denoted $P(X)$, is the collection of all subsets of $X$. Thus

$$P(\{1, 2, 4\}) = \{\emptyset, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\}.$$  

The Power Set Axiom asserts that the power set of any set is again a set. As another example, notice that

$$P(\emptyset) = \{\emptyset\}.$$  

The final two axioms tell us that if we are given a set and some property or rule, then the given set can be transformed using the property or rule into a new set. The Axiom of Separation asserts that if we collect together all members of a given set which satisfy a given property, the
resulting collection is a set. Thus, for example, we could start with the set $E$ of even numbers and form the collection of all members of $E$ that have the property of being multiples of the number 7. The Axiom of Separation guarantees that this collection is a set.

The Axiom of Separation may remind the reader of the naive notion of a set that we mentioned earlier, prevalent in Cantor’s time. In fact, the Axiom of Separation historically arose as a deliberate weakening of this naive notion, designed to avoid inconsistency.

Finally, the Replacement Axiom asserts that replacing elements of any set with other sets—according to some rule—produces a set. As an example, suppose we start with the set of natural numbers, $X = \{0, 1, 2, \ldots \}$, and we replace members of $X$ according to the following rule: Replace each number in $X$ by the set which contains both it and the number $\frac{1}{2}$. Thus, we replace 0 by $\{0, \frac{1}{2}\}$, 1 by $\{1, \frac{1}{2}\}$, and so forth. By the Axiom of Replacement, the resulting collection

$$\{ \{0, \frac{1}{2}\}, \{1, \frac{1}{2}\}, \ldots \}$$

is a set.

Collectively, these axioms about sets are very powerful; every theorem in mathematics can be translated into a statement in the language of sets, and virtually all such statements can be derived directly from the list of axioms given above. This fact provides powerful conceptual unification of the entire range of mathematics. In addition, as we have said before, the axioms give rise to a very natural universe in which all mathematical objects—circles, lines, functions, numbers, groups, topological spaces, and so on—can be located.

§4. $V$: The Universe of Sets

The universe that can be built using ZFC proceeds in stages. The zeroth stage, denoted $V_0$, is the empty set itself; of course, the Empty Set Axiom guarantees that this stage is an allowable set. The next stage, $V_1$, is the set $\{\emptyset\}$ whose only element is the empty set; $V_2 = \{\emptyset, \{\emptyset\}\}$. These two stages can also be proven to be allowable sets by using the Pairing Axiom. The pattern of unfoldment is that each later stage is obtained by collecting together all subsets of the previous stage. After we have built up $V_n$, for every natural number $n$, the axioms tell us that
we can continue building if we extend our number system beyond the natural numbers.

**Ordinal numbers** allow mathematicians to continue long constructions which extend beyond the indexing capabilities of the natural numbers. The ordinals extending past the natural numbers are given the following names, in increasing order: \( \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega, \omega + \omega + 1, \ldots, \omega_1, \omega_1 + 1, \ldots, \omega_2, \omega_2 + 1, \ldots \). The Axioms of Infinity, Power Set, Union, and Replacement in combination guarantee that this long sequence of infinite numbers exists. The axioms allow us to continue defining new, larger stages of our universe: \( V_\omega, V_{\omega + 1}, \ldots \), \( V_\omega + \omega \), and so forth. \( V_\omega \) is obtained\(^{15}\) by forming the union of all the preceding stages \( V_0, V_1, \ldots \). Then \( V_{\omega + 1} \) is the set of all subsets of \( V_\omega \), \( V_{\omega + 2} \) is the power set of \( V_{\omega + 1} \), and \( V_{\omega + \omega} \) is the union of all previous stages; proceeding beyond \( \omega + \omega \), we continue taking power sets and unions. Finally, we can declare our universe of sets to be the collection of all sets that can be found in at least one of the stages.

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**Ordinal Numbers and the Stages of the Universe**

There are two basic types of ordinal numbers that are used in different ways in the build-up of the universe through the stages \( V_0, V_1, V_2, \ldots \). A successor ordinal is an ordinal number that has an immediate predecessor; the familiar numbers 3, 5, and 393 are examples of successor ordinals (since they have predecessors 2, 4, and 392, respectively). The ordinal \( \omega + 3 \) is also a successor ordinal, having predecessor \( \omega + 2 \). On the other hand, 0 and \( \omega \) are examples of ordinals without immediate predecessors; such ordinals are called *limit ordinals*. The reader will notice that the stages of the universe are formed according to what kind of ordinal number is being used to index the stage: for instance, \( V_\omega \) is defined to be the union of previous stages, while \( V_{\omega + 1} \) is defined to be the power set of the stage immediately prior to it, \( V_\omega \). The formal definition of the stages of the universe is given by:

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\(^{15}\) In actual fact, the Axiom of Replacement is needed to form the sequence \( <V_0, V_1, V_2, \ldots > \); once this sequence has been formed, the Axiom of Union may be applied to it (actually, to its range).
$V_0 = \emptyset$

$V_{\alpha + 1} = P(V_\alpha)$

$V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$, $\alpha$ a limit ordinal

$V = \bigcup_\alpha V_\alpha$

**Figure 1** - Sequential Unfoldment of the Universe of Sets
Locating the Set of all Fractions in $V$

We examine here how to locate the set of all fractions $a/b$—where $a$ and $b$ are positive natural numbers—within the universe $V$. This exercise illustrates how any set can be formally located inside $V$. First, let us see where each natural number can be found in $V$. In set theory, the natural numbers are defined as follows:

$$
0 = \emptyset \\
1 = \{0\} \\
2 = \{0, 1\} \\
\vdots \\
n + 1 = \{0, 1, 2, \ldots, n\} \\
\vdots
$$

Notice that for each natural number $n$, $n$ is a subset of $V_n$ and is in fact a member of $V_{n+1} \setminus V_n$. It follows that the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers is a subset of $V_\omega$, and so $\mathbb{N}$ is a member of $V_{\omega + 1}$. To locate the fractions within $V$, we must find a way to code up fractions, represented in the form $a/b$, as sets, just as the natural numbers have been canonically identified with sets. The usual way to do this is to represent $a/b$ as the ordered pair $(a, b)$. Different fractions are thereby identified with different ordered pairs and each ordered pair stands for a unique fraction. The final step is to code ordered pairs of sets as other sets. Again, the usual way to do this is to represent the pair $(a, b)$ as the set $\{\{a\}, \{a, b\}\}$. It is an interesting exercise to verify that a set of this kind successfully separates the components of the ordered pair, i.e., that two sets $\{\{a\}, \{a, b\}\}$ and $\{\{c\}, \{c, d\}\}$ are equal if and only if $a = c$ and $b = d$. Now, to locate the fractions within $V$ notice that since $a \in V_{a+1}$ and $b \in V_{b+1}$ then $a, b \in V_{a+b+2}$, Thus $a/b = \{\{a\}, \{a, b\}\} \in V_{a+b+2} \subseteq V_\omega$. It follows that the set of all fractions is included in $V_\omega$ as a subset, and is therefore a member of $V_{\omega + 1}$.
As we have mentioned before, the universe $V$ is highly successful as a unifying background for mathematical research. Virtually all mathematical objects and structures can be formally located in the universe; yet paradoxical sets have been successfully excluded.

Because this mathematical universe plays the role of the fundamental wholeness underlying mathematics, it is natural at this stage to use our source of intuitive motivation, Maharishi Vedic Science, to see to what extent the fabric of $V$ reflects that of the wholeness of pure consciousness. Recall that we are seeking to modify this construction if possible because we wish to provide an account for large cardinals in mathematics. Thus, our plan is to bring the construction of $V$ into closer relationship with the structure of pure consciousness, as far as this is possible.

§5. Application of Maharishi Vedic Science to Mathematics

We provide here an overview of how we plan to use Maharishi Vedic Science in our treatment of modern set theory. Our plan, as outlined briefly in the Introduction, is to use the qualities and dynamics of wholeness, pure consciousness, as principles for guiding our intuition concerning the “right” structure of the universe of sets, considered as a wholeness. We will observe that on the one hand, many of the qualities of pure consciousness have natural correlates in the standard universe $V$; on the other hand, those qualities concerned with self-interaction of pure consciousness—specifically, fully awake within itself, self-referral, and bliss—as well as the quality of infinite correlation, appear to be entirely absent from the universe (using a reasonable interpretation of these qualities within the context of set theory).

We will also observe that the fundamental dynamics of pure consciousness, by which its infinitely expanded value collapses to its own point value, are not reflected in the structure of $V$. In order to enrich the universe so that these missing qualities and dynamics are more fully expressed, we will suggest that three features should be introduced to the structure of $V$: Some kind of truth-preserving embedding should naturally accompany the structure of $V$ (corresponding to self-interacting dynamics of consciousness); elementary (set) submodels of $V$—structures which fully reflect all first order properties of $V$—should permeate the universe (corresponding to the quality of infinite correlation); and the
consciousness-based education and mathematics

dynamics of wholeness embodied in the truth-preserving embedding ought to reflect the collapse of infinity to a point in some natural way.

We will observe that these three criteria are satisfied in a strikingly direct way by postulating the existence of a nontrivial, elementary embedding from the universe to itself—a transformation which preserves all first order properties of the universe and implies that nearly every set is itself an elementary submodel of $V$. It so happens that such an embedding represents the natural culmination of all large cardinal axioms—the very things that set theory has had such difficulty accounting for and that our new approach is designed to explain.

We will address one apparently serious technical difficulty in our approach: There is a well-known proof that seems to say the existence of such an embedding is inconsistent with set theory! To address this issue, we will show that certain assumptions (which are often not mentioned in discussions about this theorem) are required for the proof of this result to go through; and we will indicate how our approach explicitly avoids these assumptions. Having addressed the problem of inconsistency, we will assert in an axiom the existence of a certain kind of elementary embedding of the universe to itself, and add this axiom to the usual axioms of set theory. From this new expanded theory, we will indicate many of its strong consequences, among which is the fact that virtually all large cardinal axioms are derivable from this expanded set theory.

We will then discuss at some length the new dynamics that arise in the universe as a result of this new postulate. We will see how, just as the infinitely expanded value of pure consciousness collapses to a point value in the unfoldment of the Veda and creation, so, if we view sets from the perspective of $V$ as a whole, the creation of sets can be seen to arise when the large cardinal-like properties of $V$ become focused on the first point $\kappa$ to be moved by the undefinable embedding. Then, just as the Veda sequentially emerges from the collapse of $A$ to $K$, so a special sequence within the $\kappa$th stage of the universe emerges from the embedding that contains essentially all the information about the location of every set in the universe. This sequence, called a Laver magic sequence, can be shown to “give rise to” every set, much as the Veda gives rise to every detail of creation. We will pursue the analogy further by describing an analogue to the eight stages that are involved in the
collapse of \( A \) to \( K \); in particular, since the axioms defining the larger large cardinal axioms provide increasingly close approximations to our new postulate, we search for eight prominent large cardinal axioms to correspond to these eight stages described in Maharishi Vedic Science.

Our search uncovers eight especially significant large cardinal axioms that represent landmarks in any investigation of the structure of \( V \). As a further step in the analogy, we show how just as the eight stages of collapse are given expression in Rk Veda in a threefold manner, in terms of Rishi, Devatā, and Chhandas (elaborated in the 24 syllables of the first richa of Rk Veda), so we shall observe how these eight large cardinal axioms can be expressed in terms of the structure of \( V \) (corresponding to Rishi), in terms of elementary embeddings (corresponding to Devatā), and in terms of properties of a specific point in the universe, i.e., a specific large cardinal (corresponding to Chhandas).

Therefore, by introducing this new axiom, which states in mathematical terms that wholeness by its nature moves within itself and knows itself, we will find that not only is the structure of \( V \) enriched to the point of displaying nearly all the qualities and dynamics of pure consciousness, and, but the previously mysterious large cardinal properties also can be accounted for very naturally as the properties of the first point moved by our postulated undefinable elementary embedding, which represents the “unmanifest self-interacting dynamics” of the wholeness embodied in \( V \).

§6. Qualities of Pure Consciousness and the Universe \( V \)

In this section, we will examine the universe \( V \), looking to see which of the qualities ascribed to the field of pure consciousness by Maharishi Vedic Science find expression in this foundational structure. As we shall see, some of these qualities will seem to capture the very intent behind the cumulative hierarchy, while others may not seem quite so relevant. Since we are attempting to use Maharishi Vedic Science as a source of intuitive guidance, our plan is to look for ways of enriching the construction of \( V \) so that qualities which originally seemed irrelevant will become as fully embodied as the other qualities. As a starting point, we give a quick summary of some of the main qualities of pure consciousness. [Since the time this paper was first written, the list of
qualities specified by Maharishi Vedic Science has grown considerably, but this partial list still provides an excellent sampling—Ed.]

**Qualities of Pure Consciousness**

_**all possibilities**_ All activity begins from the field of pure consciousness; all laws of nature begin to operate from this level; the point $K$ represents the point of all possibilities within this field.

_**omniscience**_ The self-interacting dynamics of pure consciousness constitutes that pure knowledge on the basis of which all knowledge and existence arise. Knowing this level of life, all else is known.

_**freedom**_ Remaining ever uninvolved in its own self-referral dynamics, pure consciousness is a state of eternal freedom.

_**unmanifest**_ The self-referral dynamics of pure consciousness form the unseen government of nature. All manifest life is governed by these unmanifest dynamics.

_**simplicity**_ Pure consciousness is known when that which is foreign to the nature of the knower drops away. “The simplest form of awareness is a state of perfect order, the ground state of all the laws of nature” (Maharishi, 1991b, p. 283).

_**omnipotence**_ Pure knowledge has infinite organizing power. Pure consciousness knows no limitation in its creative expression as it unfolds sequentially within itself.

_**total potential of natural law**_ The creation unfolds and is maintained in accordance with the most fundamental laws of existence—the laws that govern the flow of pure consciousness from the Constitution of the Universe.

_**discriminating**_ The flow of pure consciousness within itself is not only highly dynamic and unrestricted, but precise and sequential in its unfoldment. Each stage of expression comes about methodically and with full awareness of all that has come before it and all that is yet to come.
infinite silence The infinitely silent quality of pure consciousness is expressed in the first letter A of Rk Veda. This quality quietly nourishes the infinitely dynamic unfoldment of pure consciousness.

infinite dynamism Being awake to itself, pure consciousness undergoes an infinity of transformations within itself; the infinite organizing power inherent in these dynamics structures the infinite diversity of creation.

pure knowledge Being awake to itself, pure consciousness knows itself. This self-knowing, a sequential flow within the unmanifest, is called pure knowledge.

infinite organizing power “Knowledge has organizing power. Pure knowledge has infinite organizing power.” —Maharishi

evolutionary The pure intelligence inherent in the infinite organizing power at the basis of creation directs life toward ever-increasing levels of progress and fulfillment.

perfect orderliness The laws governing the precise sequential flow of pure consciousness are at the basis of the orderly functioning observed in nature.

self-sufficiency Pure consciousness needs nothing outside itself for its existence, creative expression, and fulfillment. Creation unfolds and dissolves within pure consciousness.

purifying Enlivenment of pure consciousness, the ultimate reality of manifest life, brings an end to unwanted tendencies, which are foreign to life.

infinite creativity The infinite organizing power inherent in pure consciousness finds unrestricted expression in the unfoldment of creation.

integrating The wholeness of pure consciousness is maintained through the integral coexistence of opposite values, such as infinite dynamism and infinite silence.
**harmonizing** The basis of harmony is enlivenment of the infinitely harmonizing quality of pure consciousness in which extreme opposite values are simultaneously lively without conflict.

**perfect balance** “The balance inherent in the eternal continuum of the unmanifest nature of the Absolute is reflected in the balance that nature maintains—amidst the dynamism of evolutionary change.” (Maharishi 1976, p. 148)

**unboundedness** All boundaries are structured in the boundless, unlimited value of pure consciousness.

**nourishing** All stages of expression of pure consciousness are nourished by the infinitely silent value of pure consciousness.

**immortality** Birth, death, and the field of change are the creative expression of pure consciousness. Pure consciousness itself is an immortal field, beyond the manifest field of change.

**omnipresence** The self-referral dynamics of consciousness are present at every point in creation.

**fully awake within itself** Pure consciousness is by its very nature pure awakefulness.

**self-referral** Pure consciousness, through all stages of unfoldment is awake to itself; its nature and creation are, therefore, self-referral.

**invincibility** "Nothing can . . . disrupt the perfect balance . . . of this field . . . since everything is a part of its structure. (Maharishi, 1991b, p. 281).

**bliss** Self-interacting dynamics of consciousness form the unmanifest structure of bliss. " . . . the Absolute ever celebrates its own nature within its unmanifest, nonchanging Self" (Maharishi, 1976, p. 146).

**infinite correlation** Pure consciousness "is a field of infinite correlation in which an impulse anywhere is an impulse everywhere" (Maharishi, 1976, p. 150).
The alert reader will no doubt discover many ways to interpret these qualities in the context of set theory and foundations that we have not considered here. Our account may at times assert that certain qualities are absent from the set theoretic universe which the reader, taking a slightly different approach, may find abundantly present. We feel that these different viewpoints are to be expected and mark the beginning of a healthy, rigorous research program. In our own research, we found that, as we reflected on the significance of each quality, natural ways for this quality to be expressed in a foundational context became apparent to us. We asked ourselves, for example, “What would a universe have to look like in order to embody the quality of infinite correlation?” It seemed apparent to us that this quality would be most clearly embodied if all knowledge about the universe were available throughout the universe. From the perspective of set theory, we interpreted this to mean that we were looking for a universe in which a substantial proportion of sets would reflect all first-order properties of the universe. This requirement is certainly not met by the universe arising from ZFC; as we shall see, however, by suitably supplementing ZFC with an axiom about the wholeness of \( V \), this requirement expresses one of the most appealing features about the new resulting universe.

In the discussion below, we list most of the qualities and suggest the ways in which ZFC set theory, as a foundation for mathematics, exhibits these qualities. Several qualities from our table are not mentioned below; these, in our view, do not find natural expression in set theory as it is presently understood. We will discuss these at greater length later in this section, outlining the ways in which we might expect to find these qualities displayed in an enriched set theory and our reasons for believing they are absent from the present foundation.

**Qualities of Pure Consciousness in the Universe \( V \)**

*all possibilities* All models of every mathematical theory are located in \( V \); all sets needed for the development of any mathematical theory are located in \( V \).
omniscience Every mathematical fact is true in the model $V$; thus, if one could view mathematics from the vantage point of $V$, the wholeness underlying mathematics, every mathematical truth could be known.

freedom The power set axiom freely generates the set of all subsets of a given set; since no restriction is placed on the sets generated in this way, the continuum may consistently be taken to have arbitrarily large cardinality.

unmanifest $V$ is too large to be an individual set; although all properties of sets can be rigorously determined and demonstrated using the axioms of set theory, nothing can be directly proven about $V$.

simplicity A single elegant recursive rule is at the basis of the sequential and simultaneous unfoldment of all stages of the universe.

omnipotence Any mathematical truth that has ever been demonstrated can be seen as a derivation from the axioms of set theory using rules of logic, and all of these can be found in coded form within the structure of the universe itself.

total potential of natural law The laws governing a mathematical theory are expressed by axioms; the content of every axiom of set theory is fully realized in the universe of sets.

discriminating The sets which emerge in the cumulative construction of $V$ do not lead to any known paradox.

infinite silence At limit stages of the construction of the universe, no new sets are added; this silent phase of the construction creates smoothness and uniformity in the universe.

infinite dynamism In the construction of $V$, each new stage produced by the power set operator is larger than the previous stage; in particular, the power set operator produces an endless sequence of ever larger infinities.

pure knowledge The information content in ZFC is the basis for virtually all known mathematical theorems.
infinite organizing power The organizing power of a mathematical theory is expressed by its models;\textsuperscript{16} the models of set theory are infinite, complete, and all-inclusive.

evolutionary Set theory has stimulated progress in a wide range of mathematical fields.

perfect orderliness All theorems of set theory, and hence of virtually all of mathematics, can in principle be generated automatically by a computer once sufficiently many axioms have been input.

self-sufficiency All the information needed to construct the stages of the universe is coded in the first few stages of the universe; the universe can therefore reproduce itself.

purifying The recursive construction of \( V \) systematically prevents the entry of paradoxical sets.

infinite creativity All the creativity of the brightest mathematicians of recorded history can be coded up as formal theorems derivable from the simple axioms of set theory.

integrating All mathematical theories, with their own special mathematical languages, find a common basis in set theory; the interrelationships between theories are thereby more easily identified.

harmonizing Superficial differences in style between different theories are stripped away when the formal content of these theories is expressed in the language of set theory.

perfect balance Despite the differences in style and content between different theories and their models, all such models naturally emerge in the uniform and simply defined unfoldment of the stages of the universe.

unboundedness The sequence of stages of the universe \( V \) unfold without bound; the resulting universe \( V \) is so vast that it cannot be considered a set.

\textsuperscript{16} Weinless (2011) discusses at some length this notion that the organizing power of a set of axioms is expressed in its models.
Every mathematical theory has a basis in set theory; as a result, each theory can make use of the tools of set theory within its own context.

The conceptual reality developed by pure mathematicians, and uniformly codified in set theory, is time independent.

All mathematical structures can be located inside $V$.

As our list indicates, set theory with its universe $V$ exhibits a wide range of the qualities attributed to pure consciousness in Maharishi Vedic Science. In the table below, we provide the reader additional information about our point of view concerning the presence of these qualities in set theory by considering one such quality—self-sufficiency—in greater detail.

Our main concern here is with the five qualities, present in the table given earlier, that do not appear on our list. These aspects of wholeness, described by Maharishi Vedic Science, are, in our view, missing from set theory and the structure of $V$; we shall argue later that the difficulties set theory faces as a foundation are intimately tied to these omissions. The omitted qualities are infinite correlation, invincibility, fully awake within itself, bliss, and self-referral.

As we mentioned at the beginning of this section, the first of these qualities, infinite correlation, would be exhibited in a universe in which a significant proportion of its sets satisfied all the first-order properties of the universe itself. Using Gödel’s Incompleteness Theorem, however, one can easily show that it is impossible to prove from ZFC that there are any sets in $V$ which are even models of ZFC, much less sets which reflect all first-order properties of the universe!

As indicated in the table, the quality of invincibility is the characteristic of pure consciousness by which it maintains its connection to its unbounded source through all stages of expression, and therefore is not foreign or antagonistic to any aspect of its creation. In our view, this quality could be ascribed to the universe if, as in the case of infinite correlation, “nearly all” sets in the universe satisfied all first order properties of $V$ itself. In that case, clearly, the nature of wholeness would not be lost at any stage of the unfoldment of $V$. 
The next three qualities have one common property that leads us to declare that they are absent from the structure of $V$: All three arise from a fundamental self-interaction of wholeness, of pure consciousness. According to Maharishi Vedic Science, being fully awake within itself, pure consciousness is fully awake to itself; its own wakefulness results in its own self-knowing and self-interaction. This dynamic state in which pure wakefulness is awake to itself represents the eternal nature of pure consciousness to be ever in a state of self-knowing; this unchanging condition of self-knowing is called self-referral, and is another fundamental instance of self-interaction.

Finally bliss is a description of the experience of this self-referral flow of consciousness. At this level, the experience and that which is experienced are the same (recall “The experience of pure Being and the state of Being mean the same thing” cited in the list above). Thus, the subjective experience of self-referral consciousness as bliss is no different from the reality of self-referral consciousness as bliss. Again, this quality arises from the self-interaction of pure consciousness.\(^1\)

What does self-interaction mean in the context of the universe $V$? At the very least, we would expect a self-interacting universe to have some sort of transformation associated with it that would move its elements. One observation that many category theorists and physicists have made regarding $V$, both in publications (see, for example, Lawvere, 1979, and McLarty, 1990) and in lectures and discussions, is that it is unduly static; even the central concept of a function—the very essence of mathematical transformation—is formalized as a set of ordered pairs, on a par with other sets, like the sets of rationals or integers, which exhibit no essentially dynamic features. In short, the mathematical intuition of dynamism embodied in the concept of a function is not well expressed in $V$ either on the local scale (particular functions are mere sets of ordered pairs) or on the global scale ($V$ is not naturally associated with any map\(^2\) from $V$ to itself that would transform its elements). Thus,

---

\(^{17}\) Hagelin (1983) identifies the quality of bliss in pure consciousness as a quality of the unified field of natural law, as described by quantum field theory, because of this unified field’s “continuous effervescence of topological fluctuations”—a fundamental interaction of the field with itself.

\(^{18}\) Interestingly, nearly two decades after this article was first written (1993), it was discovered that the Axiom of Infinity is provably equivalent to the existence of a certain kind of structure-preserving map from $V$ to itself (Corazza, 2010). The naturalness of this phenomenon led the author to argue in favor of the Wholeness Axiom—as discussed in this paper—as a new axiom to be added to ZFC—Eds.
in order for us to declare that the universe exhibits self-interacting
dynamics comparable to those of pure consciousness, we would expect
that some natural transformation of \( V \) into itself should accompany the
construction of \( V \).

Thus, our viewpoint about these five basic qualities suggests to us
that a universe more in accord with our objective, more in accord with
the nature of the wholeness set theorists wish to capture, will display
\textit{infinite correlation} and \textit{invincibility} through the widespread presence
of sets embodying all first-order properties of \( V \), and \textit{self-interacting
dynamics} expressed perhaps by some natural map from \( V \) to itself.

Notice that if we are successful in our efforts to give expression to
these qualities in an enriched set theory, we should expect to find that
many of the other qualities on the list above will be expressed in a man-
ner even more in accord with their expression within pure conscious-
ness. For instance, the fact that pure consciousness can be described as
\textit{pure knowledge} or as \textit{omniscient} arises from the nature of its self-inter-
acting dynamics: Pure consciousness, being awake to itself, is eternally
engaged in the act of self-knowing, and all knowledge emerges from
the sequential unfoldment of this process. Our use of these qualities as
descriptions of the foundation of mathematics differs from the pattern
we find within pure consciousness, and this difference stems from the
fact that the universe \( V \), as it is presently understood, does not exhibit
any fundamental self-interaction from which “knowledge” could be
said to emerge. Thus, even though we feel these qualities are exhibited
to some extent in the present universe, once we have invested \( V \) with a
fundamental form of self-interaction, we shall expect to find the quali-
ties of \textit{pure knowledge}, \textit{omniscience}, and many others, arising from
these new dynamics.

In this section, our aim has been to identify qualities of pure con-
sciousness that appear to be absent from the structure of \( V \) so that our
intuition concerning the “right” structure for \( V \) could be suitably guided.
In the next section, when we compare the \textit{dynamics} of pure conscious-
ness with those of \( V \), the difference between these two wholenesses—pure
consciousness and the present foundation of mathematics—will become
even more apparent.
Self-Sufficiency in the Universe $V$

Here we show how all the information needed to build $V$ can be located within $V$ itself. The basic idea is that set theory is formalized within a symbolic language; the symbols of this language can be identified with sets, as can the basic rules of proof. This means that the informal reasoning we used earlier to build up $V$ using the axioms of ZFC can be formalized in symbolic logic and coded as a set. We now investigate some of the details of this coding.

The symbolic language in which formal set theory is expressed is called first-order logic. The symbols which are used in first-order logic are listed below.

<table>
<thead>
<tr>
<th>Variables</th>
<th>$x_0, x_1, x_2, \ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logical symbols</td>
<td>$\neg$ (not), $\land$ (and), $\lor$ (or), $\rightarrow$ (if... then), $\forall$ (for all), $\exists$ (there exists)</td>
</tr>
<tr>
<td>Parentheses and comma</td>
<td>( ),</td>
</tr>
<tr>
<td>Membership relation</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>Equality relation</td>
<td>$=$</td>
</tr>
</tbody>
</table>

These symbols are put together according to simple rules of formation to obtain the formal sentences of set theory. Here is an example:

$$\forall x_0 \exists x_1 (x_0 \epsilon x_1).$$

This sentence symbolically represents the assertion, “Every set is contained in some other set” (or more precisely, “for every set $x_0$, there exists a set $x_1$ such that $x_0$ is a member of $x_1$”).

The axioms of set theory can be expressed in this formal language. For instance, the Empty Set Axiom has the following symbolic form:

$$\exists x_0 \forall x_1 (x_1 \not\in x_0).$$

With our formal language in place, formal rules of proof can also be
developed which give precise criteria for deriving theorems from the ZFC axioms. Using these, one can, for example, give a formal proof of the formal sentence given above that asserts every set belongs to some other set. We now identify the basic symbols of first-order logic with sets according to the following scheme:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Set</th>
<th>Symbol</th>
<th>Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>∈</td>
<td>0 (= ∅)</td>
<td>∈</td>
<td>0 (= ∅)</td>
</tr>
<tr>
<td>=</td>
<td>1 (= [0])</td>
<td>(</td>
<td>8</td>
</tr>
<tr>
<td>¬</td>
<td>2 (= [0, 1])</td>
<td>)</td>
<td>9</td>
</tr>
<tr>
<td>∧</td>
<td>3 (= [0, 1, 2])</td>
<td>,</td>
<td>10</td>
</tr>
<tr>
<td>∨</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>→</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>∀</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>∃</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>)</td>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>,</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11 (= 11 • 2^n)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22 (= 11 • 2^1)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We can now translate any formal sentence of set theory into a set by translating symbols directly into sets using the key above. In order to preserve the order in which the symbols occur in a given sentence, we attach to the first symbol the number 0, to the second the number 1, and so forth. As an example, we can translate the formula ∀x₀ ∃x₁ (x₀ ∈ x₁) as the following set:

{(0, 6), (1, 11), (2, 7), (3, 22), (4, 8), (5, 11), (6, 0), (7, 22), (8, 9)}.

In this way, all the axioms of set theory can be located within the universe V (in fact, within V^ω). Moreover, it can be shown that all the rules of proof can also be identified with sets; hence, all provable statements and their proofs can also be located in V. In particular, all the reasoning needed to construct V from the axioms of ZFC can be coded up as a single set, which can be located as a subset of V^ω!
§7. Dynamics of Pure Consciousness and the Universe $V$

Continuing to compare the structure of the universe $V$ with the wholeness of pure consciousness described by Maharishi Vedic Science, in this section we seek to determine to what extent the *dynamics* ascribed to pure consciousness are displayed in the structure of the universe. We find a rather unmistakable difference between $V$ and pure consciousness in our comparison. The list below summarizes the important principles of these dynamics, described in Maharishi Vedic Science:

**The Dynamics of Pure Consciousness**

**Existence** The first truth about pure consciousness is that it exists.

**Nature** The nature of pure existence is pure wakefulness or pure intelligence.

**Three-in-one structure** Being awake to itself, pure existence is conscious of itself and assumes the roles of Rishi (knower), Devatā (process of knowing), and Chhandas (that which is known). Put another way, the pure intelligence of pure existence distinguishes a three-in-one structure within pure existence, the Samhitā of Rishi, Devatā, and Chhandas.

**All possible transformations** As each of Samhitā, Rishi, Devatā, and Chhandas is fully awake within itself, each is awake to each of the others. Being awake to each other transforms each. These transformed values of Samhitā, Rishi, Devatā, and Chhandas are themselves fully awake to themselves and each other, and the process of transformation continues. An infinity of transformations—all possible transformations—of pure consciousness emerge in this unfoldment.

**Pure knowledge and infinite organizing power** These transformations of pure consciousness unfold sequentially. This sequential unfoldment within pure consciousness itself is called pure knowledge. The Veda is pure knowledge together with the infinite organizing power contained within it. This organizing power gives rise to the whole creation and all the laws of nature.
Constitution of the Universe  The laws governing the sequential unfoldment of the Veda are known collectively as the Constitution of the Universe. The self-interacting dynamics of consciousness function as the primary administrator of the universe.

Collapse of A to K  Pure knowledge emerges in the collapse of the infinitely expanded value of wholeness to the fully contracted point value of wholeness; fullness, infinite silence, embodied in A, the first letter of Rk Veda, collapses to emptiness, the point value, the point of all possibilities and infinite dynamism, embodied in the second letter of Rk Veda, K.

Collapse and expansion with infinite frequency  In the unfoldment of pure knowledge, the point, embodied in K, expands to infinity. The process of collapse and expansion occurs with infinite frequency and is the theme of unfoldment of the Veda and all of creation.

Apaurusheya Bhāshya  Maharishi’s Apaurusheya Bhāshya asserts that the Veda provides its own commentary on itself. The structure of total knowledge is found in its most concentrated form in A, and in successively more elaborated forms in AK, in the first pada, the first richa, the first Sūkta, and the first mandala of Rk Veda, and finally in its most elaborated form in the entire Veda.

Eightfold collapse  The collapse of A to K is like a whirlpool that contracts to a point in eight stages. These eight stages correspond to the five Tanmatras and the three subjective principles—mind, intellect, and ego. These eight stages unfold from three perspectives: from the point of view of Rishi, Devatā, and Chhandas.

Coexistence of infinite silence and infinite dynamism  The fabric of pure knowledge is composed not only of infinite dynamism and the tendency to give rise to creation, but also infinite silence by which pure consciousness remains forever uninvolved in its own creation. Prakṛiti unfolds within Purusha; pure consciousness is both pure Samhitā and Samhitā of Rishi, Devatā, and Chhandas.
**Maintaining unity, wholeness** In its sequential unfoldment, the self-interacting dynamics of consciousness always remain infinitely correlated with the source, the Samhitā value of pure consciousness.

**Present at every point in creation** The self-interacting dynamics of consciousness, the Veda, is unmanifest and present at each point in creation.

In Maharishi’s description of the dynamics of pure intelligence, he observes that pure intelligence, being pure wakefulness, becomes aware of itself. This process of becoming aware involves a move of the fully expanded aspect of its nature, represented by the letter A, toward the fully contracted, point value of its nature, represented by the letter K. In this collapse of infinity to a point, all possible transformations of pure consciousness take place. In this collapse, the full unmanifest power of unbounded silence is imparted to the point value which then is impelled to expand its contracted nature to the fully expanded infinite value. This expansion of the point to infinity gives rise in sequential fashion to the entire blueprint of creation, the Veda, which emerges as an elaboration of the transformations occurring within the original collapse.

This description suggests to us that the construction of $V$ emphasizes only one half of the dynamics of wholeness, namely, the expansion of the point (represented by the empty set) to infinity (represented by the ever larger stages of the universe).19 Again we note that none of the ZFC axioms actually attempts to describe the nature of wholeness; instead they focus on the nature of sets. Thus the construction of the universe necessarily proceeds in a one-sided way. From the point of view of Maharishi Vedic Science, we would expect that the unfoldment of parts in any foundational system that does not maintain a connection with the nature of the whole is doomed to fall short of its goal (Maharishi, 1991):

If the expansion of Rishi, Devatā, and Chhandas into the infinite universe does not remain in contact with the source, then the goal of expansion will not be achieved.

19 See Weinless (2011) for an excellent detailed treatment of the relationship between the expansion of the universe of sets from the empty set and the dynamics of the point expanding to infinity in Maharishi Vedic Science.
In our view, this one-sided development of the universe \( V \) has created an unnecessary mystery at the root of set theory: Where do large cardinals come from? Using Maharishi Vedic Science as motivation for the intuition that the wholeness that \( V \) represents should by nature move within itself and “know” itself through its own self-interaction, we will introduce an axiom (actually, an axiom schema), which we call the *Wholeness Axiom*, which asserts in mathematical terms that \( V \) is moved within itself via a truth-preserving embedding and that these dynamics are present “at every point.” We will see that the mystery of large cardinals vanishes in the presence of this new axiom.

We should mention here that ours is not the first attempt to locate the unmanifest dynamics of pure consciousness in the structure of \( V \). Weinless (2011) suggests that the “collapse of infinity to a point”—which we have said is absent from the structure of \( V \) as it is presently defined—is to be found in the *Reflection Principle*, which states that properties true of \( V \) as a whole (“infinity”) should also hold true of certain sets (“point”) in the universe because the universe as a whole should be conceived as structurally undefinable. We shall discuss this principle at greater length in Section 10; we shall see that the introduction of a truth-preserving endomorphism of the universe is, on the one hand, motivated by considerations such as the Reflection Principle, and yet on the other hand, accomplishes in a somewhat cleaner way many of the same things as the Reflection Principle in Weinless’ treatment.  

We also mention here that there have been other attempts to extend ZFC by including axioms about \( V \) itself (see for example Gödel, 1940; Kelley, 1955, Appendix; Quine, 1951, and more recently, Maddy, 1983). However, these experts in foundations have been thwarted by a lack of reliable intuition about such a vast wholeness; moreover, such extended axiom systems have neither met with wide acceptance among set theorists nor resolved in the smallest way the issue about the origin of large cardinals.

Before introducing the Wholeness Axiom, we shall offer a brief introduction to the theory of large cardinals and the foundational challenges which accompany them; we shall see that large cardinals them-

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20 We hasten to point out that, in the same paper, Weinless (2011) himself suggests that an embedding of the sort we are describing would “provide an ideal mathematical expression of the self-interacting dynamics of the Samhitā,” but did not pursue this direction because of the well-known limitative result of Kunen; see Section 15 of the present work.
selves suggest the very foundational solution we are seeking. Since large cardinals are vast infinite sets, we begin with a discussion of mathematical infinity.

§8. Mathematical Infinity

*The whole material creation is just a sequence of quantified values of infinity.* —Maharishi 1990c

Prior to the work of Cantor, mathematicians viewed the concept of infinity as a kind of unreachable ideal that various mathematical sequences could approximate. The sequence 0, 1, 2, . . . of natural numbers, for example, was viewed as continuing indefinitely, but was never conceived as a completed collection. In studying certain problems in mathematical analysis, Cantor found it useful to consider certain infinite collections as completed wholes which could be further manipulated using techniques commonly used on finite collections. His work was at first met with skepticism but by now has come to be considered one of the great achievements of modern mathematics.

Once the concept of sets having infinite size is in place, it is natural to ask, as Cantor himself did, whether all infinite sets have the same size. In order to answer the question, Cantor first needed to describe a way of comparing two infinite sets. Certainly, the familiar method of comparing the sizes of two finite sets—namely, by counting the number of elements in each—would not apply to the case of two infinite sets (how many elements does an infinite set have?). However, another method of comparing finite sets does turn out to be useful in the context of infinite sets: Consider two fairly large finite sets $A$ and $B$ and arrange each set’s elements in a row, aligned as in the diagram below:

\[
\begin{align*}
\text{elements of } A &: \quad \ldots \ldots \\
\text{elements of } B &: \quad \ldots \ldots \ldots \ldots \ldots
\end{align*}
\]

In the diagram, it is clear that $B$ has more elements than $A$ even before we attempt to count the number of elements in each set. This is because, without concern for the actual numbers of elements involved, we can see that there is no way to match up the elements of $A$ with
those of $B$ in a one-to-one way. The same sort of reasoning shows that the sets $C$ and $D$ below do have the same size:

\[ \text{elements of } C: \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]  
\[ \text{elements of } D: \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

Thus, Cantor reasoned, two sets, whether finite or infinite, can be said to have the same size if their elements can be matched up one for one; moreover, a set $A$ can be said to be smaller than a set $B$ if the elements of $A$ can be matched with those of a subset of $B$, but not conversely.

Using this method of comparing infinite sets, Cantor showed that the even numbers 0, 2, 4, \ldots form a set that has the same size as the entire set of natural numbers, whereas the set of real numbers is strictly bigger than the set of natural numbers.\(^{21}\)

Cantor went on to make an even more startling discovery about the infinite: For any set $A$, the collection of subsets of $A$ is strictly bigger than $A$ itself. Using notation from set theory, we can say that $P(A)$ is bigger than $A$, or more briefly, $A < P(A)$, for any set $A$, where $P(A)$ stands for the set of all subsets, or power set, of $A$. In particular, we have the following endless sequence of infinite sets, each one bigger than the previous one:

\[ N < P(N) < P(P(N)) < \ldots \]

Cantor hoped in his time that the sequence of infinities given above would include all possible infinite sizes of sets. (Nowadays, it is known that this hypothesis about the sequence of infinite sizes, known as Cantor’s Generalized Continuum Hypothesis, is consistent with ZFC, but not provable from it.) Since he was unable to prove his conjecture, he devised a hierarchy of numbers—which he called transfinite numbers and which in contemporary language are called infinite cardinals—that were intended to represent all possible infinite sizes. In modern-day notation, Cantor’s infinite cardinals form a subclass of the ordinal numbers.

\(^{21}\) See Rucker (1982) for a popular treatment of this famous result; Hallett (1988) for a historical treatment; Roitman (1990) for a pedagogical treatment; and Weinless (2011) for a treatment that interfaces with principles from Maharishi Vedic Science.
discussed above; in the context of ordinals, a cardinal number can be
defined to be any ordinal which does not have the same size as any of
its predecessors. Every finite ordinal (i.e., every natural number) is also
a finite cardinal; the first few infinite cardinals are listed below:

\[ \omega, \omega_1, \omega_2, \ldots, \omega_\omega, \omega_\omega + 1, \ldots \]

In particular, if Cantor’s Generalized Continuum Hypothesis happens
to be true, we have the following neat correspondence:

- the size of \( N \) is \( \omega \)
- the size of \( P(N) \) is \( \omega_1 \)
- the size of \( P(P(N)) \) is \( \omega_2 \)
- 
- 

It is helpful for our mathematical intuition to view the progression of
the infinities of set theory from \( \omega \) through Cantor’s hierarchy as a grow-
ing approximation to a full description of the ultimate nature of the
Infinite. We shall see that as we climb the hierarchy of infinities, more
and more of the qualities of the field of pure self-referral consciousness,
as described by Maharishi Vedic Science, become embodied in these
cardinals. Thus, for example, the smallest infinite cardinal, \( \omega \), simply
embodies the quality of unboundedness in that for every number \( n \) less
than \( \omega \), \( n + 1 \) is also less than \( \omega \). The qualities of completeness, indescrib-
ability, self-referral, all-inclusiveness, self-sufficiency and others, which we
find present in the ultimate Infinite are absent from \( \omega \); however, as we
will see, these qualities begin to be expressed by cardinals higher up in
the hierarchy. Climbing to the level of large cardinals, we will find that
deep properties of the universe \( V \) as a whole begin to be reflected into
sets having large cardinal size; thus, it is natural to study large cardinals
to gain an intuitive sense of the nature of \( V \) as a whole.
§9. Large Cardinals

A *large cardinal* is a cardinal which cannot be obtained using any conceivable set-theoretic operation on the cardinal numbers below it. Each of the first few infinite cardinals (see the list above) can be obtained by applications of the axioms of set theory to cardinals which occur earlier in the list, and hence are not large. For instance, \( \omega \) is obtained explicitly from one of the axioms (the axiom says, essentially, “\( \omega \) exists”). \( \omega_{\omega} \) is obtained as the union of the cardinals which are below it: \( \omega_{\omega} = \bigcup \{ \omega_n : n < \omega \} \). The cardinality of the stage \( V_{\omega + 1} \) is the size of the power set of the previous stage \( V_\omega \). Each of the cardinals whose existence is derivable in set theory is obtained in a similar way, building up from below. But a large cardinal does not arise in this way. A famous theorem due to Kurt Gödel shows that it is impossible to prove that large cardinals exist at all.23

If large cardinals cannot be proven to exist, why haven’t mathematicians discarded the concept altogether? One major reason is that large cardinals are a central part of a number of basic results in mainstream mathematics. There are problems in measure theory, topology, algebra, and logic whose solutions involve large cardinals in an indispensable way.

To get a feeling for large cardinals, we consider the smallest of the large cardinals, *inaccessible* cardinals. One definition of an inaccessible cardinal is the following: a cardinal \( \kappa \) is inaccessible if \( \kappa > \omega \) and the stage \( V_\kappa \) has the following two properties:

1. \( V_\kappa \) is not the union of fewer than \( \kappa \) many of the earlier stages \( V_\alpha \).
2. The size of any previous stage \( V_\alpha \) is less than \( \kappa \).

We can see rather quickly that the ordinary infinite cardinals we have described so far could not possibly be inaccessible. For instance, if we consider the cardinal \( \omega_1 \), it can be shown that property (2) fails because

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22 By “conceivable set-theoretic operation,” we mean an operation that can be formalized in ZFC.
23 For an introduction to large cardinals for the nonmathematician, see Rucker (1982). For more formal treatments of this subject, including a discussion of all results mentioned in this section of the paper, see Rothman (1990); Kanamori and Magidor (1978); Drake (1974); Jech (1978); and Corazza (2000). For an excellent discussion of large cardinals and their relationship to the principles of Maharishi Vedic Science, see Weinless (2011).
the size of $V_{\omega + 1}$ is at least as big as $\omega_1$. On the other hand, if we consider the cardinal $\omega_\omega$, we can see that property (1) fails; in fact $V_{\omega_\omega}$ is the union of just $\omega$ many previous stages:

$$V_{\omega_\omega} = V_\omega \cup V_{\omega_1} \cup V_{\omega_2} \cup \ldots$$

We have established that if an inaccessible cardinal exists at all, it must be extremely big. One indication of the enormity involved is the fact that if $\kappa$ is inaccessible, $\kappa$ must have the property that

$$\omega_\kappa = \kappa.$$ 

Our experience tells us that the phenomenon indicated by (*) is very unusual: $1 < \omega_1$; $2 < \omega_1$; $\omega < \omega_\omega$, and so forth. To find a $\kappa$ with the property (*) would require a very long journey through the hierarchy of cardinal numbers (and using ZFC alone, even in an endless journey, a large cardinal would never turn up).25

As we indicated earlier, bigger infinities in the universe can be understood to be sets which embody more of the qualities of the ultimate nature of the infinite. This point can be illustrated especially well with inaccessible cardinals: Properties (1) and (2) above indicate not only that an inaccessible cardinal embodies a very strong form of unboundedness, but also that an inaccessible cardinal is truly transcendental, beyond intellectual apprehension—and these are well-known qualities of pure consciousness (Maharishi, 1969):

*The senses, they say, are subtle; more subtle than the senses is mind; yet finer than mind is intellect; that which is beyond even the intellect is be.*

—Bhagavad-Gita, 3.42

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24 Note that each natural number lies in $V_\omega$, thus each (possibly infinite) subset of the set of natural numbers is a member of $V_{\omega + 1}$. But there are at least $\omega_1$ such subsets. Hence the size of $V_{\omega + 1}$ is at least $\omega_1$.

25 It is interesting to note that a cardinal having property (*) will indeed turn up after a sufficiently long climb, but such cardinals will not be large in the technical sense. The least such cardinal can be obtained by taking the supremum of the sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ defined by $\alpha_0 = \omega$, $\alpha_{n+1} = \omega_\beta$ where $\beta = \alpha_n$. On the other hand, if there is a sufficiently large cardinal $\kappa$ (such as a measurable), nearly all cardinals below $\kappa$ have property (*)!
Yato vāho nivartante aprāpya manasā saha

From where speech returns, even with the mind it is unapproachable.

—Taittiriya Upanishad 2.4.1

Another quality of the infinite which is embodied in inaccessible cardinals is indicated by property (✱) above: if κ is inaccessible, it is its own index, and hence in a sense is known and verified only at its own level. This property of the infinite is brought out in (Maharishi, 1991b, p. 190) in comparing the field of pure consciousness with the structure of the unified field discovered by modern physics:

Ultimately, because the unified field is completely holistic in its nature and interacts with itself alone, it can be verified only at its own self-referral level.

We find the same theme expressed more succinctly in Maharishi’s (1967) commentary to the Bhagavad-Gita (p. 120):

Realization is not something that comes from outside: it is the revelation of the Self, in the Self, by the Self.

Adding large cardinals to set theory increases the power of the theory to decide a wide variety of mathematical questions and also serves to unify apparently antagonistic theories and views of foundations. When we speak of “adding large cardinals” to set theory, what we mean is “adding a large cardinal axiom to the list of ZFC axioms.” A large cardinal axiom is an assertion of the form “A cardinal number having property P exists,” where property P is some combination of properties which (consistently) imply (1) and (2) above. Adding to ZFC the axiom “There exists an inaccessible cardinal” (known as the Axiom of Inaccessibility) tremendously increases the power of set theory; new and interesting results can be proven which could not be proven in ZFC alone.

Below we give the names of many of the better known large cardinals in increasing order of strength: For instance, adding the axiom “There exists a huge cardinal” to ZFC is much stronger than adding the axiom “There is a measurable cardinal.”

inaccessible
weakly compact
The procession of large cardinals given in the list above comes to a dramatic halt when we arrive at “\(n\)-huge for every \(n\),” for if we attempt to strengthen this axiom even the least bit—say by considering the concept of \(\omega\)-huge—we run into logical contradictions. We will see that, by carefully formulating a large cardinal axiom that is slightly stronger than “\(n\)-huge for every \(n\),” but not quite as strong as “\(\omega\)-huge,” riding, so to speak, on the edge of inconsistency, we can account for all other large cardinal axioms. We will see that this new axiom will be strongly motivated by principles in both Maharishi Vedic Science and the theory of large cardinals.

§10. Early Attempts to Justify Large Cardinals

Modern set theory has been rather helpless in trying to explain the peculiar phenomenon of large cardinals. A naive solution to the dilemma would be simply to declare that large cardinals exist; we could simply add large cardinal axioms to the axioms of ZFC and the worries would be resolved. However, all the axioms of set theory have strong intuitive motivation; each axiom is a simple property of sets that really ought to be true about sets. Why should an axiom of the form “There exists a large cardinal” be true?

One of the most successful efforts to motivate such large cardinal axioms involves a concept known as the Reflection Principle. The Reflection Principle asserts that any property which is true of the universe \(V\) as a whole should be true of some set; in addition, any property which is true of the class of ordinals as a whole should be true of some particular ordinal. The reason the Reflection Principle is reasonable is that the universe and the class of all ordinals represent a kind of absolute
infinity which is too vast to be captured by a single property; if for some property \( R \), the universe \( V \) were the only collection which had the property \( R \), i.e., no set had this property, then \( V \) could actually be defined as the unique collection which satisfies the property \( R \). This sort of conclusion is intuitively unappealing; \( V \) ought to be somewhat more rich than the property \( R \) is able to express. In some sense, being so vast, \( V \) ought to be “indescribable.” Thus, the Reflection Principle makes sense, and appeals to the intuition of many set theorists.

Although the universe \( V \) was not formally defined in Cantor’s time, Cantor had an intuitive conception that, beyond all sets of all possible infinite sizes, there must lie an Absolute Infinite beyond which no larger infinity could be conceived, an Infinite whose properties no mere set could ever begin to approximate (Hallett, 1988):

The Absolute, says Cantor, is the ‘veritable infinity’ whose magnitude is such that it ‘...cannot in any way be added to or diminished, and it is therefore to be looked upon quantitatively as an absolute maximum. In a certain sense, it transcends the human power of comprehension, and in particular is beyond mathematical determination.’ (p. 13)

Further (Hallett, loc. cit.):

What surpasses all that is finite and transfinite...is the single completely individual unity in which everything is included . . . . (p. 13)

Cantor’s intuition about the Absolute Infinite was the original motivation for the work done in the last quarter century on the Reflection Principle. Moreover, recent research by Jensen (\( \infty \)), Friedman (1993), and others, provides impressive ways of demonstrating the transcendent vastness of \( V \); they show that it is impossible to prove that \( V \) can be obtained by expanding any of the known highly structured well understood models of set theory by means of standard expansion techniques. 

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26 See (Hallett, 1988) for a historical discussion of Cantor’s original notion of the Absolute Infinite. See (Reinhardt, 1974) for a justification of the Reflection Principle based on Cantor’s notion of the Absolute Infinite. See Weinless, (2011) for a detailed study of the Reflection Principle and its relationship to the principles of Maharishi Vedic Science.

27 In particular, they show that \( V \) cannot be proven to be a locally Cohen generic extension over \( L \), nor, assuming large cardinals, over any known form of the core model \( K \). See Section 20 of this paper or Weinless (2011) for a discussion of \( L \) and of forcing extensions of a model of set theory.
Before explaining how the Reflection Principle is used to justify large cardinals, let’s consider a simpler application of this principle. Consider the following property:

\[ R(\alpha) : \alpha \text{ has infinitely many predecessors.} \]

\( R \) is a property which is true of the class \( ON \) of all ordinals: if we think of \( ON \) as the largest of all ordinals, then \( R(ON) \) is true because \( ON \) does indeed have infinitely many predecessors. The Reflection Principle then tells us that there must be an actual ordinal number \( \alpha \) (i.e., an ordinal not equal to \( ON \)) which also has the property \( R \). The simplest example of such an \( \alpha \) is the least infinite ordinal \( \omega \).

Let us now turn to the justification of the existence of inaccessibles offered by the Reflection Principle. The property \( R(\alpha) \) we wish to consider is

\[ R(\alpha) : (1) \ \alpha \text{ is a cardinal} > \omega, \text{ and } V_\alpha \text{ is a stage of the universe which is not the union of fewer than } \alpha \text{ previous stages, and} \]
\[ (2) \text{ the size of } V_\alpha \text{ is less than } \alpha \text{ for all } \alpha < \alpha. \]

If we assume for the moment that \( V = V_{ON} \) as if \( V \) were obtained as its own last stage, then \( V \) is not the union of fewer than \( ON \) many stages, so part (1) of \( R(V) \) holds. As for (2), it is also clear that the size of any stage is an actual cardinal number, hence less than \( ON \) itself. Thus (2) holds as well. By the Reflection Principle, there must be an ordinal \( \kappa \) such that \( R(\kappa) \) holds. Hence, there is an inaccessible cardinal in the universe.

The Reflection Principle goes a long way toward justifying the presence of large cardinals in mathematics, but is not entirely successful.\(^{28}\) First, as far as anyone knows, the very largest of the large cardinals cannot be justified with these reflection arguments. Second, and more importantly, the Reflection Principle is not entirely precise in its formulation—which “properties” \( R(\alpha) \) are we allowed to use? If, for example, we try the property

\[ R(\alpha) : \text{ Every set is a member of } \alpha, \]

\(^{28}\) See Weinless (2011) for a justification of ineffable cardinals using the Reflection Principle and Reinhardt (1974) for a justification of measurables and extendibles.
we are faced with the undesirable fact that although $R(V)$ is true, $R(A)$ is false for every set $A$. To fully understand the origin of large cardinals, we need a deeper and more exact principle than the Reflection Principle.

Although the Reflection Principle does not give a complete solution to the problem of large cardinals, it does give us a significant hint: Large cardinals exhibit properties of the universe as a whole. As we observed earlier, the most noticeable omission in the development of modern set theory—at least from the point of view of Maharishi Vedic Science—is the lack of an axiom describing the nature of the universe $V$ as a whole. The Reflection Principle tells us that the information about the nature of the wholeness of $V$ is revealed ever more fully in the properties of ever larger large cardinals.

Our plan is therefore to consider the very largest of the large cardinals and see what properties they exhibit; these properties should suggest to us what $V$’s “nature” is, as a wholeness. We will see that the strongest of these large cardinal axioms assert that the universe $V$ can be embedded in another model of set theory in a highly coherent way. To grasp the subtleties involved, we must introduce the concept of a model of a theory.

§11. Mathematical Theories and Their Models
A mathematical theory, like set theory, is a collection of all the mathematical statements that can be proven from a set of basic axioms for the theory. Models of a theory are the concrete structures in which the dynamics inherent in the theory are fully realized. Using terminology from Maharishi Vedic Science, Weinless (2011) observes that the axioms for a theory correspond to the concept of “pure knowledge,” and its models are the expression of its organizing power. At the same time, as we shall see, models, their elements, and a fundamental logical relationship between them provide a direct parallel to principles in Maharishi’s theory of knowledge.

Models of a theory, even though concrete realizations of exactly the same set of axioms, may differ radically in their structure. To illustrate

29 Here we are not intending to make precise use of the converse of the Reflection Principle ("large cardinal properties which hold true of $\kappa (V)$ also hold true of ON ($V$)"). We intend only to assert that experience with the Reflection Principle suggests that larger large cardinals reflect ever more fully the properties of the universe as a whole.
this crucial point, we consider a simple set of axioms for the operation of addition and note the wide range of models admitted by the theory. The basic properties of the operation of addition are that it is commutative, associative, and has an additive identity which is usually denoted by ‘0’. Let us assume we have in our formal language the symbol 0; we can state these three properties as basic axioms:

**Axioms for Addition**

1. (Additive Identity) For all \(x\), \(x + 0 = x\)
2. (Commutativity) For all \(x, y\), \(x + y = y + x\)
3. (Associativity) For all \(x, y, z\), \((x + y) + z = x + (y + z)\)

The model we have in mind when we set forth these axioms is the structure \(\mathbb{N}\) of the natural numbers, that is, the set \(\{0, 1, 2, \ldots\}\) together with the usual operation of addition. And it is certainly true that all three axioms hold in the model \(\mathbb{N}\); we say that \(\mathbb{N}\) is the intended interpretation of the theory. Often, however, much of the richness of a theory is discovered in studying the models of the theory that are quite different from the intended one; this is certainly the case in set theory.

Examples of other models of this “theory of addition” abound; consider the set \(\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}\) of integers or the set \(\mathbb{Q}\) of all rational numbers, each with its usual operation of addition. Both these structures satisfy all three axioms; yet each satisfies unique properties that do not hold in our intended interpretation \(\mathbb{N}\). For instance, \(\mathbb{Z}\) has the property:

4. There are \(x, y\), both not equal to 0, such that \(x + y = 0\);

and \(\mathbb{Q}\) has the property:

5. For any \(z > 0\), there are \(x, y > 0\) such that \(z = x + y\).

Neither property holds in the structure \(\mathbb{N}\).\(^{30}\) Property (4) makes use of the presence of the negative integers in \(\mathbb{Z}\); property (5) makes use of

\(^{30}\) Property (4) fails because the only solution to the equation \(x + y = 0\) in \(\mathbb{N}\) is \(x = 0, y = 0\). Property (5) fails because there do not exist two positive integers having a sum of 1.
the fractions in $\mathbb{Q}$. Yet, both $\mathbb{Z}$ and $\mathbb{Q}$ are perfectly valid models of our theory of addition even if they exhibit properties that may be unexpected from the point of view of our intended interpretation.

§12. Models of Set Theory
The simple situation described in the last section parallels the state of affairs in set theory. We have a set of intuitively appealing axioms for describing the behavior of sets, namely, the axioms of ZFC; and we have an intended interpretation of ZFC, namely, the universe $V$. As we mentioned earlier, $V$ is often called the standard universe of sets. Nonetheless, ZFC admits other interpretations; that is, it is quite possible to have a wide range of models of ZFC. Each model, since it satisfies all the basic axioms of sets, is a suitable universe of sets in its own right; each can be considered an adequate background for all of mathematics. Yet, models of ZFC may be radically different in many respects. There are many mathematical statements which cannot be either proven or disproven from the axioms of ZFC; these will hold true in some models of set theory and be false in others. Such statements are said to be independent of ZFC.

The most famous independent statement is known as Cantor’s Continuum Hypothesis. Cantor’s Continuum Hypothesis is a conjecture that states that the size of the set of real numbers is precisely the cardinal number $\omega_1$, just one level of infinity greater than the size of the set of natural numbers.31 He was never able to prove this result, and some 70 years later, Paul Cohen demonstrated why: Cohen produced models of set theory in which Cantor’s Continuum Hypothesis was false (the size of the real line turns out to be $\omega_2$ in one model, $\omega_1$ in another, and $\omega^{\omega_1}$ in yet another); some years earlier, Kurt Gödel had shown that the Continuum Hypothesis was true in certain other models. The two results together demonstrate that the Continuum Hypothesis is independent of the axioms of ZFC set theory. (See Jech, 1978, for a modern-day treatment of these results.)

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31 Cantor was able to prove unequivocally that the set $\mathbb{R}$ of real numbers is larger than the set of natural numbers; the question was, how much bigger? See Hallett (1988) for the historical development leading to Cantor’s Continuum Hypothesis; see Kunen (1980) and Jech (1978) for a modern-day formal treatment of statements that are independent of ZFC; see Weinless (2011) for a discussion of independence results in the context of Maharishi Vedic Science.
Set theorists look at this multiplicity of possible universes for mathematics as different points of view about the nature of sets; if you look through the glasses provided by one model of set theory, sets appear to have one set of properties; through glasses provided by another, a different set of properties emerge. In each universe, the basic laws given by the ZFC axioms remain true, but independent statements like the Continuum Hypothesis will be settled in different ways.

This attitude that different models of ZFC represent different views of the universe represents an important approach to research among set theorists. To illustrate this approach, consider the following perplexing fact: If ZFC has a set model, then one can actually find (using techniques familiar to logicians) a model $M$ of ZFC which has the same size as the set of natural numbers. Yet, being a model of ZFC, it must contain the set of all real numbers, a set which is bigger than the model itself! The paradox is resolved by observing that the model $M$ “doesn’t know” that it has the same size as the set of natural numbers; the one-to-one match-up between $M$ and the natural numbers is not available to $M$’s “world-view”. Being a model of ZFC, $M$ “knows” that the set of reals is bigger than the set of natural numbers, and “believes” that its own structure is much vaster than that of the reals, expansive enough to include all cardinal numbers. From the perspective of the real world $V$, $M$’s version of the real line is only countable and so $M$’s view of the world is somewhat distorted.

This manner of ascribing the subjective qualities of “knowing” and “believing” to models of set theory is very natural and parallels to a high degree Maharishi’s principle that knowledge is different in different states of consciousness. Recall that according to Maharishi Vedic Science, as more of the value of pure consciousness becomes available to the experiencer, the nature of his knowledge of any object of knowing also changes, reflecting a more holistic and comprehensive appreciation of whatever is known. Now, just as varying the state of consciousness produces different knowledge about the world, so varying the model of set theory produces different truths about sets. And, just as there is a level of awareness$^{32}$ which automatically perceives the ultimate truth of the world, so the universe $V$ may be seen as an absolute reference frame in which truth is absolute truth.

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$^{32}$ Maharishi calls this level of consciousness *Unity Consciousness*; see Maharishi (1972).
We can take the analogy further, and it will be useful to do so to lay the groundwork for later work. As the reader will recall, Maharishi points out that there are three components to the emergence of knowledge: the knower (Rishi), the object of knowledge (Chhandas), and the process of knowing (Devatā).\footnote{Cf. Chandler (1987, 5–26).} So far, we have identified models of set theory as analogous to the Rishi aspect and individual sets as analogous to the Chhandas value. Interestingly, Weinless (2011) observes the same analogy with somewhat different motivation. What corresponds to the Devatā value? The way that statements about sets are determined to be true or false in a given model $M$ is by means of the satisfaction relation ($\models$), a logical relation that is designed to systematically\footnote{The word “systematically” is not intended to mean “algorithmically” here.} determine the truth or falsity of a given statement relative to $M$. Consider for instance the statement

\[ \text{The real line } \mathbb{R} \text{ has size } \omega_5. \]

Since this statement depends on the two sets $\mathbb{R}$ and $\omega_5$, we can name the above statement $\phi(\mathbb{R}, \omega_5)$. ($\phi$ is called a formula with parameters $\mathbb{R}$ and $\omega_5$.) If $M$ is a model of set theory in which $\mathbb{R}$ does indeed have size $\omega_5$, we would write:

\[ M \models \phi(\mathbb{R}, \omega_5). \]

In this type of expression, common in the literature in set theory, we see the analogues to Rishi, Devatā, and Chhandas clearly displayed: $M$ corresponds to Rishi; the sets in $M$—in this case $\mathbb{R}$ and $\omega_5$—correspond to Chhandas; and the satisfaction relation $\models$ corresponds to Devatā. Recall that in Maharishi’s theory of knowledge, knowledge is what emerges in the relationship, or “togetherness,” of Rishi, Devatā, and Chhandas; likewise, the relationship of $M$, $\models$, and the sets $\mathbb{R}$ and $\omega_5$ results in $M$’s (partial) “knowledge” of these sets.

To sum up, we see that models of set theory are closely related to Maharishi’s theory of knowledge: A model of set theory together with its satisfaction relation ($\models$) and its members, i.e., sets, correspond to the basic principles of Rishi, Devatā, and Chhandas; moreover, just as different levels of consciousness provide the knower with different truths
and knowledge about reality, so different models of set theory provide
different views of sets and their relationships.

We next turn to a consideration of the relationship between different
models of set theory; elementary embeddings offer the most interesting
of these possible relationships by providing a natural analogue to
self-knowledge. As a preliminary to the notion of elementary embed-
ddings, we introduce its conceptual components: elementary submodels
and isomorphisms.

§ 13. Elementary Submodels and Isomorphisms
Whenever mathematicians encounter a proliferation of differences, like
the variety of models of set theory, one question that naturally arises
is, “In what ways are these different objects actually the same?” In the
context of models of set theory, this question is answered in two ways:
Models are basically the same either if they are isomorphic or if one is an
elementary submodel of another.35 We will see later that both these con-
cepts are intrinsic features of the fundamental nature of $V$ as it moves
within itself and “knows” itself.

First, let’s consider elementary submodels. Suppose we have two
models of set theory, $M$ and $N$, where $M$ is a subset of $N$. It is conceiv-
able that, concerning the sets that both models know about, namely,
the sets inside $M$, both models could have exactly the same knowl-
edge, and believe exactly the same statements. When this phenomenon
occurs, $M$ is said to be an elementary submodel of $N$ and we write

$$M \prec N.$$  

Stated more precisely, we say $M$ is an elementary submodel of $N$ if
$M \subseteq N$ and for any sets $A_1, A_2, \ldots, A_n$ in $M$ and any relationship
$\phi(A_1, \ldots, A_n)$ between them, $\phi$ is true in $M$ if and only if $\phi$ is true in
$N$. To state the matter another way, $M$’s knowledge of the sets $A_1, \ldots,
A_n$ is exactly the same as $N$’s knowledge of them; borrowing terminol-
ogy from Maharishi Vedic Science, we could say that $M$ and $N$ exhibit

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35 Actually, there are other notions of sameness between models. One notion of equivalence that
is at least as prevalent in the work of model theorists as elementary submodels and isomorphism is
that of elementary equivalence; two models are elementarily equivalent if they satisfy the same first
order sentences. If two models are isomorphic, or if one is an elementary submodel of the other,
then the models are elementarily equivalent. See Chang and Keisler (1973).
“infinite correlation” in terms of their knowledge of the sets in $M$, even though, in many ways, $M$ and $N$ may appear different (for instance, the two models might have radically different sizes).

Next we consider isomorphic models. Suppose again that $M$ and $N$ are two models of set theory. $M$ and $N$ are said to be isomorphic if there is a way to transform $M$ into $N$ so that all relationships and differences among sets in $M$ are preserved in $N$. More precisely, $M$ and $N$ are isomorphic if there is a way to match up each set $A$ in $M$ one-for-one with a set $B$ in $N$ so that for any formula, $\phi(A_1, \ldots, A_n)$ is true of the sets $A_1, \ldots, A_n$ in $M$ if and only if $\phi(B_1, \ldots, B_n)$ is true of the sets $B_1, \ldots, B_n$ in $N$. Intuitively, $M$ and $N$ exhibit the same properties and dynamics qua universes of sets; each has its own version of the empty set; each has its own version of the real line $\mathbb{R}$ and of the cardinal numbers. If, for example, $M$ thinks that its own version of $\mathbb{R}$ has size equal to its own version of $\omega_5$, then $N$ will believe that the object in $N$ matched with the $\mathbb{R}$ in $M$ has size equal to the cardinal in $N$ matched with the $\omega_5$ of $M$.

Elementary submodels and isomorphic models exhibit a preservation of fundamental structure in the face of certain types of transformation. An elementary submodel of a model can be said to have maintained the structure of the large model in the face of “miniaturization.” An isomorphic copy of a model $M$ can be said to have maintained the structure of $M$ in the face of a “name reassignment.”

These kinds of preservation are central in the study of set theory. Moreover, they exhibit a basic property of the dynamics of pure consciousness itself, as described by Maharishi: the many stages of expression that emerge from the self-interacting dynamics of pure consciousness always remain connected with the holistic Samhitā value of consciousness; this feature is the basis for the qualities of infinite correlation and invincibility that are ascribed to pure consciousness.

In the next section, we will encounter the surprisingly powerful consequences that result from combining the concepts of elementary submodel and isomorphism in the context of models for ZFC.

§14. Elementary Embeddings of the Universe

Having considered the concept of a model of set theory and the possible structure-preserving relationships between models, we can return to
our study of the strongest large cardinal axioms to see what conclusions can be drawn about the nature of $V$ as a whole.

In the hierarchy of large cardinals, those at the upper end, like measurable, strong, supercompact, and huge cardinals, are defined in terms of a special kind of transformation $j$ called a nontrivial elementary embedding of the universe. A typical embedding of this kind is given by an expression like the following:

$$j : V \rightarrow M,$$

where $M$ is a transitive model of set theory containing all the ordinals. The behavior of $j$ can be considered in two steps. First, $j$ isomorphically transforms $V$ into another model $V'$ where $V'$ forms a subcollection of $M$. Secondly, the model $V'$ is an elementary submodel of $M$. See Figure 2.

Figure 2. An elementary embedding of the universe $V$.

From our discussion in the last section, it follows that $V, V'$ and $M$ are all extremely similar in their structure. In fact, $V$ and $M$ bear the following relationship:

For all sets $A_1, \ldots , A_n$, and for any relationship $\phi(A_1, \ldots , A_n)$ between them, $\phi(j(A_1), \ldots , j(A_n))$ holds true in $M$.

$^36$ $M$ is transitive if for any $y \in M$ and any $x \in y$, we have $x \in M$. The transitive models tend to be easiest to understand because their elements are “normal” sets.
This property of \( j \) is simply a combination of the fact that \( V \) and \( V' \) are isomorphic via the transformation \( j \) and that \( V' \) is an elementary submodel of \( M \).

To illustrate the property, suppose it is true that in \( V \), the size of \( \mathbb{R} \) is \( \omega_5 \). Then given \( j \) as above, it follows that in \( M \), the size of \( j(\mathbb{R}) \) is \( j(\omega_5) \). Moreover, since \( V \) believes \( \mathbb{R} \) is the real number line, \( M \) will believe that \( j(\mathbb{R}) \) is the real number line. And since \( V \) believes \( \omega_5 \) is the fifth uncountable cardinal, \( M \) will believe that \( j(\omega_5) \) is the fifth uncountable cardinal. See Figure 3.

![Figure 3](image-url)  
**Figure 3.** Preservation of relationships under elementary embeddings.

The embedding is called *nontrivial* to eliminate the possibility that \( j \) is merely the identity function from \( V \) to itself, the function which assigns to each set \( A \) in \( V \) the set \( A \) itself; the identity function, though important in its own way, does not have any powerful mathematical consequences. In particular, if \( j \) is a nontrivial elementary embedding, some set in \( V \) must be sent to a set different from itself in \( M \); we say that some set is *moved* by \( j \). See Figure 4.
As we mentioned above, elementary embeddings of the universe $V$ give rise to large cardinals. It can be shown that if a set is moved by $j$, some ordinal must also be moved. The least ordinal moved is called the **critical point of $j$**. This ordinal is typically denoted by a Greek letter; in this paper, we use the letter $\kappa$ (pronounced “kappa”). Moreover, this critical point is necessarily a large cardinal (at least a measurable cardinal). (See Jech, 1978.) Thus, the first ordinal moved by this very natural–seeming transformation of $V$ into another universe is infused with extraordinary properties of infinity.

The stronger large cardinal axioms assert the existence of nontrivial elementary embeddings of various kinds. As we remarked earlier, experience with the Reflection Principle suggests that the larger large cardinals reveal properties of the universe as a whole. We suggest therefore that $V$ “tends” to move within itself: It is a characteristic of the structure of $V$ to be moved into a universe of sets (this universe could be $V$ itself or some other universe) via an elementary embedding. In this way, large cardinals are generated that allow us to see, in the realm of sets, what is true about a wholeness—$V$—that is beyond our ability to know.

Moreover, it turns out that the stronger the large cardinal generated by an embedding (or class of embeddings) of the form $j: V \to M$, the more closely $M$ must resemble $V$ in its structure. To illustrate this

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**Figure 4.** Nontrivial elementary embeddings vs. the identity map.
pattern, let us contrast the definition of a measurable cardinal with that of the much stronger concept of a strong cardinal. We begin with the formal definition of a measurable cardinal:

**Definition (Measurable Cardinals).** $\kappa$ is a measurable cardinal if there is a nontrivial elementary embedding $j : V \rightarrow M$ having critical point $\kappa$.

Figure 5 displays the behavior of such an embedding. No special extra properties have been imposed on $j$ or $M$; the presence of any nontrivial elementary embedding of the universe is sufficient to give rise to a measurable cardinal.

As Figure 5 indicates, in addition to resembling $V$ in all the ways that an elementary embedding requires, $M$ also resembles $V$ in that its first $\kappa$ stages are the same as the first $\kappa$ stages of $V$. One writes:

$$V_\kappa = V^{M}_\kappa.$$

**Figure 5.** Measurable cardinals and strong cardinals

**Definition (Strong Cardinals).** $\kappa$ is a strong cardinal if for every cardinal number $\lambda \geq \kappa$, we can find a universe $M_\lambda$ and a nontrivial elementary embedding $j_\lambda : V \rightarrow M_\lambda$ having critical point $\kappa$, such that
What is new here is that an entire class of embeddings is guaranteed to exist and the corresponding universes $M_\lambda$ resemble $V$ more and more completely as $\lambda$ increases; as $\lambda$ increases, more and more stages of $V$ are required to be identical to those in the image model. In particular, for each $\lambda$, the first $\lambda$ stages of $V$ are identical to the first $\lambda$ stages of $M_\lambda$.

Still stronger large cardinal axioms require, sometimes in subtler ways, that the image models $M$ resemble $V$ even more closely.

If stronger and stronger large cardinal axioms assert the existence of embeddings $j : V \rightarrow M$ with $M$ resembling $V$ more and more closely, could it happen that $M = V$? An axiom that asserts the existence of a nontrivial elementary embedding $j : V \rightarrow V$ would represent the natural culmination of all previous large cardinal axioms; one would expect that the large cardinal that would emerge from such an embedding as its critical point would have the strongest properties of all.

Moreover, let us consider the implications of such an embedding $j : V \rightarrow V$ in light of our earlier analogy between models of set theory and Maharishi’s theory of knowledge. As the reader will recall, $V$ “knows” about the properties of the sets it contains via the satisfaction relation; here, $V$ plays the role of knower; sets, the known; and the satisfaction relation, the process of knowing. These dynamics parallel the familiar process of gaining knowledge of the outer world. However, this same process of knowing can be applied to consciousness itself and the result, as described in Maharishi Vedic Science is the dynamics of self-knowing that constitute the eternal unmanifest activity of pure consciousness at the basis for all activity in the manifest world.

Likewise, in the presence of $j$, a fundamental dynamism is introduced that places $V$—representing the Rishi or knower—in relationship with itself. First, $V$ is transformed within itself to $V'$, the image of $V$ under $j$. This transformation of $V$ is completely structure-preserving: All truths about the structure of $V$ are preserved under this transformation. Then, although in certain respects $V'$ appears different from $V$, $V'$ remains infinitely correlated with $V$ in the sense that $V'$ is an elementary submodel of $V$. Thus, as $V$ interacts with itself via the embedding $j$, the structure of $V$ remains intact throughout the phases of transformation.

We find the dynamics embodied in $j$ similar to the dynamics of self-knowing attributed to the wholeness of pure consciousness or
Samhitā: Recall from Maharishi Vedic Science that in order for wholeness to know itself, the fundamental unity of Samhitā appears as three; Samhitā must assume the roles of knower, object of knowledge, and the relationship between them in order for Samhitā to know itself. Samhitā, remaining ever the same, exhibits these divisions within its own nature. Likewise, the nature of $V$ as a whole becomes known when $V$ moves within itself via the embedding $j$. Without $j$, $V$ remains a transcendental wholeness beyond the realm of sets; the embedding $j$, however, dynamically relates $V$ to itself, placing it in different roles in relationship to itself while preserving its fundamental structure: On the one hand, it plays the role of pure Samhitā, the unified value of wholeness; on the other hand, it assumes the roles of Rishi (“knowing” as it does the various truths about its own structure) and Chhandas (as that which is being known). We will see in the next section that new knowledge about $V$’s fundamental structure emerges from this interaction.

Figure 6. An elementary embedding from the universe to itself.

Despite the naturalness of the large cardinal axiom “There is a non-trivial elementary embedding from $V$ to $V$,” K. Kunen (1971) proved, under certain assumptions, that the existence of such an embedding would lead to an inconsistent set theory. In the next section, we discuss this dilemma at some length and suggest an attractive solution. The starting point is the intuition, motivated by Maharishi Vedic Science,
that the dynamics of $V$, which represents wholeness in mathematics, should mirror the dynamics of wholeness itself, the dynamics by which creation itself emerges. This intuition suggests that some kind of truth-preserving embedding from $V$ to itself ought to exist. This intuition will motivate us to look more carefully at the assumptions underlying Kunen’s theorem and its proof.

§15. Attempts to Bypass Kunen’s Theorem

The naturalness of a nontrivial elementary embedding $j : V \to V$ has not gone unnoticed by set theorists. Kunen’s proof that such embeddings do not exist has been studied from a number of different angles to see if some weaker form of elementary embedding retaining the flavor of $j : V \to V$ could still be consistent.

One line of thought that has resulted in deep work by H. Woodin (1989) begins with the observation that Kunen’s proof relies heavily on the Axiom of Choice (see Weinless, 2011, for a discussion of this axiom)—so much so that the proof will not work if the Axiom of Choice is replaced by any of the better known weakenings of this axiom. Consequently, it is quite likely that there is a universe $V$ in which Choice fails but one of these weakenings of Choice still holds, and there is a nontrivial elementary embedding $j : V \to V$. Woodin has shown (unpublished) that such a “choiceless” embedding is still strong enough (in the presence of a weaker choice principle) to consistently imply all known large cardinal axioms; in particular, he has shown how, starting from such a $j$, to build another model $M$ in which the Axiom of Choice holds and all known large cardinal axioms are true.

Woodin’s result is a masterpiece of mathematics, but we do not feel that a universe in which the Axiom of Choice fails is the right starting point for mathematics; nor is it intuitively desirable to have to step into the relativized world of Woodin’s forcing model to gain access to large cardinals.

Another lesser known angle has been to weaken the definition of elementary embedding, the insight being that perhaps “elementary embedding” is simply too powerful a concept to be the “right” notion. Work is still underway in this area by a handful of researchers. The most enticing result so far has involved weakening “elementary embedding” to “exact functor”; an exact functor from the universe to itself.
is one that preserves certain simple functional relationships between finite collections of sets;\textsuperscript{37} it is an especially natural concept in the context of the geometry of sheaves. A. Blass (1976) showed that there is a nontrivial\textsuperscript{38} exact functor from the universe\textsuperscript{39} to itself if and only if there is a measurable cardinal.

This line of research is very promising; so far, however, functors of this kind have not produced cardinals even as large as extendible. Moreover, if such a functor could be devised, it remains to be seen if its properties will be as geometrically natural as “exactness.”

Yet another observation has been that Kunen’s proof does not forbid elementary embeddings from a stage $V_\lambda$ to itself when $\lambda$ is a limit; such an embedding forces $V_\lambda$ to be a model of set theory. One could then ask if such a $V_\lambda$ would be the right foundation for all of mathematics. One might expect the answer to be “no” because $V_\lambda$ fails to include most of the stages of $V$ (namely, those that come after $V_\lambda$), but this problem is not so serious as one might expect. This approach has interesting parallels with Maharishi Vedic Science and plays an important role in the approach that we propose in this paper; we therefore postpone further discussion for a later section (see Section 19).\textsuperscript{40}

§16. The Wholeness Axiom

A closer look at Kunen’s proof reveals another assumption implicit in the reasoning: In order to arrive at an inconsistency, it must be assumed that the elementary embedding $j$ is \textit{weakly definable}\textsuperscript{41} in $V$. Intuitively, this means that $V$ “knows about” the embedding in much the same way it “knows about” sets. If our intuition about $j$ is to be guided by the principle that $j$ corresponds to the fundamental dynamics of wholeness moving within itself, as described by Maharishi Vedic Science,

\begin{itemize}
  \item \textsuperscript{37} Cf. Mac Lane (1971) for a precise definition of \textit{exact functor}.
  \item \textsuperscript{38} In this context, “nontrivial” means “not naturally isomorphic to the identity functor”.
  \item \textsuperscript{39} In this context, the universe is understood in the context of category theory; $V$ is taken to be the category of all sets together with all functions. See Weinless (2011) and (Mac Lane 1971) for further discussion.
  \item \textsuperscript{40} A nontrivial elementary embedding from a stage $V_{\lambda+1}$ to itself for some limit $\lambda$ is also not known to be inconsistent, but Kunen’s proof forbids such an embedding from $V_{\lambda+2}$ to itself. See Kunen (1971).
  \item \textsuperscript{41} Kunen’s proof forbids more than just \textit{definable} elementary embeddings from $V$ to itself; a class $C$ in $V$ is weakly definable if, treating $C$ as an extra predicate in the language, all instances of Replacement in the expanded language hold true.
\end{itemize}
and that sets correspond to manifest existence, we would not expect the dynamics of \( j \) to be on a par with the dynamics of sets. Moreover, we would expect that, if possible, \( j \) ought to be in some way unmanifest, hidden from the more “expressed” activity of sets. Here is what Maharishi says about the dynamics within wholeness:

In the state of one-being-three we have the state of complete unified wakefulness. In this is the first value of transformation in the unmanifest value. When we say ‘transformation’, we still mean this level is unmanifest. Samhitā in terms of Rishi, Devatā, and Chhandas, and Rishi, Devatā, and Chhandas in terms of Samhitā: this is the fundamental transformation, the fundamental relationship. (1990b)

Supreme intelligence does not partake of activity. It is so exalted and powerful that by virtue of its very being it is the field of all possibilities, the source of all action. It is so unlimited that it can function without functioning—its very presence regulates activity so that it is spontaneously right. (1976, p. 131)

Guided by this intuition, we suggest that the “right” axiom for describing the fundamental dynamics of \( V \) should involve an elementary embedding \( j \) which is not (weakly) definable in \( V \). Technically, \( j \), as a “function” from \( V \) to itself, is a subcollection of \( V \), but there is no first-order formula which defines this subcollection.\(^{42}\) We say that \( j \) ought to be a transcendental\(^{43} \) elementary embedding.

\(^{42}\)The definition of “function” must be modified somewhat to be applicable in this context. The most obvious problem is that, because \( j \) is not a set, it cannot properly be called a function either. One may still consider \( j \) to be a vast subcollection of \( V \) consisting of ordered pairs, \( j = \{(x,y): y = j(x)\} \), but this formulation is also not correct because the definition of \( j \) as an elementary embedding requires that the codomain of \( j \) be specified. Thus, to be precise, if we let \( j_0 = \{(x,y): y = j(x)\} \), then we may formally define \( j \) to be the disjoint union of the collections \( j_0 \) and \( V \).

\(^{43}\)Sometimes such embeddings are called “external”; we have chosen not to use this terminology because it incorrectly suggests that \( j \) lies outside of \( V \). Certainly \( j \) is not an element of \( V \) and is not definable in \( V \), but, as we have seen, \( j \) does form a subcollection of \( V \). The point is that \( j \) lies within \( V \) but is not “graspable” within \( V \) in the usual ways; Maharishi Vedic Science provides excellent intuition for this phenomenon.
Definability and Weak Definability

Intuitively speaking, the sets, functions, and other mathematical concepts that mathematicians typically work with are definable. More precisely, if the objects constituting a set (or the ordered pairs comprising a function) are precisely those objects which satisfy some formula (in a given model $M$), then that set (or function) is said to be definable (in $M$).

For example the set $\{0, 2, 4, \ldots \}$ of even natural numbers is definable in the model $\mathbb{N}$ of natural numbers: Consider the formula

$$
\phi(x): \text{there exists } n \text{ such that } x = n + n.
$$

The natural numbers which can be used to replace the variable $x$ to obtain a true sentence are precisely the even numbers.

Next, we consider an example of a definable function in the context of sets rather than natural numbers. Let $F$ be the function defined on sets which assigns to each set $A$ the singleton set $[A]$, so $F(A) = [A]$. To see that $F$ is definable (in $V$), consider the formula

$$
\phi(x, y): \text{the only member of } y \text{ is } x.
$$

Now, we can see that for all sets $A, B$, $F(A) = B$ if and only if $\phi(A, B)$ holds in $V$.

The function $F$ provides a typical, though simple, example of definable functions: The definability of such a function guarantees that there is a uniform procedure for obtaining the output, given any input.

What about weak definability? The difference between these two notions is subtle. For most purposes, the concepts are the same and so we will loosely proceed in this paper as if they were the same.

From the point of view of Maharishi Vedic Science, we would not expect that the behavior of a function intended to represent the unmanifest dynamics of existence could be uniformly described with a single formula; therefore, it is natural to expect that, if a nontrivial embedding $j : V \to V$ exists at all, it must be undefinable (indeed, not even weakly definable).
We must be careful, however, not to remove \( j \) too radically from the world of sets in \( V \). From the mathematical point of view, to insist that \( j \) be transcendental without any other conditions would significantly weaken the axiom—so much so that the resulting axiom would be weaker than a measurable cardinal!

From the perspective of Maharishi Vedic Science, we need to consider somewhat more deeply our analogy between \( j, V \), and sets on the one hand, and the dynamics of pure intelligence, wholeness, and manifest existence, respectively, on the other hand. Maharishi explains that the fundamental self-interacting dynamics of pure intelligence form the blueprint of creation itself, the Veda, and that this field of life is the primary administrator of all of creation (Maharishi 1976):

\[
\ldots \text{consciousness is the prime mover of life and administrator of all action, and} \ldots \text{anyone who develops in himself the full potential of consciousness enjoys a natural authority over the whole field of action and achievement. (p. 123)}
\]

This fundamental field of life has two important attributes:

1. Its activity is hidden from view, unmanifest.
2. Its activity is intimately integrated with creation; in fact, it is present at every point in creation.

Maharishi elaborates on this second point in the *The Science of Being* (1966):

\[
\text{It has been said that Being is the ultimate reality of creation and that It is present in all strata of creation. It is present in all forms, words, smells, tastes and objects of touch; in everything experienced; in the senses of perception and organs of action; in all phenomena; in the doer and the work done; in all directions—north, south, east and west; in all times past, present and future; It is uniformly present. It is present in front of man, behind him, to left and right of him, above him, below him and in him. Everywhere and in all circumstances Being, the essential constituent of creation, permeates everything. (p. 29)}
\]

Maharishi explains that this twofold reality of pure consciousness—that it is both unmanifest and present at every point in creation—is not merely an abstract truth of transcendental existence, but can be made a
living reality in individual life through the development of consciousness (Maharishi 1976, p. 132):

When consciousness is so developed that it can make everything its own, it flows into the channels of relative life while at the same time maintaining its own transcendental, absolute state, non-channelled and all-pervading.

Using these two points to guide the development of our axiom about $j$, we see that our decision to require $j$ to be transcendental corresponds to the first of these points (that the activity within pure consciousness is unmanifest), but that we need to formulate another condition corresponding to the second point (that this activity is present at every point in creation).

A very natural way to frame the second point in mathematical terms emerges when we look to see why a bare transcendental elementary embedding is so weak: The problem is that when we attempt to form sets in the universe using $j$, the collections we form turn out not to be sets at all (since $j$ is transcendental) but, like $j$, remain “hidden from view,” forbidden from interacting with other sets in the usual way.

We may eliminate this discoordinating effect of having an undefinable embedding by requiring that $V$ be fully $j$-closed. This means that whenever we define a subcollection of a given set using $j$ (or one of its iterates $j^n$), the subcollection turns out to be a real set in the universe. This requirement corresponds very nicely to our second point: When $V$ is fully $j$-closed, $j$ is permitted to participate in set formation as a parameter in formulas in exactly the same way individual sets can; since sets represent the point-value of the universe $V$, we can say that full $j$-closedness permits $j$ to play the role of a point-value in the universe.

We can now state our axiom. For completeness, we first give the following formulation of the concept of $j$-closedness:\footnote{The definition and formulation of the concept of $j$-closedness and of the Wholeness Axiom given here are made more technically precise in the mathematical literature (Corazza, 1994, 2000, 2006).}
**Definition:** Suppose \( j : V \rightarrow V \) is a transcendental elementary embedding. Then \( V \) is fully \( j \)-closed if for every set \( A \), every natural number \( m \), every formula \( \phi(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) \) and all sets \( B_1, \ldots, B_n \), the collection

\[
\{ a \in A : \phi(B_1, \ldots, B_n, A, j^m) \}
\]

is a set, where \( j^m \) is the \( m \)-fold composition of \( j \) with itself.

**Wholeness Axiom**

There is a transcendental (nontrivial) elementary embedding \( j : V \rightarrow V \) such that \( V \) is fully \( j \)-closed.

Before exploring the interesting consequences of the Wholeness Axiom, let us summarize what we have accomplished so far. We began our study by observing that, while set theory with its universe of sets has been extremely successful as a foundation for mathematics, the fact that large cardinals, which arise naturally in many areas of mainstream mathematics, cannot be accounted for by ZFC impels one to search for a satisfactory intuition by which to strengthen the present axiom system and thereby determine which large cardinals should be allowed in the universe.

We chose to use principles of Maharishi Vedic Science to clarify our intuition about the nature of wholeness, believing that the wholeness set theorists are attempting to express in the concept of the universe of sets has been examined thoroughly in the Vedic tradition of knowledge. In reviewing the basic qualities of wholeness, as described by Maharishi Vedic Science, we found that a few such qualities—infinitesimal correlation, awake within itself, self-referral, and bliss—were not adequately expressed in the construction of \( V \); in particular, we observed that in order for a universe of sets to exhibit these qualities adequately, it should have an abundance of sets that reflect all the first order properties of the universe, and there should be, associated with it, some natural kind of transformation so that it exhibits a form of self-interaction.

In addition, analysis of the dynamics of pure consciousness revealed that, using ZFC alone, sets emerge from the empty set in a way that is strongly analogous to just one-half of the dynamics of wholeness moving within itself—namely, those dynamics concerned with the expan-
sion of the point value of wholeness to the fully expanded infinite value of wholeness. Thus, we were led to seek a universe exhibiting several new qualities and displaying new dynamics such as “infinity collapsing to a point.” From the point of view of mathematics, we took a hint from the Reflection Principle about where to look for new axioms that talk about the nature of the universe as a whole. This principle suggested to us that large cardinal properties are powerful properties of the infinite that actually ought to be considered properties of the wholeness of $V$ itself. This intuition suggested that, since all the strongest large cardinal axioms are framed in terms of the existence of elementary embeddings of the universe with image models increasingly similar to $V$ itself, we should consider it to be the very nature of the wholeness of $V$ to be moved by such an embedding, and the most natural of such embeddings should have codomain $V$. In light of Kunen’s theorem, we clarified the requirements of the embedding so that it would not be definable (or even weakly definable) but, at the same time, remain coordinated with the structure of $V$ (requiring that $V$ be fully $j$-closed).

We proceed now to show that our efforts have been successful. We will indicate how the creation of sets in the universe, in light of our new axiom, embodies the new qualities and dynamics we have been seeking. We will also see that virtually all known large cardinals can be accounted for in our new set theory.

To begin the discussion, we will first gain a feel for the new dynamics that the Wholeness Axiom introduces.

§17. Simple Consequences of the Wholeness Axiom

The Wholeness Axiom, as we have indicated, has many powerful consequences. However, in this section we focus instead on the new style of reasoning that arises in applications of the axiom. We shall see how proofs from the axiom involve repeated swings between particulars about individual sets on the one hand, and awareness of the nature of $V$ as a whole on the other hand. This continual calling of attention to the fact that $V$ is our underlying model and that $j$ is moving $V$ within itself produces a new dimension of profundity to the mathematical arguments.

In order to appreciate the new feature that arises in reasoning with $j$, we first observe that statements in mathematics are assertions that
certain properties hold with respect to certain sets (even a mathematical computation can be viewed in this way, where the property involved is “equality”). In the language of Maharishi Vedic Science, the particular sets that we reason about in a mathematical argument are “point values”—specific points in the universe. Our attention is focused on manipulating and relating points in the universe; it is not focused on the underlying wholeness in which these activities operate. The new feature that emerges in working with \( j \) is that our attention must swing between properties about points to awareness of \( V \) as a whole, and then back again to points. As we discussed earlier, the creative activity within pure consciousness unfolds through the repeated collapse of infinity to a point and expansion of point to infinity. The fact that a similar phenomenon occurs in working with \( j \) provides further confirmation that addition of the Wholeness Axiom to ZFC introduces dynamics into the foundation of mathematics that mirror those of nature’s functioning.

To illustrate this new feature, let us first recall the characteristic feature of our embedding \( j \): Being an elementary embedding from \( V \) to itself means that for any property \( P \) that talks about sets \( A_1, A_2, \ldots, A_n \), two things must happen:

1. if \( P \) is a true property of the sets \( A_1, A_2, \ldots, A_n \), then \( P \) must also be a true property of the image sets \( j(A_1), j(A_2), \ldots, j(A_n) \);
2. if \( P \) is false with respect to \( A_1, A_2, \ldots, A_n \), then \( P \) must also be false with respect to \( j(A_1), j(A_2), \ldots, j(A_n) \).

In this way, the embedding \( j \) can be seen to be a truth-preserving and truth-reflecting map; every conceivable relationship between sets is maintained both by \( j \) and its “inverse”.

We will now apply these properties of \( j \) in a simple example and point out the new features of the argument. Let us prove that \( j(\emptyset) = \emptyset \) (where, recall, \( \emptyset \) denotes the empty set). First, let us observe that \( \emptyset \) is the unique set that has no element. We have just described a property \( P \):

\[
P(x): \ x \text{ has no element;}
\]

moreover, this property \( P \) is true with respect to the empty set \( \emptyset \); that is, the formula becomes true when \( \emptyset \) is substituted for \( x \). In other words,
“\(P(\emptyset): \emptyset\) has no element” is true.

Now we can apply (1) to conclude that \(P\) is also true with respect to \(j(\emptyset)\); in symbols:

“\(P(j(\emptyset)): j(\emptyset)\) has no element” is true.

Of course now, since \(\emptyset\) is the only set that has no element, it follows that \(j(\emptyset) = \emptyset\).

The flow of the argument develops in three stages. First, there is an assertion concerning the empty set; this assertion has the same characteristic as most other mathematical statements in that it is concerned about localized sets and not at all about \(V\). Second, we take stock of the property asserted to hold of \(\emptyset\) and formulate it precisely; this property comes to be viewed as a point of knowledge (that \(V\) has concerning one of its sets) to which the global functioning of \(j\) may be applied. When we make the move from “\(P(\emptyset)\) holds” to “\(P(j(\emptyset))\) holds,” applying \(j\) in this way to the parameter \(\emptyset\) of the formula, we are engaging the dynamics of the universe’s move within itself; our focus has expanded to a global one. Finally, having applied \(j\), we return to the localized world of small sets and evaluate the character of the new set \(j(\emptyset)\), discovering that because “emptiness” uniquely characterizes \(\emptyset\), \(j(\emptyset) = \emptyset\).

A similar pattern is evident whenever we attempt to use \(j\) in our arguments: At certain points in our reasoning, we must expand our local context to the strictly unlocalized behavior of \(j\); having applied \(j\), we again return to our local context for further reasoning. In practice, the contrast between local and global can be quite startling and often leads to elegant proofs.

Recall that in set theory, the natural number 0 is identified with the empty set \(\emptyset\); thus the argument above shows that \(j(0) = 0\). Similar reasoning shows that for each natural number \(n\), \(j(n) = n\). It turns out that for every set \(A\) that commonly occurs in mathematical practice, \(j(A) = A\). As a second example of this latter fact (omitting certain details), let us consider the circle \(C\) with radius 1 and origin \((0, 0)\), and compute \(j(C)\). Of course, if \(j\) were an ordinary function, \(j(C)\) could be anything—certainly there would be no reason to expect \(j(C)\) to have any of the characteristics of a circle. But since \(j\) “preserves all properties,” we
should expect to find a strong resemblance between $C$ and $j(C)$. Let’s begin the computation by stating precisely how $C$ has been defined:

“$C$ is a circle with radius 1 and center $(0, 0)$.”

This is a statement involving the set parameters $C$, 1, and $(0, 0)$; applying $j$ tells us that $j(C)$ is a circle with radius $j(1)$ and center $j((0, 0))$. We have already seen that $j(1) = 1$. Let’s evaluate $j((0, 0))$ by considering the property $P(x, y, z)$ that asserts that $z$ is the ordered pair with components $x$ and $y$:

$$P(x, y, z): z = (x, y).$$

Then $P(0, 0, (0, 0))$ is true. Applying $j$, we conclude that $P(j(0), j(0), j((0, 0)))$ is also true; in other words, $j((0, 0))$ is the ordered pair with components $j(0)$ and $j(0)$. Since $j(0) = 0$, $j((0, 0)) = (0, 0)$.

We may now conclude that $j(C)$ is a circle whose radius is 1 and center is $(0, 0)$; it follows, therefore, that $j(C) = C$.

In this argument, we expanded to an unlocalized view at least twice. The first time involved applying $j$ to $C$, 1, and $(0, 0)$ in order to find out what properties $j(C)$ would have. The second time involved applying $j$ to 0 in order to evaluate $j((0, 0))$.

As a final example, we shall prove a proposition that will be quite useful later on: Not only is it true that the critical point $\kappa$ is the first ordinal number moved by $j$, but in fact, no set (ordinal or otherwise) that occurs in any of the first $\kappa$ stages of $V$ is moved by $j$. In other words:

**Proposition** For all sets $A$ in $V_\kappa$, $j(A) = A$.

**Proof** First, let us observe that any set $A \in V_\kappa$ is actually in some $V_\alpha$, $\alpha < \kappa$. We first show that $A$ and $j(A)$ must lie in exactly the same stages $V_\alpha$ for $\alpha < \kappa$. First let us notice that, for such $\alpha$, $j(V_\alpha) = V_\alpha$: Notice that “$V_\alpha$ is the $\alpha$th stage” is a property true in $V$; applying $j$, we conclude that “$j(V_\alpha)$ is the $j(\alpha)$th stage” is also true. Thus, $j(V_\alpha) = V_{j(\alpha)}$. But now since $\alpha < \kappa$, it follows that $j(\alpha) = \alpha$, and so

$$j(V_\alpha) = V_{j(\alpha)} = V_\alpha.$$
We wish to show that for any $A$ and any $\alpha < \kappa$, $A \in V_{\alpha}$ if and only if $j(A) \in V_{\alpha}$. So, suppose $A \in V_{\alpha}$. Notice that “$A \in V_{\alpha}$” is a property true in $V$. Applying $j$ (by (1) above), we have that $j(A) \in j(V_{\alpha})$; now since $j(V_{\alpha}) = V_{\alpha}$, we conclude that $j(A) \in V_{\alpha}$.

Conversely, suppose $j(A) \in V_{\alpha}$. We wish to show that $A \in V_{\alpha}$. Since $j(V_{\alpha}) = V_{\alpha}$, it follows that $j(A) \in j(V_{\alpha})$. Therefore, “$j(A) \in j(V_{\alpha})$” is a property that is true in $V$. Applying (2) above to this property, we conclude that $A \in V_{\alpha}$. Summing up, for any set $A$ and any $\alpha < \kappa$, we have seen that $A \in V_{\alpha}$ if and only if $j(A) \in V_{\alpha}$.

We have completed the first phase of the proof by showing that $A$ and $j(A)$ must lie in the same stages of $V$. To show that the two sets are equal, we use an inductive argument. We begin with the definition of the rank of a set: the rank of a set $X$ in $V$ is the least ordinal $\gamma$ for which $X \subseteq V_{\gamma}$. We will argue by induction on the rank of $A$. Let $\beta = \text{rank}(A)$, and (arguing by induction), let us assume that for any set $B$, if the rank of $B$ is less than $\beta$, then $B = j(B)$. We show that $A = j(A)$ by showing that $A \subseteq j(A)$ and $j(A) \subseteq A$. Given a set $B \in A$, since $B$ is of lower rank than $A$, $j(B) = B$. But “$B \in A$” is a property true in $V$; applying $j$ gives us the true property “$j(B) \in j(A)$.” Therefore, since $j(B) = B$, we conclude that $B \in j(A)$. We have shown $A \subseteq j(A)$.

For the other direction, assume $B \in j(A)$. Since $j(A)$ has the same rank as $A$ (as we showed in the first phase of the proof), $B$ must be of rank lower than that of $A$, and so, again, $B = j(B)$. Now, the property “$j(B) \in j(A)$” is a property that holds in $V$; applying rule (2) again leads to “$B \in A$.” We have therefore shown $j(A) \subseteq A$, and we are done. End of Proof

With this glimpse of the new character of proofs using the Wholeness Axiom, we move on to examine the new features that arise in the structure of $V$ as a result of postulating this axiom.

§18. The New Dynamics of $V$

We recall that one of the shortcomings of the universe $V$ (as constructed from ZFC) as an analogy for wholeness as described by Maharishi Vedic Science was that the unfoldment of $V$ exhibited only “one half” of the dynamics found within pure intelligence: The principle of expansion of the point to infinity is actualized in the unfoldment...
of all sets from the empty set, but the collapse of infinity to a point does not appear to have a parallel in the dynamics of $V$.\footnote{Actually, one can argue, as Weinless (2011) does, that this direction of \textit{Akshara} is expressed, to some extent at least by the Reflection Principle.} In this section, we will see how the addition of the Wholeness Axiom to ZFC results in a new way to view the unfoldment of sets, now from the perspective of the \textit{wholeness} of $V$ as it gives rise to each set individually; we will see that these new dynamics provide a striking parallel to the dynamics of the unfoldment of the Veda emerging from the collapse of infinity to a point within pure intelligence.

We begin with a discussion of several points on Maharishi Vedic Science. Maharishi explains that being wholeness, pure intelligence has within it both the fully expanded infinite value of wakefulness and the fully contracted point value of wakefulness (Maharishi, 1991a):

\begin{quote}
A will not be fully awake without its own point.
\end{quote}

By the nature of these opposite values, the fully expanded value of wholeness is drawn to the fully contracted value and there is a move of pure intelligence within itself. Maharishi (1974a) explains that this fully expanded value of wholeness is embodied by the Sanskrit letter A, the first letter of the Rk Veda; pronunciation of this letter is done with the throat fully open, without stops or modifications of any kind. A also represents infinite silence. The fully contracted value of wholeness is embodied in the letter K, the second letter of the Rk Veda, pronounced with a fully closed throat—the ultimate value of “stop.”

According to Maharishi (1974a), in the move of the fully expanded value A, to the fully contracted value K, awareness of the move within pure intelligence happens when the point value K is reached; prior to the emergence of K, awareness of the move is not available. When A is stopped at K, then infinite dynamism is imparted to this point value K in preparation for the full unfoldment of the Veda and the creation.

It is in the experience of stop that we gain knowledge of the move. If we continue A we wouldn’t know that it is moving. So the move which is the concentration of all principles is from the experience of stop. (1974a)
In $K$ is the awakening of knowledge of move. This knowledge of move is that package of knowledge which is the fountainhead of all principles of knowledge and creation. (1974a)

*Richo akshare parame vyoman*

The hymns of the Rk Veda emerge in the collapse of $A$.

(Rk Veda 1.164.39)

Another feature of this collapse of $A$ to $K$ is that all possible transformations of pure intelligence within itself occur in this transition; all transformations in the unfoldment of creation can be located in the collapse of $A$ to $K$. These transformations can be understood as arising from the infinite silence of wholeness by virtue of the fact that pure intelligence, Samhitā, is by its nature pure wakefulness; being awake within itself it assumes the roles of knower, known and process of knowing (Rishi, Chhandas, Devatā respectively). As each of these phases of pure intelligence is fully awake within itself, each can become awake to the other, and new values of each emerge. In this way, an infinity of transformations of pure intelligence emerge by virtue of its fundamental nature to be awake. Thus, the entire range of self-interacting dynamics of consciousness can be seen to be the process of wholeness knowing itself.

Moreover, this move of $A$ to $K$ and the consequent dynamics of unfoldment of the Veda constitute the foundation of creation; Maharishi calls the laws governing these dynamics the *Constitution of the Universe*. The administration of all the affairs of nature is conducted by means of these unmanifest dynamics (Maharishi, 1992):

The laws governing the self-interacting dynamics of the Unified Field can therefore be called the Constitution of the Universe—the eternal, nonchanging basis of Natural Law and the ultimate source of the order and harmony displayed throughout creation.

**Overview of the New Dynamics of $V$**. With these dynamics of wholeness in mind, let us turn to the unfoldment of the sets in the universe, now from the perspective of the wholeness of $V$ itself using our new Wholeness Axiom. The Wholeness Axiom tells us that $V$ moves within itself and, in a sense, “knows” itself, by virtue of the existence of $j : V \rightarrow V$. 

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The first phase of the behavior of $j$ is silent: until its critical point $\kappa$ is reached, $j$ behaves like the identity, and no set is moved. Now the identity map $i : V \rightarrow V$ represents the nonactive, silent, unmoving value of $V$; hence we may say that, prior to moving the first point $\kappa$, $j$ embodies the absolute silence represented by the identity map $i : V \rightarrow V$.

The significance of $j$ as a nontrivial elementary embedding arises when the $\kappa$th stage is reached and $j$ assigns a value to $\kappa$ (see Figure 7).

![Figure 7. The embedding $j$: silent below $\kappa$, dynamic at $\kappa$.](image)

As we remarked earlier, a great deal of theoretical power is generated by the existence of this type of embedding, and this power is as if directed into a single point in the universe—the critical point $\kappa$. By being the first ordinal moved by $j$, $\kappa$ inherits many of the powerful properties of $V$ (in fact, as we shall see, the stage $V_\kappa$ inherits all first-order properties of $V$).

Our analogy so far associates $V$ with the quality of fully expanded wholeness; the embedding $j$ with the basic move of wholeness within itself; and the critical point $\kappa$ with the fully contracted value of wholeness, represented by $K$.

**All Possible Transformations Coded into a Magic Sequence.** After $\kappa$ is moved by $j$, several stages of unfoldment occur, eventually giving rise to the Laver magic sequence. First, if we follow the behavior of $j$ past
κ, we find that, once κ has been moved, every set that occurs past κ in the universe is also moved higher up in the universe. In particular, \( \kappa < j(\kappa) < j(j(\kappa)) < \ldots \). In the process of moving κ, j(κ), and so on, we find that j “interacts” with itself through composition. For instance, \( j(j(\kappa)) \) is obtained by applying \( j \circ j \) to κ. We use the notation \( j^2 \) to denote \( j \circ j \), \( j^3 \) to denote \( j \circ j \circ j \), and so forth.

Meanwhile, as j moves sets stage by stage through the universe, a number of special sets are created. For each \( \alpha \geq \kappa \), let us denote the set of all subsets of \( \alpha \) of size less than \( \kappa \) by \( P^{\kappa}_\alpha \). One can define, using \( j, j^2, j^3, \ldots \), a certain “measure,” which partitions the subsets of \( P^{\kappa}_\alpha \) into “large” and “small” subsets. The large subsets form a set denoted \( U_\alpha \) which is known as the supercompact ultrafilter over \( P^{\kappa}_\alpha \). This canonical collection of ultrafilters thereupon gives way to an explosion of additional supercompact ultrafilters throughout the universe. In fact, once we have \( U_\alpha \) for \( \alpha \geq |PP^{\kappa}_\lambda| \), one can show that \( P^{\kappa}_\alpha \) bears the maximum possible number of supercompact ultrafilters. Each such supercompact ultrafilter over \( P^{\kappa}_\alpha \) gives rise to a new canonical elementary embedding \( i \) (called a supercompact embedding) having critical point \( \kappa \), domain \( V \), and codomain a new model of set theory \( M \); all possible supercompact embeddings with these properties must have one of the canonical embeddings as a factor.\(^{46}\) These canonical embeddings are selectively coded into a special sequence \( S = \langle X_0, X_1, \ldots, X_\alpha, \ldots \rangle_{\alpha < \kappa} \) of subsets of the stage \( V_\kappa \).

The sequence \( S \) is known as a “magic sequence” because of its unusual properties. One of these properties is that every set in the universe can be located using \( S \): Given any set \( A \) in the universe, there is an ordinal \( \lambda \geq \kappa \) such that \( A \) can be located as the \( \kappa \)th term of the image of \( S \) under a canonical supercompact ultrafilter \( U \) over \( P^{\kappa}_\lambda \); in symbols,

\[
A = i(S)(\kappa).^{47}
\]

\(^{46}\) If \( h : V \to N \) is any supercompact embedding with critical point \( \kappa \) with \( h(\kappa) > \lambda \), then if we let \( U \) consist of all sets \( \mathcal{A} \subseteq P^\lambda_\mathcal{A} \) such that \( \mathcal{B}^\lambda \mathcal{A} = h(\mathcal{A}) \), then \( U \) is a supercompact ultrafilter and gives rise to a canonical supercompact embedding \( i : V \to M \); one can then show that there is an elementary embedding \( k : M \to N \) such that \( k \circ i = h \). Hence every supercompact \( h \) has a canonical \( i \) as a factor.

\(^{47}\) In fact, for each set \( A \) and each \( \lambda \) at least as large as the transitive closure of \( A \) (the smallest transitive set that includes \( A \)), there is a supercompact ultrafilter \( U \) on \( P^{\kappa}_\lambda \) such that if \( i \) is the corresponding embedding, \( i(S)(\kappa) = A \).
Figure 8. $j$ gives rise to a Laver magic sequence from which all sets in $V$ can be located.

In these dynamics, the vast collection of all possible supercompact embeddings (with critical point $\kappa$) emerging from the iterates of $j$ corresponds to the infinity of transformations of pure intelligence within itself in the collapse of $A$ to $K$, in which every possible transformation of Samhitā, Rishi, Devatā, and Chhandas, one into another, takes place in sequential fashion. Notice here that the knowledge about these embeddings does not arise until the critical point $\kappa$ has been moved; likewise, knowledge about the infinity of transformations from $A$ to $K$ emerges as a commentary to this fundamental collapse. As Maharishi explains, the entire Veda, and the creation itself, serve as an elaborated commentary on these fundamental dynamics. Then, just as an infinity of transformations in the collapse of $A$ to $K$ structures the sequential unfoldment of the Veda which gives rise to every detail of manifest creation, so we find that these supercompact embeddings structure a compact sequence from which every set in the universe can be located.

The Analogy Between Laver’s Magic Sequence and the Veda. The similarity between the magic sequence defined above and the Veda goes

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48 See Maharishi (1991a).
much further. Not only does the Veda give rise to every detail of manifest creation, but it does so, according to Maharishi Vedic Science, by virtue of its nature as simultaneously infinitely dynamic and infinitely silent; all opposite values find their lively integration within this field. This high degree of integration is due to the fact that at each stage of unfoldment, the Veda remains completely self-referral and united with itself; the parts of expression never dominate the original underlying wholeness. This integration within the Veda is responsible for all unity and coherence displayed in manifest existence.

A striking feature of the structure of the Veda is the fact that the totality of Veda is fully present in increasingly elaborated “packets” as the Veda expands from its first letter A to its first word, hymn, and mandala. Maharishi describes this structure of the Veda as a “self-created commentary” (Apaurusheya Bhāṣya) since later stages of unfoldment express in ever greater detail the totality of knowledge inherent in the earlier stages.

We have seen that our magic sequence S gives rise to every set in the universe by way of canonical supercompact embeddings. We shall see that this extraordinary fact is due to the internal structure of S which exhibits to a high degree the same qualities that are characteristic of the Veda, including the stage-by-stage unfoldment that is central to the Veda’s structure.

If we peer into S—which, as the reader will recall, is a \( \kappa \)-sequence of elements of \( V_\kappa \)—we first notice repeated occurrences of familiar sets. We find, for example, that the number 0, the set \( \mathbb{N} \) of natural numbers, the real number line \( \mathbb{R} \), and the set of all mathematical structures ever used in physics all occur \( \kappa \) many times in the sequence S. In fact, every set \( A \in V_\kappa \)—and such sets account for all mathematical objects used in ordinary mathematics—occurs stationarily often (this is even stronger than saying that each occurs \( \kappa \) many times). This phenomenon directly accounts for the fact that every \( A \in V_\kappa \) occurs as the \( \kappa \)th term of the image sequence \( i(S) \) for some canonical supercompact embedding \( i \). As we discuss below, similar though more complex dynamics are responsible for the full Laver property, that every set \( A \) is \( i(S)(\kappa) \) for some \( i \). This phenomenon is also reminiscent of the fact that the Veda, rather than being separate from or external to creation, is in fact the very dynamics and life breath of creation. The creation can be located in the
Veda just as a tree can be located in a seed: if one sees clearly enough the fine mechanics of transformation within the seed, the tree in all its detail can be said to be fully present within the seed.

The unifying character of $S$ is more completely revealed when we attempt to locate within the structure of $S$ the dynamics which allow us to capture not only every set in $V_\kappa$, but all sets. To understand these dynamics, let us start with a set $A$ that we wish to capture. To find the right supercompact embedding $i$ to capture $A$ as the $\kappa$th term of the sequence $i(S)$, we need to be sure that $i(\kappa)$ is larger than the transitive closure of $A$; this will guarantee that the image model $M$ contains $A$. Once we have the model $M$, the set $A$ is associated in $M$ with a function $g$ defined on $P_\kappa\lambda$ (where $\lambda$ is at least the size of the transitive closure of $A$). Now the magical trait of $S$ is that, stationarily often, the value of $g$ on a set $P$ is the same as the value of $S$ on the set $P \cap \kappa$; this guarantees that $i(S)(\kappa) = A$. Thus, the magic sequence serves to "harness" and coordinate the great dynamism of the huge collection of supercompact embeddings acting on and transforming $V$. We can say that every set is captured by $S$ because the internal, infinitely diverse, highly coherent structure of $S$ focuses the actions of these embeddings so that each point in the universe eventually occurs as an output.

The sequence $S$ not only unifies the dynamism of these supercompact embeddings, but displays infinite dynamism within its own structure. Dynamism in mathematics and in nature is often expressed through rapidly growing functions, such as exponential functions. For instance, the exponential function which takes a real number $x$ to $2^x$ eventually dominates every polynomial. On a stationary set, a Laver sequence exhibits similar behavior in a more dramatic way. To set up a revealing example, let us define, for any ordinals $\alpha$ and $\beta$, the cardinal number $\beth(\alpha, \beta)$ by induction: $\beth(0, \beta) = \beta; \beth(\alpha + 1, \beta) = 2^{\beth(\alpha, \beta)}; \beth(\lambda, \beta) = \bigcup_{\alpha < \lambda} \beth(\alpha, \beta)$ for $\lambda$ a limit. Hence, for example,

$$\beth(0, \omega) = \omega; \beth(1, \omega) = 2^{\omega}; \beth(2, \omega) = 2^{2^\omega}.$$ 

Now, as an example of a rapidly growing function, consider $g : \kappa \to \kappa$, defined by letting $g(\alpha) = \beth(\alpha, \alpha)$ for each $\alpha < \kappa$. Thus, for example:

49 The transitive closure of a set $A$ is $A \cup (\bigcup A) \cup (\bigcup \bigcup A) \cup \ldots$. 

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The dynamism of a magic sequence $S$ becomes apparent when we consider the fact that there is a stationary set $B \subseteq \kappa$ such that for all $\alpha \in B$, $|S(\alpha)| > g(\alpha)$. In fact, it can be shown that such a stationary set $B$ can be found for virtually any\(^{50}\) function definable in $V\kappa$. Such results show that if we use rapidly growing functions as a measure of dynamism, a magic sequence exhibits a strong type of dynamic behavior within its own structure.

At the same time, $S$ exhibits infinite silence in the following way: Stationarily often, the $\alpha$th value of $S$ is $\alpha$ itself; such $\alpha$ are not moved by $S$. It is interesting to note that this feature of the internal structure of $S$ comes into view when we attempt to capture $\kappa$ itself by $S$; that is, attempt to find $i$ such that $i(S)(\kappa) = \kappa$. It can be shown that in order for such an $i$ to exist, $S$ must not move stationarily many $\alpha$. Therefore, the truth of the self-referral expression “$i(S)(\kappa) = \kappa$” depends on the pervasiveness of “silence” in $S$.

The magic sequence $S$ that we have defined using the Wholeness Axiom\(^ {51}\) also has an internal structure in which earlier “stages” of the sequence have the same essential properties as the sequence as a whole, reminiscent of the fact that the Veda unfolds in discrete stages, each elaborating in greater detail the same totality of knowledge inherent within earlier stages. It can be shown that for stationarily many $\alpha$, the restriction $S|\alpha$ is itself a magic sequence at $\alpha$. Thus, the special properties which define $S$ as a whole are found everywhere permeating the structure of $S$.

As a final point linking $S$ to the structure of the Veda, we point out that a magic sequence exhibits as one of its properties a kind of incorruptibility in that we may alter as many as $\kappa$ many terms of $S$ (as long as those $\kappa$ many terms form a “thin” enough subset of $\kappa$) without changing its status as a magic sequence. If, for example, we decide to replace the first $\omega\omega$ terms of $S$ with the number 0, this altered sequence

\[
g(\omega) = \bigcup[\omega, 2^\omega, 2^{2^\omega}, \ldots]
\]

\(^{50}\) The result is given precise formulation in Corazza (2011).

\(^{51}\) The existence of a magic sequence $S$ at $\kappa$ with the property that the set of all $\alpha < \kappa$ at which there is another magic sequence has normal measure 1 does not follow from the existence of a supercompact cardinal—although a single supercompact is sufficient to obtain an ordinary magic sequence. It can be shown that this stronger property actually implies that $\kappa$ is the $\kappa$th supercompact cardinal, and much more.
still gives rise to every set in the universe exactly as before. This sug-
gests that the reality expressed by a particular magic sequence may still
be accurately expressed even if many of the details of expression are
changed. Likewise, the eternal reality of the Veda does not lie at the
level of the various interpretations of the Vedic literature that may be
possible; rather, as the Ṛk Veda itself declares, the Richas or hymns of
the Veda exist in the immutable transcendental field; recall the verse
richo akshare parame vyoman, mentioned earlier. The profound internal
dynamics of the Veda are as if hidden from view; so likewise does the
“magic” of a magic sequence reside at a more holistic level of the struc-
ture of the sequence, since changes to individual parts of the sequence
do not alter its fundamental properties.

Collapse and Expansion with Infinite Frequency We recall that the
theme of unfoldment of the Veda consists in the infinitely frequent
oscillation of “collapse of infinity to a point and expansion of point to
infinity.” By Maharishi’s Aparajita Bhashya, these dynamics can be
located in AK, the first syllable of Ṛk Veda, in which all possible trans-
formations of pure consciousness occur in seed form as Rishi, Devata,
Chhandas, and Samhitā interact amongst themselves.

We find this theme of unfoldment expressed in the new dynamics of
V in the following way: The vast infinity of transformations that arise
from j (in the form of supercompact embeddings) are actually struc-
tured on the basis of an infinity of new point values that arise from κ
(namely, the supercompact ultrafilters over all possible index sets $P_\kappa$)
and that provide new focal points for an infinite variety of “collapses” of
V to other new models of set theory. These new embeddings correspond
to new values of Rishi, Devata, Chhandas, and Samhitā that emerge
from the original three-in-one structure represented by our embedding
$j: V \rightarrow V$.

We will now explain in greater detail how the emergence of an
infinity of new models of set theory in the presence of these derived
supercompact embeddings can be understood as a repeated “collapse of
infinity to a point and expansion of point to infinity.”

As we have seen, the maximum possible number of supercompact
ultrafilters over each $P_\kappa$ ($\lambda \geq \kappa$) arise from the fundamental move
due to $j: V \rightarrow V$. Each such ultrafilter $U$ may be viewed as a new “point
To effect this collapse, we first form a new “variable” universe $V^{P,\lambda}/U$; the elements of this “universe” consist of all functions from the index set $P,\lambda$ to $V$. We call this vast collection of functions a variable universe because each function $f \in V^{P,\lambda}$ may be thought of as representing a set that varies according to a parameter that ranges through all elements in $P,\lambda$. For instance, suppose $A, B \in P,\lambda$. Then we could consider $f$ to be the set $f(A)$ relative to $A$ and $f(B)$ relative to $B$.

Now, technically, $V^{P,\lambda}$ is not a universe of sets as it does not satisfy all the axioms of ZFC (indeed, the natural way of thinking of $V^{P,\lambda}$ as a model at all is to treat it as having infinitely many truth values in addition to the usual values of “true” and “false”). We obtain an ordinary universe of sets by “collapsing” this variability (and “collapsing” the infinity of truth values) by way of the point $U$ in the following way: Two functions $f, g$ in $V^{P,\lambda}$ will be called equivalent mod $U$ if they agree pointwise on a set in $U$. The equivalence classes that arise from this equivalence relation form the elements of a new class $V^{P,\lambda}/U$. By amalgamating functions into their equivalence classes, the variable quality of $V^{P,\lambda}$ is eliminated. Because we use an ultrafilter $U$ to define the equivalence classes, which partitions all sets in $P,\lambda$ into just two classes (“large” and “small”), the number of truth values associated with the new model is reduced to the two usual values: “true” and “false.” Since the formation of $V^{P,\lambda}/U$ from $V^{P,\lambda}$ occurs by treating equivalence classes as points—relative to the reference point $U$—in a new model, the process of formation of the new model is clearly analogous to “collapse of infinity to a point.”

The class $V^{P,\lambda}/U$ turns out to be a new model of set theory (new in that it is not identical to $V$); unfortunately, since its elements are equivalence classes, its membership relation must be nonstandard. This minor inconvenience can be corrected by “reshuffling” the elements of $V^{P,\lambda}/U$...
the model using a standard technique that is based on the Mostowski Collapsing Theorem. The result is a standard (transitive) model $M$ of set theory that is isomorphic to $V^{P, \lambda}/U$.

In this form, the unfoldment of sets from the empty set closely parallels the unfoldment of sets in $V$ itself; the difference is that in the unfoldment within $M$, $M$’s version of the power set operator is used. As we saw for $V$, the resulting sequential unfoldment of sets provides an analogue for “expansion of point to infinity.” Thus, the formation of the model $M$ can be seen as the expression of “collapse of infinity to a point” as well as “expansion of point to infinity.”

We can summarize the entire process of building the model $M$ by describing the supercompact embedding $i$ that naturally arises from the construction. The embedding $i$ is a composition $\pi \circ \eta \circ e$ where

1. $e(x)$ expands the set $x$ to the constant function $\epsilon_x : P, \lambda \rightarrow V$ where $\epsilon_x(A) = x$.
2. $\eta : V^{P, \lambda} \rightarrow V^{P, \lambda}/U$ maps $f$ to $[f]$.
3. $\pi$ is the Mostowski collapsing isomorphism that transforms $V^{P, \lambda}/U$ to the model $M$.

Thus, the construction of $M$ may also be understood as the outcome of a fundamental transformation of the wholeness $V$, represented by the embedding $i$.

Next, we examine how the emergence of these supercompact embeddings also provides a parallel to the infinity of transformations of Rishi, Devatā, Chhandas, and Samhitā amongst themselves in the sequential unfoldment of the Veda. Recall that the flow of the Veda may be considered a flow of an infinite variety of frequencies or sounds.

As an analogy, each supercompact embedding $i$ may be considered as a particular frequency extracted from its original source in the embedding $j$ on the basis of the point value $U$. All such embeddings $i$ taken together may be considered again by analogy to express the entire range of “frequencies” contained in seed form within the embedding $j$ (where again we consider $j$ as an analogue to the flow of the Veda). In the Veda, frequencies arise in the interaction of Rishi, Devatā, and Chhandas within Samhitā, which produce an infinity of derived values of Rishi, Devatā, Chhandas, and Samhitā.
Recall from earlier sections that we naturally associate the fundamental embedding $j$ with the Devatā value and that, since $V$ is interacting with itself, $V$ is associated both with Rishi and Chhandas. Moreover, since the embedding behaves entirely within $V$, $V$ plays the role of the wholeness that unites the three as well, namely, Samhitā.

Now we can further observe that the vast collection of supercompact embeddings display “derived” values of Rishi, Devatā, Chhandas, and Samhitā in that each embedding emerges from $j$ (thus, is “derived” from $j$) and each embedding $i : V \rightarrow M$ displays a three-in-one structure of knowledge, where $V$ plays the role of Rishi, $i$ the role of Devatā, and $M$ the role of Chhandas, and $V$ itself also plays the role of Samhitā since all the dynamics of the embedding occur within $V$.

Thus, the unfoldment of supercompact embeddings from $j$ corresponds not only to the repeated collapse of infinity to a point and expansion of point to infinity (the theme for the expansion of the Veda), but also to the infinity of transformations of Rishi, Devatā, Chhandas, and Samhitā, which constitute the very fabric of the Veda.

**Summary.** We summarize the correspondences discussed so far in a chart followed by a diagram illustrating the stages of unfoldment under the Wholeness Axiom:

**Table 1. The analogy between the dynamics of the universe $V$ and the dynamics of wholeness**

<table>
<thead>
<tr>
<th>Dynamics of $j$ Moving $V$ Under the Wholeness Axiom</th>
<th>Dynamics of Pure Intelligence Moving Within Itself</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$, the universe of sets as a whole</td>
<td>Fully expanded value of wholeness, represented by $A$</td>
</tr>
<tr>
<td>$j : V \rightarrow V$</td>
<td>The move of pure intelligence within itself</td>
</tr>
<tr>
<td>$\kappa$, the critical point of $j$</td>
<td>The fully contracted value of wholeness, represented by $K$</td>
</tr>
<tr>
<td>The class of all possible supercompact embeddings (having critical point $\kappa$) arising from $j$ and its iterates</td>
<td>All possible transformations emerging in the collapse of $A$ to $K$</td>
</tr>
</tbody>
</table>
The magic sequence
\[ S = \langle X_0, X_1, \ldots, X_\alpha, \ldots \rangle_{\alpha<\kappa} \]

<table>
<thead>
<tr>
<th>The Veda</th>
<th>The Veda</th>
</tr>
</thead>
<tbody>
<tr>
<td>The repeated collapse of the infinite dynamism of ( V ), embodied in classes of the form ( V^{P,\lambda} ), focused at new point values ( P, \lambda ) (for all ( \lambda \geq \kappa )) and ( U ) derived from ( \kappa ), resulting in the unfoldment of fully expanded universes ( M ), with all dynamics embodied in the canonical supercompact embeddings—all giving rise to Laver's magic sequence</td>
<td>Collapse of infinity to a point and expansion of point to infinity occurring with infinite frequency, giving rise to the self-interacting dynamics of pure intelligence</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sets</th>
<th>Manifest creation</th>
</tr>
</thead>
<tbody>
<tr>
<td>The initial embedding ( j : V \rightarrow V ) giving rise to all possible supercompact embeddings, each in turn giving rise to a canonical sequence of additional embeddings that can be seen as natural and inevitable modifications of the original</td>
<td>Samhitā assuming the roles of Rishi, Devatā and Chhandas, interacting with themselves to give rise to all possible transformations</td>
</tr>
</tbody>
</table>

**Figure 9.** Emergence of a Laver magic sequence from \( j \).
**New Qualities in V.** Our analysis suggests that adding the Wholeness Axiom to ZFC brings the fundamental dynamics of the universe in much closer accord with the dynamics of pure intelligence, as described by Maharishi Vedic Science. Notice that, from this new perspective, $V$ very naturally exhibits the qualities of pure intelligence that seemed to be missing before: Now, $V$ is understood by its very nature to move within itself and “know” itself through its own self-interaction; hence, by our earlier discussion, $V$ can be said to more fully exhibit the qualities of self-referral, awake within itself, and bliss.

To see how the fourth quality, infinite correlation, is enlivened by the presence of the Wholeness Axiom, we need to mention yet another new feature of the structure of $V$. Starting with our wholeness operator $j : V \rightarrow V$ and its critical point $\kappa$, we will use Kunen’s inconsistency proof in a new way to show that the sequence of ordinals we obtain by repeatedly applying $j$ to $\kappa$,

$$\kappa, j(\kappa), j(j(\kappa)), \ldots$$

(called the critical sequence of $j$) is unbounded in the universe! In other words, we will show that there is no ordinal in the universe that is simultaneously larger than every term of the critical sequence. To see this, first, notice that the terms of the sequence are strictly increasing:

$$\kappa < j(\kappa) < j(j(\kappa)) < \ldots .$$

This can be shown using the elementarity of $j$ (since $\kappa < j(\kappa)$, applying $j$ yields that $j(\kappa) < j(j(\kappa))$, and so forth). Now suppose there actually is an ordinal that exceeds all the ordinals $\kappa, j(\kappa), j(j(\kappa)), \ldots$; let $\lambda$ denote the least such ($\lambda$ is called the supremum of the sequence). It can be shown that $j$ takes each set in the stage $V_{\lambda+2}$ back into $V_{\lambda+2}$. Thus the restriction of $j$ to $V_{\lambda+2}$ is an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$. But now, as we mentioned in an earlier section, we can carry out Kunen’s proof and arrive at a contradiction! Thus, assuming that there is an ordinal lying above $\kappa, j(\kappa), j(j(\kappa)), \ldots$ leads to an inconsistent set theory. For ordinary sequences of ordinals, the assumption that the sequence has a supremum would be warranted; but here, the sequence is defined from
our \textit{transcendental} elementary embedding \( j \). Consequently, the sequence has no supremum.\footnote{Intuitively, the fact the ordinals \( \kappa, j(\kappa), j(j(\kappa)), \ldots \) lacks a supremum seems odd, especially because this sequence is so short: notice that these ordinals are matched one-for-one with the natural numbers 0, 1, 2, ... (simply match 0 with \( \kappa \), 1 with \( j(\kappa) \), 2 with \( j(j(\kappa)) \), and so on). Thus, they form the shortest possible infinite sequence, and yet they extend all the way through the universe. One possible intuition, taken from Maharishi Vedic Science, which may help to explain this phenomenon comes from considering the nature of evolution to enlightenment. We can think of climbing upward through the universe \( V \), stage by stage, as analogous to a pathway to the ultimate realization of wholeness in life. Without a proper technique, such a path will be truly endless. In the language of ZFC, any path passing all the way through \( V \) must necessarily be of greater length than any infinite cardinal. In the language of Maharishi Vedic Science, the vastness of the creation simply cannot be fathomed; attaining the knowledge of the field of action that brings the ultimate knowledge of life requires a technique. Commenting on the discourse of Lord Krishna in verse 17, Chapter 3, of the Bhagavad-Gítā, Maharishi (1967) remarks:}

\begin{center}
\begin{quote}
The Lord has said that knowledge of action is necessary and yet, the course of action, being unfathomable, knowledge of it must remain incomplete. He therefore brings to light a technique by which the effects of knowledge will be gained without the necessity for gaining the knowledge. (p. 278)
\end{quote}
\end{center}

With a suitable technique, an individual may begin at whatever level of involvement in relative existence he may find himself, and quickly awaken to the wholeness of life within his own awareness (Maharishi 1966):

\begin{center}
\begin{quote}
This practice [Transcendental Meditation] is pleasant for every mind. Whatever the state of evolution of the aspirant, whether he is emotionally developed or intellectually advanced, his mind, by its very tendency to go to a field of greater happiness, finds a way to transcend the subtlest state of thinking and arrive at the bliss of absolute Being. (pp. 55–56)
\end{quote}
\end{center}

By directly enlivening within individual awareness the dynamics of pure intelligence—and recall that these dynamics, in our analogy, correspond to the action of \( j \)—the path to full enlightenment becomes relatively short. Likewise, although no ordinary sequence of sets is sufficiently long to pass through the entire universe, still, with reference to the embedding \( j \), an extremely short path \( \kappa, j(\kappa), j(j(\kappa)), \ldots \) passing beyond every stage, is as if carved out of the vastness of \( V \).
$\mathcal{V}_\alpha$ is an elementary submodel of $\mathcal{V}$.\footnote{For each $n$, it can be shown that the set of all ordinals $\alpha$ such that $\mathcal{V}_\alpha$ is an elementary submodel of $\mathcal{V}_{\mathcal{J}(\kappa)}$ is closed and unbounded in $\mathcal{J}(\kappa)$.} This result is extremely powerful; it says that full information about the nature of $\mathcal{V}$ as a whole is available throughout the universe. Recall from our discussion of elementary submodels that one model is an elementary submodel of another, the two models are “infinitely correlated” in the sense that they satisfy exactly the same properties. Thus, the fact that we find such a pervasive occurrence of elementary submodels of the universe suggests that the Wholeness Axiom has enlivened the quality of infinite correlation in the structure of $\mathcal{V}$.

**Totality of Mathematical Knowledge in One Step.** We can make one final point about the dynamics of $\mathcal{V}$ under our new axiom, motivated by a remark by Maharishi (Hagelin, 1992). Maharishi points out that the ideal expression of the totality of knowledge should involve no steps, as in the first letter, $A$, of Rk Veda. But, to express this totality of knowledge through a discipline based on steps, the most compact expression that could be hoped for would be some analogue to the transformation from $A$ (the first vowel of Rk Veda) to the expression $\mathcal{A}\mathcal{A}$ (the last vowel) capturing the totality of knowledge in one step. To some extent at least, this goal is realized in the statement of our Wholeness Axiom, which can be represented in the following diagram:

![Figure 10](image.png)

**Figure 10.** The totality of knowledge in one step.

As the diagram suggests, the fundamental move of the wholeness represented by $\mathcal{V}$ involves a single step in which $\mathcal{V}$ moves in a very
coherent way within itself. From this single move, all the dynamics of sets, and hence all mathematics, is generated.

We have seen that new qualities and dynamics in the universe arise in the presence of the Wholeness Axiom and that these mirror, to a great extent, those of pure consciousness as it moves within itself. Thus, our attempt to include more of the properties of our intuitive model from Maharishi Vedic Science into our construction of $V$ has been successful. In the next section, we show that our efforts have useful mathematical consequences; most importantly, we will show how we can derive large cardinals from our new axiom.

§19. The Origin of Large Cardinals
With the addition of the Wholeness Axiom to ZFC, we are now in a position to understand the origin of large cardinals. Since we obtained this axiom as the culmination of the strongest known large cardinal axioms, it may seem almost obvious that our axiom will succeed in providing the necessary derivations. However, the fact that we have insisted that the elementary embedding $j$ be transcendental and that $V$ be fully $j$-closed often introduces a twist in the expected reasoning.

More interesting than the proofs, however, is the realization that all large cardinal properties can be understood as the properties of the critical point of this basic elementary embedding $j$, that large cardinal properties arise as the properties of the focal point of the fundamental move of the wholeness of the universe within itself. This perspective provides not only an account of the origin of all large cardinal properties but, as well, an intuition about why these properties have proven to be so powerful.

In this section, we will carry out three of the derivations of large cardinals from the Wholeness Axiom; in particular, we will show that the Wholeness Axiom implies the existence of inaccessible, measurable, and extendible cardinals. We will also consider in this section the relationship between our axiom and the existence of an elementary embedding from a stage $V_\lambda$ to itself; we will see that this extremely strong axiom is naturally related to ours. The reader who wishes to avoid mathematical technicalities may wish to skip to the next section.

The proof of the existence of inaccessibles will use the same sort of techniques we used in Section 17. It will also make central use of the Principle of Countable Unboundedness, introduced in the last section.
The proof of the existence of measurables will illustrate the mathematical impact of our axiomatic assumptions about \( j \). The verifications that other large cardinals can be derived from the Wholeness Axiom have a similar flavor.

The proof of the existence of extendibles, on the other hand, is trivial and illustrates how large cardinal axioms may very naturally be viewed as approximations to an elementary \( j : V \rightarrow V \). Because it is so brief, we proceed with the proof here: A cardinal \( \kappa \) is said to be extendible if, for each ordinal \( \beta \), there are an ordinal \( \zeta \) and an elementary embedding \( i_{\beta} : V_{\kappa + \beta} \rightarrow V_{\zeta} \) with critical point \( \kappa \), satisfying

\[
\kappa + \beta < i_{\beta}(\kappa) < \zeta
\]

To obtain the needed embeddings \( i_{\beta} \), we can simply use the iterates of \( j \): Given \( \beta \), let \( n \) be least such that \( j^n(\kappa) > \kappa + \beta \); if we let \( i_{\beta} = j^n | V_{\kappa + \beta} \), then \( i_{\beta} : V_{\kappa + \beta} \rightarrow V_{j^{n}(\kappa) + \beta} \) is elementary with critical point \( \kappa \) and satisfies \((\ast)\). We proceed to the other proofs:

**Theorem.** The Wholeness Axiom implies that there exists an inaccessible cardinal.

**Proof.** To begin, let us recall the definition of inaccessible cardinal given earlier: a cardinal \( \gamma \) is inaccessible if \( \gamma > \omega \) and the stage \( V_{\gamma} \) has the following two properties:

1. \( V_{\gamma} \) is not the union of fewer than \( \gamma \) many of the earlier stages \( V_{\alpha} \).
2. The size of any previous stage \( V_{\alpha} \) is less than \( \gamma \).

We also need to recall that in Section 17 we showed that

- (a) \( j(\omega) = \omega \);
- (b) if \( \kappa \) is the critical point of \( j \), then \( j \) does not move any set in \( V_{\kappa} \), that is, for all sets \( A \in V_{\kappa}, j(A) = A \).

Therefore, we begin the proof by letting \( j : V \rightarrow V \) be the wholeness embedding having critical point \( \kappa \). We show \( \kappa \) is inaccessible. From (a) we may infer that \( \kappa > \omega \); thus, the critical point of \( j \) is already showing
signs of being quite big. To establish (1), suppose on the contrary that $V_\kappa$ is the union of $\delta$ many of the earlier stages $V_\alpha$, where $\delta < \kappa$. A chart will be useful here:

<table>
<thead>
<tr>
<th>Formula true in $V$</th>
<th>Formula after applying $j$ (also true in $V$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_\kappa$ is the union of the following list of $\delta$ many earlier stages: ${V_{\alpha_0}, V_{\alpha_1}, V_{\alpha_2}, \ldots}$</td>
<td>$j(V_\kappa)$ is the union of the following $j(\delta)$ many earlier stages: ${j(V_{\alpha_0}), j(V_{\alpha_1}), j(V_{\alpha_2}), \ldots}$</td>
</tr>
</tbody>
</table>

The lower right box can be simplified: As we showed before (Section 17), for any ordinal $\gamma$, $j(V_\gamma) = V_{j(\gamma)}$. Also, since $\delta < \kappa$, $j(\delta) = \delta$. Likewise, $j(\alpha_0) = \alpha_0$, $j(\alpha_1) = \alpha_1$, and so forth. The expression in the lower right box becomes:

$V_{j(\kappa)}$ is the union of the following $\delta$ many earlier stages:

$\{V_{\alpha_0}, V_{\alpha_1}, V_{\alpha_2}, \ldots\}$

Thus, both $V_\kappa$ and $V_{j(\kappa)}$ are the union of the stages $\{V_{\alpha_0}, V_{\alpha_1}, V_{\alpha_2}, \ldots\}$ and so $V_\kappa = V_{j(\kappa)}$. But this is impossible because $\kappa < j(\kappa)$. Hence, our assumption that $V_\kappa$ could be expressed as the union of fewer than $\kappa$ many earlier stages has proven to be incorrect, and we have thereby established (1).

We proceed to (2): Let us assume that some stage $V_\alpha$, with $\alpha < \kappa$, has size greater than or equal to $\kappa$. The Principle of Countable Unboundedness guarantees that the size of $V_\alpha$ cannot simultaneously exceed every one of the cardinals $\kappa, j(\kappa), j(j(\kappa)), \ldots$. Let $n$ be large enough so that the $n$th iterate $j^n(\kappa)$ of $j$ applied to $\kappa$ is greater than the size of $V_\alpha$; assume further that $n$ is least for which $j^n(\kappa)$ is greater than the size of $V_\alpha$. We may use a chart to arrive at a contradiction:

<table>
<thead>
<tr>
<th>Formula true in $V$</th>
<th>Formula after applying $j$ (also true in $V$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The size of $V_\alpha$ is $\geq j^{n-1}(\kappa)$.</td>
<td>The size of $j(V_\alpha)$ is $\geq j^n(\kappa)$.</td>
</tr>
</tbody>
</table>
Simplifying the right box again, we see that \( j(V_\alpha) = V_{j(\alpha)} = V_\alpha \) and
\[ j(j^{-1}(\kappa)) = j^n(\kappa). \]
Thus, the right box says:

the size of \( V_\alpha \) is \( \geq j^n(\kappa) \),
and we have a contradiction. Thus, for all \( \alpha < \kappa \), the size of \( V_\alpha \) is less than \( \kappa \). **End of Proof**

We now turn to the proof of the existence of a measurable cardinal.

**Theorem.** The Wholeness Axiom implies there is a measurable cardinal.

**Proof.** To begin, we need to recall that \( \kappa \) is measurable if \( \kappa \) is the critical point of a nontrivial elementary embedding \( i: V \rightarrow M \) that is definable in \( V \). As a first attempt to prove the theorem, we might try using \( j \) as our embedding; but because \( j \) is not definable in \( V \), we are forced to proceed along less direct lines.

Still, there are fairly standard procedures one can follow to obtain \( i \). If we can find a way to divide up all the subsets of \( \kappa \) into two classes, “big” and “small,” then a well-known\(^{56} \) procedure called the ultrapower construction (which was described briefly in the previous section) will produce the model \( M \) and the embedding \( i: V \rightarrow M \) that are required. The hard part, then, is to divide up the subsets of \( \kappa \) into these two classes. Once “big” has been properly defined, the meaning of “small” will be clear (namely, a set will be small if it is not big!). Thus, our task is to delineate the “big” subsets of \( \kappa \). In order for our construction to work, our notion of “big” must meet the following requirements:

1. \( \kappa \), as a subset of itself, is “big”;
2. all “big” subsets must have size \( \kappa \);
3. if a subset \( A \) includes a “big” set \( B \) as a subset, then \( A \) must itself be “big”; and
4. the intersection of fewer than \( \kappa \) many “big” sets is again “big.”

Elegantly enough, if we consider all subsets \( A \) of \( \kappa \) with the property that \( j(A) \) contains \( \kappa \) as an element, these sets satisfy all four require-

\(^{56}\) See Jech (1978, Chapters 27–28).
ments, where $j : V \to V$ is a wholeness operator. The reason that this method works, intuitively speaking, is that $\kappa$ is not actually a member of the image $V'$ of $V$ by $j$ and plays the role of a random point among the ordinals below $j(\kappa)$. In order for the image $j(A)$ of the set $A$ to contain this random point, $A$ must be quite large as a subset of $\kappa$.

We will denote by $U$ the collection consisting of all these “big” sets; more precisely,

$$U = \{ A \subseteq \kappa : \kappa \in j(A) \}.$$

The next step is to define $i$ using $U$, following standard procedures. However, in order to do so, we need to know that $U$ is a set. Notice that $U$ has been defined using $j$; if $V$ were not $j$-closed, there would be no hope of demonstrating that $U$ is set. However, because the Wholeness Axiom tells us that $V$ is $j$-closed, and because $U$ is a subcollection of a known set (namely, the power set of $\kappa$) defined using $j$, we may conclude that $U$ is indeed a set.

Thus the standard ultrapower construction can be carried out to produce the model $M$ and the required elementary embedding $i : V \to M$. Therefore, the critical point of a wholeness operator is a measurable cardinal. **End of Proof**

Finally, let us return to some remarks we made earlier about various weakenings of the inconsistent notion of a nontrivial definable elementary embedding of the universe to itself. We mentioned that there were two possible consistent weakenings which took the form “there is an elementary embedding from some $V_\lambda$ to itself”; these axioms have been given the names $I_1$ and $I_3$ in the literature:57

$p$ | There are ordinals $\kappa < \lambda$ and an elementary embedding $i : V_{\lambda+1} \to V_{\lambda+1}$ with critical point $\kappa$ such that $\lambda = \sup\{\kappa, i(\kappa), i(i(\kappa)), \ldots\}$.}

57 The ‘$I$’ in $I_1$ and $I_3$ stands for “inconsistent.” Kunen discovered these axioms as a corollary to his proof that, in Kelley-Morse set theory, there is no nontrivial elementary embedding $j : V \to V$. Kunen observed that the axioms $I_1 - I_4$, so close to inconsistency, could not quite be proved inconsistent using the techniques of his paper. See Kunen (1971).
There are ordinals $\kappa < \lambda$ and an elementary embedding $i : V_\lambda \rightarrow V_\lambda$ with critical point $\kappa$ such that $\lambda = \operatorname{sup}\{\kappa, i(\kappa), i(i(\kappa)), \ldots \}$.

Each of these axioms is strong enough to imply the consistency of virtually all large cardinal axioms, just as the Wholeness Axiom does.\textsuperscript{58} It is natural to wonder about the relationships among these powerful axioms.

Before giving an answer to this natural question, we first notice that there is one feature of our Wholeness Axiom which deserves clarification: If $j : V \rightarrow V$ is supposed to be transcendental to $V$, and $V$ includes “everything,” where is $j$ supposed to exist? As we mentioned before, $j$ can be coded as a subcollection of $V$ which does not happen to be definable. Using our model of wholeness from Maharishi Vedic Science for intuition, this picture of the universe makes sense: The embedding $j$, representing the unmanifest dynamics of wholeness interacting with itself, is neither an element of $V$ nor definable within $V$ even though it lies within $V$ as a subcollection.

On the other hand, it is possible to picture the situation in another way: We may wish to think of both $j$ and $V$ as existing as elements within a greater wholeness, a wholeness which includes them both but does not participate directly in their activities. The axiom $I_3$ suggests this intuition.

In the diagram, if we think of $V_\lambda$—which can be shown from $I_3$ to be a model of set theory—as being the actual universe $V$, and the real universe as representing some sort of superuniverse, then we have a concrete realization of the intuition just described. In this context, both $j$ and the universe are mere sets in this vaster superuniverse.

\textsuperscript{58}The critical points of $I_1$ and $I_3$ are quite strong, but do not have all the large cardinal properties that the critical point of the wholeness operator has. This is because these axioms have a restricted range of influence; they say nothing about the structure of the universe above $\lambda$; there may not even be an inaccessible above $\lambda$. Nevertheless, these axioms do imply the existence of models of all the strongest large cardinal notions, and for this reason they are considered stronger as axioms than other large cardinal axioms.
Figure 11. The external embedding $j : V_\lambda \to V_\lambda$ as a point in a superuniverse.

Using the axiom $I_1$ yields a similar result, although now we must picture the ordinary universe as having a top layer, $V_\lambda^{\omega+1} - V_\lambda$, with $j$ occurring as an element of some superuniverse at the stage $V_\lambda^{\omega+1}$.

This picture of the universe very naturally corresponds with a basic distinction that is drawn in Maharishi Vedic Science between the two essential natures of wholeness. In this paper, we have emphasized the nature of pure intelligence to be awake to itself; this aspect is the basis for all the dynamism of existence. Maharishi has called this aspect of the nature of wholeness variously pure intelligence (1972, Lesson 8), Samhitā of Ṛishi, Devatā and Chhandas, and Šamhitā (1990b), or the eight-fold creative nature of Samhitā (Weinless, 2011). However, there is another aspect to wholeness: the value of wholeness which is one without diversity, which Maharishi calls variously pure existence (1972, Lesson 8) or pure Samhitā (1990b). In the Bhagavad-Gītā, the distinction is expressed as follows (Maharishi, 1969):

\begin{quote}
\textit{bhumir apo 'n alo vayuh}
\textit{kham mano buddhir}
\textit{evaca ahankara itiyam me}
\textit{bhinna prakritir ashtadha}
\end{quote}

Models of Bernays–Gödel set theory or Morse–Kelly set theory have this feature. See Jech (1978, p. 76).
Earth, water, fire, wind,  
space, mind, intellect, and ego;  
this is the eightfold  
division of my nature. (7.4)

apareyam itas tvanyam prakritim  
viddhi me param jivabhutam

This is my lower nature.  
Know my other, higher,  
transcendental nature, the Self. (7.5)

It is as if the dynamism underlying the universe is just a fraction of the total reality of wholeness. Maharishi makes this observation in another way in the *Science of Being* (1966):

The realisation that eternal Being is the one ultimate, supreme reality of existence shows that the cause of creation, or almighty creativity, is latent in the very nature of Being and that It expresses Itself in the form of creation. So we find that absolute, attributeless, eternal Being is the ultimate reality of existence, and that by virtue of Its own nature the process of creation, evolution, and dissolution continues eternally without affecting the absolute status of eternal Being. This is a complete picture of absolute, eternal Being in relation to Its own almighty, creative intelligence, or universal mind, as well as to the individual mind. (p. 46)

Thus, just as the move of wholeness within itself is simply a play within the vastness of pure Being, so the fundamental move of the universe of mathematics by the embedding \( j \) can be seen as the interaction of mere sets within the context of a much vaster superuniverse.

§20. Eightfold Collapse of Infinity to a Point

So far, we have used Maharishi Vedic Science to motivate a new axiom of set theory that would introduce new qualities and dynamics into the universe, aligning its structure more with that of the wholeness of pure
consciousness. The program has succeeded at accomplishing its goals. In this section we examine the dynamics of wholeness more deeply and find that the parallels between the universe of sets and pure consciousness extend considerably further than we have indicated so far.

As we have seen, according to Maharishi Vedic Science, all the self-interacting dynamics of pure consciousness can be located in the collapse of fullness, represented by $A$, to the point value, represented by $K$. These dynamics are elaborated in sequential fashion as the verses of the Veda unfold; in fact, the entire Veda can be seen as a commentary on the expression $AK$. According to Maharishi’s *Apaurusheya Bhāshya*, the Veda unfolds in packets of knowledge, with each successive stage being a commentary and elaboration of earlier stages. Thus the first syllable of the Veda contains the totality of knowledge; in progressively more elaborated form, the first word, the first *Pāda* (consisting of the first eight syllables of the Rk Veda), the first *Rīcha* (the first 24 syllables of the Rk Veda, consisting of three *Pādas*), and the first *Sūkta* (consisting of eight *Rīchas*) are all expressions of the totality of knowledge.

Maharishi (1991a) likens the collapse of $A$ to $K$ to a whirlpool, spiraling from its fullest value, corresponding to $A$, to its point value, corresponding to $K$. He explains that this whirlpool effect unfolds in eight stages. These stages are separately elaborated in the eight syllables of the first *Pāda*; these eight syllables correspond to the eight *Prakritis*, or fundamental qualities of consciousness, namely, the five *Mahabhutas*—*Prithivi* (earth), *Jala* (water), *Agni* (fire), *Vāyu* (air), and *Ākāsha* (space)—and the three subjective principles, *Manas* (mind), *Buddhi* (intellect), and *Ahamkāra* (ego). The first *Rīcha*, consisting of 24 syllables, provides a further elaboration of these first 8; the eight-syllable structure of the first *Pāda* appears three times in the first *Rīcha*, the first time from the point of view of the knower (*Rishi*), the second, from the point of view of the process of knowing (*Devatā*), and the third, from the point of view of the object of knowledge (*Chhandas*).

We are able to locate eight stages of elaboration of the dynamics of the collapse of $A$ to $K$ in the dynamics of the universe $V$ in the presence of the Wholeness Axiom. We recall that an elementary embedding from the universe to itself represented a principle that was “approxi-

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60 In fact, the Veda can be seen as a commentary on $A$ itself since all the dynamics in $AK$ are contained in $A$—see Maharishi (1991a).

61 See Maharishi (1992) for a further elaboration of these stages of unfoldment.
mated” with ever greater accuracy by the known large cardinal axioms. Among these large cardinal axioms, there are eight that stand out as milestones in the climb to the Wholeness Axiom. These eight naturally are partitioned into two groupings; the first group of five gives more “objective” information about the size of $V$; the second group of three provides information about $V$ as it “reflects upon itself”—information which has a more subjective flavor. These eight can be viewed from the perspective of the knower, which, as we have seen, corresponds to the universe $V$; the process of knowing, corresponding to elementary embeddings; and the object of knowledge, which we associate here with the critical points of these elementary embeddings, i.e., the “point values.”

For purposes of discussion, we summarize these eight stages, cast in three perspectives, in Table 2. Each of the large cardinal properties represented in the table can be appreciated in terms of the properties of a particular cardinal number (right column—Chhandas value); in terms of the associated elementary embedding (center column—Devatā value); and in terms of the new structural features of $V$ that become apparent in the presence of the given large cardinal properties (left column—Ṛishi value). The relationship between the right and center columns is easy to understand—the cardinal number given in the right column is generally the critical point of the embedding given in the center column. The structural features of $V$ that we have placed in the left column represent, on the one hand, a catalogue of deep insights into the structure of the universe and, on the other hand, a bridge that connects large cardinal properties to the eight stages of collapse of infinity to a point in Maharishi Vedic Science.

The first five large cardinal properties listed in the chart demonstrate with increasing cogency the truly unlimited nature of the structure of $V$. As these are properties of the structure of $V$, it is natural to connect them with the structural, or objective, principles of creation: earth, water, fire, air, and space. The stages of development of these large cardinal properties correspond to the history of models of set theory, as we shall discuss shortly. The last three large cardinal properties reveal the universe’s “ability” to interact with itself in powerful ways; this self-
Table 2. Eight-fold collapse of infinity
to a point in the context of large cardinals

<table>
<thead>
<tr>
<th>8-fold Prakriti</th>
<th>rishi (the totality $V$)</th>
<th>devata (elementary embeddings of the universe)</th>
<th>chhandas (point value of the embeddings)</th>
</tr>
</thead>
<tbody>
<tr>
<td>prithivi (earth)</td>
<td>$V_\kappa$ is a model of ZFC, and for nearly all ordinals $\alpha &lt; \kappa$, $V_\alpha$ is also a model of ZFC</td>
<td>There is a $\lambda &lt; \kappa$ such that the identity $id: V_\lambda \rightarrow V_\kappa$ is an elementary embedding</td>
<td>$\kappa$ is an inaccessible cardinal</td>
</tr>
<tr>
<td>jal (water)</td>
<td>$V \neq L$ (not $CP(L)$)</td>
<td>There is an external elementary embedding $j: L \rightarrow L$</td>
<td>$\emptyset$ exists</td>
</tr>
<tr>
<td>agni (fire)</td>
<td>$V \neq K$ (not $CP(K)$)</td>
<td>There is a nontrivial elementary embedding $j: V \rightarrow M$</td>
<td>Measurable cardinals exist</td>
</tr>
<tr>
<td>vaayu (air)</td>
<td>$V \neq L[A]$ for any set $A$</td>
<td>For each $\lambda$, there is $j_\gamma: V \rightarrow M_\gamma$ such that $V_\lambda \subseteq M_\gamma$</td>
<td>Strong cardinals exist</td>
</tr>
<tr>
<td>akash (space)</td>
<td>AD holds in $L(R)$</td>
<td>There is a $j: V \rightarrow M$ with critical point $\kappa$ and there are cardinals $\delta_0 &lt; \delta_1 &lt; \ldots &lt; \delta_n &lt; \ldots &lt; \kappa$ such that for each $n$ and each $f: \delta_n \rightarrow \delta_n$ there is a $\lambda_n &lt; \delta_n$ such that $f^{\delta_n} \subseteq \lambda_n$ and there is $j_\gamma: V \rightarrow M_\gamma$ with critical point $\lambda_\gamma$ and $V_\gamma \subseteq M_\gamma$ where $\gamma = (i_\gamma)(\lambda_\gamma)$</td>
<td>There are $\omega$ Woodin cardinals with a measurable above</td>
</tr>
<tr>
<td>manas (mind)</td>
<td>A magic sequence can be defined from which all sets can be located</td>
<td>For each $\lambda$, there is $j_\gamma: V \rightarrow M_\gamma$ with critical point $\kappa$ such that all $\lambda$-sequences from $M_\gamma$ are members of $M_\lambda$</td>
<td>Supercompact cardinals exist</td>
</tr>
<tr>
<td>buddhi (intellect)</td>
<td>Almost all cardinals are large cardinals</td>
<td>For each $\alpha$, there is $j_\alpha: V_{\text{crit}} \rightarrow V_\alpha$ with critical point $\kappa$</td>
<td>Extendible cardinals exist</td>
</tr>
<tr>
<td>ahamkar (ego)</td>
<td>There exist arbitrarily close approximations to a countable sequence unbounded in $V$</td>
<td>For each $n$, there is $j_n: V \rightarrow M_n$ such that $M_n$ is closed under $\kappa_n$-sequences (where $\kappa_n$ is the $n$th iterate of $j_n$ applied to $\kappa$)</td>
<td>Cardinals $\kappa$-huge for every $n$ exist</td>
</tr>
</tbody>
</table>

interaction is suggestive of a sort of subjective quality present in the universe, and again, it is natural to connect these with the subjective principles of creation: mind, intellect, and ego.
We begin with the first five properties in the chart. These are statements that assert in an increasingly powerful way that none of the structurally complete and well-behaved models of set theory—which have been “built from below” to anticipate and resolve as many independent mathematical statements (like the Continuum Hypothesis) as possible—are simply not vast enough to be the universe $V$ itself.

The first of these properties arising from the existence of an inaccessible cardinal asserts that $V$ is much more than merely a model of ZFC: In the presence of an inaccessible, a model of ZFC is quite a common phenomenon in the sense that nearly every stage $V_\alpha$, for $\alpha < \kappa$, is such a model.\(^{62}\)

The second through fifth of these properties represent historical developments in a field that has come to be known as inner model theory. These developments began with Kurt Gödel’s (1938) proof that the Generalized Continuum Hypothesis was consistent with ZFC. His method was to construct a model of set theory, known as the constructible universe and denoted $L$, in which each successor stage is built by collecting together only those subsets of the previous stage that are definable in the previous stage. (Recall that to form a successor stage in the construction of $V$, all subsets of the previous stage are used.) This method drastically limits the number of sets that are introduced at each stage—so much so that the power set of any infinite set is guaranteed to be precisely the next larger cardinal number (the smallest value such a power set could have). One of the remarkable consequences of Gödel’s model is that virtually all statements known to be independent of ZFC can be decided under the assumption that $V = L$; that is, given virtually any statement in the language of set theory that is known to be independent of ZFC, a proof is also known that demonstrates that the statement is either true or false under the additional assumption that $V = L$. The reason for this remarkable empirical fact is that, built into the design of $L$ is a wide range of powerful combinatorial tools—tools that emerge from the severe restriction on the number of sets allowed to enter at each stage and that give the mathematician extraordinary control over the behavior of mathematical objects. Hence, problems about sets, the real line, trees, infinite abelian groups, and general topological spaces

\(^{62}\) In the present context, “nearly all $\alpha$” means that the set of all such ordinals forms a closed unbounded set in the regular cardinal $\kappa$. 

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that were impossible to resolve using the tools available in $V$ could be decided using the tools in $L$.

A natural question to entertain in light of Gödel’s discovery—and which forms the basis for the second entry in the chart above—is whether our universe $V$ could really be nothing other than $L$. It is consistent with ZFC for this to be so. The universe according to $L$ would be extremely precise and well-formed, but extremely restrictive. This extreme restrictiveness has led set theorists, including Gödel, away from the belief that $V = L$. There is common agreement that the “absolute” universe ought to be somewhat more vast and expansive than $L$.

It turns out that there is a precise point—known as $0^*$ (pronounced “zero sharp”)—in the ascent through large cardinal axioms at which the fundamental structural differences between $V$ and $L$ become apparent. R. Solovay (1967), using other large cardinals, discovered a real number which contained enough information (in coded form) to demonstrate that $L$ was a model of set theory radically different from $V$; he called this real number $0^*$. As an example of the impact of $0^*$, one of its consequences is that $L$’s set of real numbers is (from the point of view of $V$) no larger than the set of natural numbers! Results of this kind immediately demonstrated that in the presence of $0^*$, $V \neq L$ and suggested that the world according to $L$ is in fact somewhat distorted; thus, while the axiom $V = L$ has unquestionable value for establishing formal consistency results, it should not be taken as an intuitively clear assumption about the structure of $V$.

After Solovay’s discovery of $0^*$, Kunen showed that the axiom “$0^*$ exists” could be seen as a large cardinal axiom in its own right by showing that it was equivalent to the existence of an external (to $L$) elementary embedding from $L$ to $L$.

Perhaps the deepest work done in this area is due to Jensen. He showed that the axiom “$0^*$ exists” is the precise point in the hierarchy of large cardinal axioms at which the structures of $L$ and $V$ radically diverge. In particular, he showed that if $0^*$ does not exist, then $L$ and $V$ are very similar in the sense that every uncountable set of ordinals (that is, every set bigger than the set of natural numbers) lies in a set in $L$ of the same size; conversely, if $0^*$ does exist, then some large set of ordinals in $V$ is not contained in any set of ordinals in $L$ of the same size. This result has come to be known as the covering lemma for $L$. When
a model $M$ satisfies the property that every uncountable set of ordinals in $V$ is contained in a set in $M$ of the same size, $M$ is said to have the \textit{covering property} and we write $CP(M)$. Thus, Jensen’s covering lemma for $L$ states that the nonexistence of $0^\sharp$ is equivalent to $CP(L)$. As we shall see, efforts have been made to generalize Jensen’s covering lemma to models besides $L$.

One important thread of research which emerged from Jensen’s work, and which leads us to the third property listed in our chart, was the search for a more expansive model than $L$ which retained the richness of $L$’s combinatorial tools, but which did not so readily diverge from the structure of $V$. The objective was to find a model with a fine structure like that of $L$, but which satisfied the covering property even in the presence of $0^\sharp$ and possibly much larger large cardinals as well. T. Dodd and R. Jensen (1982) published extraordinary results achieving this objective by introducing the \textit{core model} denoted by $K$. The core model retains all the most desirable combinatorial properties of $L$ but has sufficient flexibility to satisfy the covering property in the presence of $0^\sharp$ and much larger cardinals as well, such as Erdös cardinals and Ramsey cardinals. The precise point of failure of $CP(K)$ turns out in this case to be a measurable cardinal: $CP(K)$ holds if and only if there is no (inner) model of a measurable cardinal. In particular, in the presence of a measurable cardinal, the structures of $K$ and $V$ can be seen to be radically different.\footnote{Jensen’s discovery of these turning points for $L$ and $K$ in the large cardinal hierarchy has had a far greater significance in mathematics than simply to exhibit points of divergence between various models of set theory. A major consequence of his work is that it makes the powerful combinatorial tools present in $L$ available for use in ordinary mathematics. One recent application of these tools has been in the determination of the exact strength of the Normal Moore Space Conjecture in general topology.}

Another attempt to expand Gödel’s $L$ to include more large cardinals but retain strong combinatorial properties led to the development of the class of models $L[A]$ for various sets $A$; these models explicitly expand $L$ in its stage-by-stage definition so that at each successor stage, the new sets that are introduced are those definable from the previous stage and from the set $A$ itself. This construction generally results in a model bigger than $L$. The most important application of this construction was to expand $L$ so that it would not conflict with measurable cardinals (recall that $V$ cannot equal $L$ or $K$ if there is a measurable
cardinal). The solution to the problem was simple and elegant: if $\kappa$ is a measurable cardinal and $U$ is an ultrafilter on $\kappa$ which demonstrates that $\kappa$ is measurable, the model $L[U]$ retains the fine structure of $L$ and at the same time retains the knowledge that $\kappa$ is measurable and that $U$ (actually, $U \cap L[U]$) is the corresponding ultrafilter.

Because of the elegance of this discovery, a program of research emerged that sought to obtain ever richer models of the form $L[A]$ in which ever larger large cardinals could be found. One hope among some of these researchers was that if enough large cardinals could be represented in such a model, then either the model itself or some sort of corresponding core model (bearing the same relationship to the model $L[A]$ as $K$ bears to the model $L[U]$\textsuperscript{64}) could then be taken to be the “real” universe of sets.

In pioneering work, W. Mitchell (1974) had the idea to build models $L[A]$ where $A$ was a sequence of ultrafilters; this technique resulted in nice inner models for large cardinals known as hypermeasurable—much stronger than ordinary measurable cardinals.

The first major stumbling block in this research program (corresponding to the fourth property on our chart) was that, beyond a certain point in the large cardinal hierarchy, it is no longer possible for $V$ to equal $L[A]$ if $A$ is a set. The point in the hierarchy at which this phenomenon is first encountered is a strong cardinal. Thus, if there is a strong cardinal, we find a radical divergence between the structure of $V$ and the structure of all models of the form $L[A]$, $A$ a set.

Mitchell and others were able to overcome this difficulty by considering sequences (and directed systems) which were themselves proper classes\textsuperscript{65} instead of sets; using these methods, they obtained combinatorially rich inner models of strong cardinals, and other larger large cardinals.

A more significant obstacle to this program arose in a rather unexpected way through work by another group of researchers who also were developing an extension of $L$ that could assume the role of the

\textsuperscript{64} Unlike $L$ and $L[U]$, the core model is a highly variable model whose structure depends on what large cardinals actually exist. If $0^\#$ does not exist, then $K = L$. If $0^\#$ does exist, but $0^{\##}$ does not exist ($0^{\##}$ is the real number that bears the same relationship to $L[0^\#]$ as $0^\#$ does to $L$) then $K = L[0^\#]$. $K$ assumes its most expansive form in the presence of a measurable cardinal; in this case it is the intersection of all iterated ultrapowers of $L[U]$—the “core” of $L[U]$, just barely avoiding the existence of a measurable cardinal.

\textsuperscript{65} See Weinless (2011) for a discussion of proper classes.
“real” universe. Their work began with the observation made earlier that, even in the presence of mild large cardinals, the real line in $L$ is only countable. Since the reals are such an important part of mathematics, it was natural to try to expand $L$ so that combinatorial properties are preserved and yet the real line $\mathbb{R}$ retains its status (as the “real” real line). The resulting model was $L(\mathbb{R})$; this model is constructed by beginning at stage 0 with $\mathbb{R}$ itself (actually, the transitive closure of $\mathbb{R}$), instead of the empty set, and then proceeding as in the construction of $L$. As desired, $L(\mathbb{R})$ contains the “real” real line and does indeed retain many of the nice combinatorial features of $L$. However, these nice combinatorial features turn out to be available only in the “upper” reaches of the universe and not in the realms of ordinary mathematics. Perhaps worse, the model (typically) failed to satisfy the Axiom of Choice. Researchers sought to replace the Axiom of Choice with another axiom that could serve to restore rich combinatorics down low in $L(\mathbb{R})$. The axiom that emerged was called the Axiom of Determinacy or AD for short.

A dedicated group of researchers in Descriptive Set Theory developed the mathematical theory based on the axiom $V = L(\mathbb{R})$ and the assumption that $AD$ holds in $L(\mathbb{R})$. For fifteen years, this group continued working out the theory, undaunted by the rather unsettling fact that it was not known whether $AD$ held in $L(\mathbb{R})$—in fact, it was not known whether $AD$ was consistent at all!

Toward the end of the 1980’s, Martin, Steele, and Woodin impressively demonstrated the consistency of $AD$ assuming large cardinals. Woodin eventually showed that the exact large cardinal strength of the consistency of $AD$ is $\omega$ Woodin cardinals. Moreover, assuming $\omega$ Woodin cardinals plus a measurable cardinal above them all, he showed that $AD$ holds in $L(\mathbb{R})$.

Using hindsight, many feel that the dedication of the original group of Descriptive Set Theorists to their unproven intuition about the truth of $AD$ in $L(\mathbb{R})$ is strong evidence for the naturalness of this universe, despite the conspicuous absence of the Axiom of Choice.

Woodin’s result—indicated as the fifth property on our chart—represents another significant structural breakthrough in the ascent through large cardinal axioms. Not only did his result provide the missing link for an important research program in Descriptive Set Theory; not only
did it serve to reveal the universe to be unexpectedly harmonizing in bringing validation to a choiceless universe within a universe in which the Axiom of Choice is valid,\textsuperscript{66} but also, as we will now see, Woodin’s result marks an upper bound to the program of inner model theory described above.

As we mentioned before, inner model theorists sought to expand Gödel’s constructible universe $L$ in such a way that combinatorial properties were preserved and yet large cardinals could be included. The combinatorial properties that researchers sought to preserve were well agreed upon and included the following:

(1) The Generalized Continuum Hypothesis
(2) The Diamond Principle\textsuperscript{67}
(3) The existence of a “nice” (projective\textsuperscript{68}) well-ordering of the reals.

But one immediate consequence of Woodin’s result is that if there are $\omega$ Woodin cardinals with a measurable above, then no “nice” well-ordering of the reals exists at all! Thus, the program of inner model theory, using the above criteria, must come to an end at the point in the hierarchy of large cardinals at which there are $\omega$ Woodin cardinals with a measurable above; in the presence of such large cardinals, the structure of the universe $V$ diverges from the structure of inner models constructed according to the general criteria given above. (In contemporary research, different notions of “canonical inner model” have emerged that are not limited by the presence in the universe of $\omega$ Woodin cardinals with a measurable above.)

As we have seen, each of the first five properties given in our chart above marks a turning point in knowledge about the structure of $V$: inaccessibles mark the point at which it becomes clear that $V$ is far more than merely a model of set theory; $0^+$ marks the realization that $V$ is radically different from $L$; measurable cardinals mark the realization

\textsuperscript{66} Cf. Weinless (2011) for an interesting discussion

\textsuperscript{67} The Diamond Principle asserts that there is a special sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ of subsets of $\omega_1$ such that for each $\alpha$, $A_\alpha \subseteq \alpha$, and for any subset $A$ of $\omega_1$, there are stationarily many $\alpha$ for which $A \cap \alpha = A_\alpha$.

\textsuperscript{68} A well-ordering of the reals $\mathbb{R}$ is a subset of $\mathbb{R} \times \mathbb{R}$. The simplest subsets of $\mathbb{R} \times \mathbb{R}$ are the open sets and the closed sets. A subset of $\mathbb{R} \times \mathbb{R}$ is projective if it is a continuous image of a closed set, or the complement of such a set, or a continuous image of such a complement, or the complement of such a set, and so forth.
that \( V \) is also radically different from the core model \( K \); strong cardinals mark the stage at which \( V \) is known to be different from models of the form \( L[A] \), \( A \) a set; and finally, \( \omega \) Woodin cardinals with a measurable above marks the point at which the traditional program of inner model theory to capture \( V \) with a model of the form \( L[C] \), \( C \) a proper class, finally breaks down, and also marks the point in the hierarchy in which \( AD \) is seen to hold in \( L(R) \) and, philosophically speaking, where Choice and Determinacy are found compatible.

We now turn to the last three properties on our chart which describe more “subjective” features which seem to arise in the universe in the presence of even larger large cardinals. Each of these large cardinal properties marks a stage at which the universe becomes, as if, “aware” of its own nature, structure, and “abilities.”

At the level of a supercompact cardinal (corresponding to the sixth level of our chart) we find that the remarkable magic sequence, described earlier, can be defined. In the absence of a supercompact cardinal, the sets in the universe are located in the usual way: Sets unfold, stage by stage, starting from the empty set, by means of the power set and union operations. But in the presence of a supercompact cardinal, a magic sequence becomes available by which every set in the universe can be located from the perspective of the wholeness of \( V \) (recall that if \( S \) is a magic sequence coded by a class of embeddings \( \{i_\alpha : \alpha \in ON\} \), then any set \( X \) can be located as the \( \kappa \)th term in the sequence \( i_\alpha(S) \), for some ordinal \( \alpha \)). In a certain sense, the magic sequence allows \( V \) to become “aware” of its constituent sets.

Extendible cardinals (at the seventh tier of our chart) are the first in the large cardinal hierarchy that imply that the universe is pervaded with other large cardinals. As we have seen, large cardinals represent the universe’s ability to reflect its properties of wholeness into its own sets; from the point of view of Maharishi Vedic Science, this process of

\[\text{Ref.}\]

\[69\] One shows that the existence of a supercompact cardinal is equivalent to the existence of a magic sequence as follows: First note that, using the proof of Kunen’s theorem, for any elementary embedding \( i : V \rightarrow M \) having critical point \( \kappa \), the set \( \{\kappa, i(\kappa), i(i(\kappa)), \ldots\} \not\in M \). Thus, given any set \( \{i_\alpha : \alpha < \rho\} \) of elementary embeddings of the universe, each with critical point \( \kappa \), the disjoint union \( A \) of the sets \( A_\alpha = \{\kappa, i_\alpha(\kappa), i_\alpha(i_\alpha(\kappa)), \ldots\} \) is not in the union of any of the image models \( M \). Thus, for any \( f : \kappa \rightarrow V \) and any \( \alpha < \rho, i_\alpha(f)(\kappa) \not\in A \). (It is now known that a notion of Laver sequence can be proven to exist using only a strong cardinal; such Laver sequences play the same role as those described in this paper. Vedic researchers will need to find the best way to revise the sixth tier here. —Ed.)
reflection is reminiscent of the wholeness of pure consciousness awak-
ening the point values of life to their fully expanded state.

Most of the large cardinals smaller than extendible imply the exist-
ence of many large cardinals below. For instance, if $\kappa$ is a Mahlo
cardinal, many of the cardinals below $\kappa$ must be inaccessible. If $\kappa$
is measurable, nearly all cardinals below $\kappa$ are Mahlo. If $\kappa$ is super-
compact, nearly all cardinals below $\kappa$ are measurable. As remarkable
as these results are, none of these large cardinals implies that even a
single inaccessible exists above them. Thus, their range of implication is
restricted to a small portion of the universe.

On the other hand, an extendible cardinal implies that arbitrarily
large measurable cardinals exist above it. As we just observed, nearly
all cardinals below a measurable are Mahlo (and hence inaccessible). It
follows that the existence of an extendible implies that nearly all cardi-
nals in the universe are large! The range of influence of an extendible is
therefore global and suggests a radical shift in our knowledge about the
structure of $V$: An extendible tells us that at nearly every stage of the
universe, we can find the lively presence of properties of $V$ as a whole.

Finally, let us consider the eighth level of our chart, occupied by
large cardinals which are $n$-huge for every $n$. Such cardinals hover at
the verge of inconsistency. We start with the definition of $n$-huge for
every $n$:

**Definition.** A cardinal $\kappa$ is $n$-huge for every $n$ if for each natural num-
ber $n$, there is an elementary embedding $j_n : V \rightarrow M_n$ with critical point
$\kappa$ such that every sequence of length $j_n(\kappa)$ (where $j_n$ is the $n$th iterate
of $j_n$) lies in $M_n$.

To see how close this definition takes us to inconsistency, suppose
that in the definition, we require infinitely many of the $j_n$ to be the
same embedding—call it $j$. Then one may show that the correspond-
ing image model—call it $M$—must in fact be closed under sequences
of length $\lambda$, where $\lambda = \sup\{\kappa, j(\kappa), j(j(\kappa)), \ldots \}$. But now we can use $\lambda$
to carry out Kunen’s inconsistency proof to conclude that $0 = 1$! Thus,
from one viewpoint, cardinals that are $n$-huge for every $n$ approximate
inconsistency arbitrarily closely.
Another way to view the impact of these large cardinals is to say that a cardinal which is \( n \)-huge for every \( n \) gives rise to arbitrarily long (definable) finite sequences of the form \( \langle \kappa; j(\kappa), j(j(\kappa)), \ldots, j^n(\kappa) \rangle \) which approximate a (undefinable) countable sequence unbounded in the universe, as described by our Principle of Countable Unboundedness. Thus, an alternative viewpoint concerning the presence of these large cardinals in the universe is that they arbitrarily closely approximate the Wholeness Axiom using definable concepts.

Maharishi points out that the very structure of pure consciousness is upheld by the coexistence of opposite values; that its very nature is at once infinite dynamism and infinite silence; and that one of the distinguishing features of an enlightened individual is the ability to live and integrate opposite—even contradictory—values in a state of balance and harmony.\(^{70}\)

In light of Maharishi Vedic Science, the presence of cardinals which are \( n \)-huge for every \( n \) is reminiscent of the pinnacle of subjective development in which the full extent of contradictory values is reconciled in the full awakening of wholeness.

Thus, the last three levels of our chart suggest the unfoldment of more subjective qualities within the universe: Supercompact cardinals mark a new awareness of the origin of sets within the universe; extendible cardinals bring with them an awareness of the omnipresence of large cardinals, which in turn, bring into the realm of sets central properties of the wholeness of the universe; and cardinals which are \( n \)-huge for every \( n \) provide a strong analogue to that state of awareness which is on the brink of awakening to the full value of wholeness, in which even the most contradictory values are unified.

The eight stages of collapse of infinity to a point, as described by Maharishi Vedic Science, is an unchanging pattern lying within the blueprint of creation; for this reason, we expect to find this pattern at work within the foundation of every discipline. Our chart and subsequent discussion suggest that the eight large cardinal axioms we have identified represent natural “power” points in the ascent through the large cardinal hierarchy, marking the confluence of diverse and seemingly unrelated results as well as radical changes in the complexion and structure of the universe. For this reason, we feel these eight, cast in

\(^{70}\) See Maharishi (1991a).
the three-fold framework described in our chart and emerging in the context of the new dynamics of wholeness provided by our Wholeness Axiom, give expression to the fundamental pattern of eightfold collapse described in Maharishi Vedic Science.

Of course, it is the nature of western science to refine itself continually. While the dynamics of wholeness embodied in the Wholeness Axiom and its ramifications do appear to give expression to the fundamental dynamics of the wholeness of pure consciousness as described by Maharishi, one would expect that as the Foundations of Mathematics evolve, new, fuller expressions of these fundamental patterns of nature will inevitably emerge. We view our work here as part of an ongoing program to give ever fuller expression to the deepest dynamics of consciousness within the foundation of mathematics, for the sake both of perfecting mathematics and of bringing fulfillment not only to the mathematician but to all who come in contact with the field of mathematics.

§ 21. Conclusion

We began our study of ZFC and the universe of sets with the observation that, while it succeeds in providing a foundation for most of modern mathematics, this foundational structure fails to account for the presence of large cardinals within mainstream mathematics. None of the heuristic devices used by mathematicians so far to account for these infinities (or to remove them) succeeds in accounting for all large cardinals, and none is very compelling. This state of affairs makes apparent the need for a single set-theoretic principle which at once accords with the fundamental intuition of set theorists and accounts for large cardinals.

Typically, mathematical intuition derives from observation of nature and from mathematical experience. In the case of large cardinals, mathematical experience suggests that large cardinal properties are actually intimately tied to metatheoretic properties of the universe as a whole. The Reflection Principle uses this connection to provide justification for many of the smaller large cardinals, while nearly all the larger large cardinals can be framed in terms of the universe's interaction with other universes, in terms of elementary embeddings of the universe. Thus, mathematical experience with large cardinals suggests that
we look for our justification for large cardinals by gaining a clear intuitive sense of the “true” nature of the universe $V$ as a whole.

Interestingly, the same conclusion results from any reasonable attempt to look to nature’s functioning for intuitive guidance concerning such foundational concerns as the origin of large infinities. Certainly, nature’s behavior on the superficial level of finite collections of objects interacting with each other is of little value in motivating concepts pertaining to the infinite. On the other hand, one might expect that nature’s behavior at its roots would suggest the “right” picture for the foundation of mathematics. On a parallel track, recent work in quantum field theory (Hagelin, 1987) tells us that, at small time and distance scales, all force and matter fields can be seen as expressions or precipitations of a single, unified, self-interacting superfield. Unified field theories provide an unprecedented unification of widely diverse phenomena and physical theories, and suggest to us the principle that, in the presence of a “theory of wholeness,” theoretical explanations for various other phenomena become more available.

For these and other reasons, we have investigated the nature of the universe of sets as a whole, seeking a basic unifying principle. We observed that certain great mathematicians and philosophers in history—Plato, Cantor, and Gödel to name three—have claimed to have a more or less clear intuition of “mathematical reality,” and that their view of their work as an attempt to give expression to this deeper reality has led to important discoveries (such as transfinite cardinals and the completeness and incompleteness theorems). We expressed the belief that this sort of intuition does not appear to be awake equally in all mathematicians and suggested the need to revitalize the intuition of all practising set theorists to gain a more uniform view of the reality glimpsed by some of the mathematical giants.

We have claimed that the basic reality, which has been intuited more or less clearly by some mathematicians and which has been uncovered to some extent through objective means in research on completely unified field theories, has in fact been thoroughly studied through subjective technologies in many traditions of knowledge throughout the world, the most ancient being the Vedic tradition. For this reason, we have appealed to Maharishi Vedic Science—constituting a modern-day systematic treatment of this ancient wisdom, complete with effective
inner technologies for exploring the realities proclaimed in the ancient texts—as an attempt to deepen our intuition about the wholeness we are trying to capture through the concept of a universe of sets.

Reviewing the qualities of the field of wholeness, pure consciousness, as described in Maharishi Vedic Science, we found certain deficiencies in the structure of $V$ as a model for wholeness. These deficiencies suggested new features of the universe we might wish to include in a unifying axiomatic principle, such as a natural transformation from $V$ to itself to capture the quality of self-interacting, and the presence of many sets reflecting all first-order properties of $V$ to capture the quality of infinite correlation. A study of the dynamics of pure consciousness revealed a more compelling deficiency, namely, that the dynamics of unfoldment of the universe is unilateral, expanding from the empty set to all sets in the universe, and not, as we find within the wholeness of pure consciousness, collapsing from the fully expanded value to a point value. This deficiency again suggested another dynamic to incorporate into our new axiomatic principle.

Appealing to mathematical experience, we also sought intuition from the statements of the strongest large cardinal axioms. These assert the existence of elementary embeddings from the universe $V$ to models resembling more and more closely the structure of $V$ itself. The natural limit to the large cardinal properties is the existence of an elementary embedding from $V$ to itself. One easily verifies that the features of the universe, suggested to us by our analysis of the qualities and dynamics of pure consciousness, become evident in the presence of such an embedding. Although Kunen showed that no such embedding could be (weakly) definable in the universe, we observed—treating such an embedding as an analogue to the unmanifest dynamics of pure consciousness moving within itself and knowing itself—that it would be more natural to require that such an embedding $j$ be undefinable (more precisely, not weakly definable) or transcendental. Realizing that pure consciousness is not only unmanifest but also present at each point in creation, we further required that the universe be fully $j$-closed. In this way we arrived at our Wholeness Axiom, asserting that such an embedding $j$ exists and that $V$ is fully $j$-closed. This approach avoids the contradiction produced by Kunen’s proof and at the same time successfully accounts for virtually all large cardinal properties as the properties
of the critical point of a single, original embedding \( j \) of the universe to itself.

Examining the dynamics arising from set theory enriched by the Wholeness Axiom, we found deep and unexpected parallels with Maharishi Vedic Science. On the one hand, the collapse of the infinitely expanded value of wholeness, represented by \( A \), to its fully contracted value, represented by \( K \), imparting \( K \) with infinite dynamism at the basis of the formation of the Veda and all creation is paralleled, on the other hand, in the first stirring within \( V \) resulting from the action of \( j \): the first set \( \kappa \) moved by \( j \) stands as a focal point, embodying the properties of the wholeness of \( V \) and imbued with the truly infinite dynamism expressed as a vast class of supercompact embeddings which are selectively coded into a compact magic sequence, which gives rise to every set in the universe.

We observed that the infinity of transformations that emerge in this collapse of \( A \) to \( K \) finds its analogue in the fact that \( j \) very naturally gives rise to all possible supercompact embeddings—each occurring as a factor of one of the canonical supercompact embeddings derived from the myriad supercompact ultrafilters that unfold from \( j \). We found that the interplay of collapse of infinity to a point and expansion of point to infinity with infinite frequency, occurring at each moment as the fundamental dynamics of pure consciousness, again finds a parallel in the infinitely often repeated construction, through expansion and collapse, of the canonical models via supercompact ultrafilters.

We noticed also that the magic sequence, which emerges naturally in the dynamics of \( j \), embodies many qualities and dynamics of the Veda. By coordinating and unifying the action of the vast class of supercompact embeddings—having \( j \) as their basis—every set can be accounted for by virtue of the internal structure of the magic sequence; yet this sequence at the same time exhibits a quality of infinite silence, since it is the identity on a large subset. Finally, we observed that the eightfold structure of the collapse of \( A \) to \( K \), in its three phases of Rishi, Devatā and Chhandas, finds a striking parallel in eight large cardinal principles that increasingly approximate, in consistency strength, the Wholeness Axiom.

We have discovered how the dynamics of pure consciousness are a fundamental reality of nature. In the spirit of J. Hagelin’s paper “Is
Consciousness the Unified Field?” (1987), we feel that, just as modern unified field theories are an attempt to model this reality with the tools of quantum field theory, so our efforts in this paper—and so the efforts of all contributors to set theory whether intentionally or not—are an attempt to model this reality with the tools of set theory. The fact that so many strong analogies can be found between the structure of $V$ and the structure of pure consciousness as a consequence of adding a single axiom suggests that our efforts have been successful.

By giving a unified and compelling account of the origin of large cardinals, we feel we have shown that our efforts to model the dynamics of pure consciousness within set theory have a genuine mathematical value. We anticipate that further investigations of this kind will turn up a rich assortment of mathematical results. This sort of relationship between a mathematical field and another science has always been fruitful in mathematics’ long history. The study of problems in physics and biology—even gambling—has resulted in the creation of whole new branches of mathematics that have occupied the careers of many bright minds.

Most recently, the tools of category theory have been brought to bear on modeling problems in computer science; the relationship between these fields in the past decade has brought advances not only in computer science—since this was the area in which researchers were seeking solutions—but in category theory as well. Category theorists, in using the tools of their trade in new ways, have encountered a whole new class of pure mathematical problems that have stimulated considerable research. We feel that a relationship of this kind is beginning to emerge between Maharishi Vedic Science and a number of sciences, including mathematics. The fact that the former provides a natural framework in which to view the origin of large cardinals, as we have discussed here, is, we feel, just the beginning. When we are asked to look at the universe of sets from the point of view of attempting to model the wholeness of pure consciousness, new features of the landscape of set theory become important and new questions arise. For instance, consider the following:

1. In our study of the eightfold collapse of infinity to a point, we singled out certain large cardinal axioms as especially signifi-
cant turning points in the ascent up the large cardinal hierarchy. Are these the best choices? One criterion of "best" might be whether the properties associated with them in the Rishi column of the chart are in fact equivalent to the large cardinal property. We observed, for example, that the existence of a magic sequence is equivalent to the existence of a supercompact cardinal. But if we reword the definition of a magic sequence so that it does not appear to depend so heavily on the concept of supercompactness, do we preserve the equivalence? Suppose we call a function $f : \kappa \to V_\kappa$ a magic sequence if for each set $A$ there is some elementary embedding $i : V \to M$ with critical point $\kappa$ for which $i(f)_\kappa = A$. Is this concept equivalent? Suppose we call a function $f : \kappa \to V_\kappa$ a magic sequence if for each set $A$ there is some elementary embedding $i : V \to M$ with critical point $\kappa$ for which $i(f)_\kappa = A$. Is this concept equivalent? Suppose we call a function $f : \kappa \to V_\kappa$ a magic sequence if for each set $A$ there is some elementary embedding $i : V \to M$ with critical point $\kappa$ for which $i(f)_\kappa = A$. Is this concept equivalent?

2. Likewise, is the notion of extendible cardinal the proper choice in our eightfold structure? It is clear that the fact that there is a proper class of measurables above an extendible does not characterize extendibility, but what structural property of the universe does characterize this large cardinal? In comparing the dynamics that arise from the Wholeness Axiom with those attributed to pure consciousness, we saw that a Laver magic sequence—selectively coding information concerning the proper class of canonical supercompact embeddings arising from $j$—plays a role that is in many ways quite similar to the role of the Veda in the unfoldment of creation: A Laver sequence is a highly compact expression by means of which every set in the universe can be located.

An ongoing aspect of our present research is to determine how far this analogy goes. We described earlier certain striking analogies between these structures; however, there are still many details to the structure of the Veda that we expect will be reflected in the structure of these magic sequences, or of some related structures.

As an example of work in this direction, it has been shown (Corazza 2011) that the magic sequence $S$ that we derived from $j$ has the property that, stationarily often, the restriction $S \upharpoonright \alpha$ is also a magic sequence at $\alpha$. We might also expect that sta-

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71 This notion is now known to be equivalent to the existence of a strong cardinal (and such cardinals are known to be much weaker than supercompact). See (Corazza 2000).
tionarily often $S(\alpha)$ is also a magic sequence at $\alpha$, but this does not appear to be true; at present we can prove only that, stationarily often, $S(\alpha)$ is a magic sequence at $\alpha$ of degree $2^\alpha$, because of certain technical limitations imposed on us by the ultrapower construction. This may suggest that a stronger notion of “magic sequence” could be defined and proven to arise from the Wholeness Axiom.\textsuperscript{72}

3. With our Wholeness Axiom, we have successfully accounted for all large cardinals except one. In proving that AD holds in $L(R)$, Woodin originally assumed the following axiom:

$$I_0: \text{There is an elementary embedding } j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1}).$$

Later, he was able to get by with a much weaker axiom, but no inconsistencies arose in his work with $I_0$. Now $I_0$ stands as the strongest of large cardinal axioms but, so far, we do not have a satisfactory account for this axiom using the Wholeness Axiom. Perhaps a stronger version of the Wholeness Axiom can be defined which captures even more completely the spirit of our present axiom.

Even as questions of mere technical interest, the mathematical problems that arise in this study justify further research into the connections between set theory and Maharishi Vedic Science. We feel, however, that this sort of research, as it directly familiarizes the researcher with the deepest dynamics of nature’s functioning, makes a much more significant contribution to the field than simply a new set of problems to work on. We feel that once set theory—or any scientific discipline—is organized around the very fundamentals of nature, nature will yield its secrets far more effortlessly. And, certainly, once these dynamics are awakened not merely in the objective work of the scientists but in the inner life of the scientist as well, then the activity of the discipline will not only reap the objective rewards of expanded knowledge but also bring to the life of the scientist the much greater fulfillment that arises from aligning individual endeavors with the deeper purpose behind creation itself and that extends beyond one’s professional life to transform all areas of individual and collective concern.

\textsuperscript{72} This hypothesis has been demonstrated to be correct. See Corazza (2000; 2011).
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Categories and Toposes:
Dynamism at the Foundation of Mathematics

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Michael Weinless, Ph.D., received his B.S. from the Massachusetts Institute of Technology (M.I.T.) in 1964 and his Ph.D. in mathematics from M.I.T. in 1968. He went on to Harvard University where he held the positions of Benjamin Pierce Lecturer and Assistant Professor of Mathematics from 1968 to 1971. In 1972, Dr. Weinless became one of the founding faculty members of Maharishi International University (renamed Maharishi University of Management in 1995), where he pioneered the development of a Consciousness-Based mathematics curriculum, integrating principles of the Science of Creative Intelligence and Maharishi Vedic Science with the traditional mathematical content of the courses. Dr. Weinless was chairman of the Department of Mathematics from 1972 to 1990.
The most widely accepted modern-day foundation for mathematics is set theory. All fields of mathematics can be expressed in the language of sets, and all theorems from those fields can be derived from the axioms of set theory. Nevertheless, some mathematicians have felt that the static, set-theoretic way of formalizing the dynamic aspect of mathematics, represented by the notion of a transformation or function, fails to capture the essence of the concept, though formally it does succeed.

This concern and others have led to the development of an alternative foundation—category theory—which uses as its starting point the concept of transformation or arrow. In this field, the alternative to a set-theoretic universe of sets is called a topos; it is in fact a generalized universe of sets. Toposes can differ from the usual universe of sets in a number of fundamental ways but, nevertheless, have the remarkable property that set theory can be developed internally—within any topos.

The great power of topos theory has been, in fact, to provide models of a generalized type of set theory, models that have extraordinary properties not available in the familiar universe of sets. The more flexible range of theories modeled by topos theory has led to the discovery of the extraordinary ability of topos theory to unify foundational theories, including set theory and category theory: they are seen to be different ways of viewing the same underlying reality.

At the same time, topos theory has been found to unify classical mathematics with intuitionistic mathematics, the school of mathematics founded by the Dutch mathematician L. E. J. Brouwer. Brouwer’s intuitionism rejected outright the set-theoretic foundation, claiming that the infinitary methods at the heart of set theory were not ultimately meaningful. Topos theory has succeeded in integrating intuitionism into the framework of traditional mathematics by showing how to construct models that provide a classical interpretation of the fundamental intuitionistic theories. This synthesis is achieved on the basis of a self-referral structure of knowledge—inhertent in a topos—whose objects simultaneously play the role of “knower” (more precisely, the stages of knowing in the Kripke-Joyal semantics of its internal language) and “object of knowledge” (namely, the objects of the topos, viewed as sets in the universe).

The role of topos theory in modern mathematics thereby displays a striking parallel to the role of Vedanta in Maharishi Vedic Science, in which the
different intellectual viewpoints on reality presented by Vedic literature are synthesized in the transcendental unity of pure consciousness. This unified wholeness of consciousness is described as containing a self-referral structure of knowledge, in which the knower and known are identical.

Our goal in this article is to show how the dynamics of the synthesis achieved by Vedic Science find expression in topos theory, in its role as a grand unifier of foundational theories. Our approach will be to comfortably lead the reader to this point of synthesis of mathematical knowledge, so that he or she may appreciate the unity that has recently been achieved by the discipline of mathematics.

Introduction

This article is an elementary introduction to the categorical approach to the foundations of mathematics. The past twenty years have witnessed an extraordinary maturation of this field, culminating in the recent applications of the topos-theoretic method of forcing to systematically construct models of higher-order intuitionistic theories. This has integrated essentially all of intuitionistic mathematics into the classical framework; on this basis, the discipline of mathematics today enjoys a degree of integration and unity undreamed of several decades ago. It is unfortunate that so few mathematicians are even aware of these developments; the author hopes that this article will help to make them accessible to a wider audience.

The article itself has evolved from lecture notes for an introductory course on categories and toposes taught at Maharishi International University in 1984. The course had no formal prerequisites and most students had extremely modest mathematical background; the course was therefore designed to be as self-contained as possible. The first chapter is an introduction to abstract algebra in the context of commutative rings: it is important for the student to be comfortable with the abstract algebraic viewpoint before introducing the more abstract category-theoretic viewpoint, which in turn must be assimilated before embarking upon topos theory.

We do not attempt to give a rigorous, axiomatic development of the subject matter. Our approach has been to introduce the abstract concepts in as simple a way as possible and to make them concrete by examining many simple examples. This seemed the most reasonable
approach for students with modest background and limited time to learn the material. More detailed developments of category theory and topos theory, at an elementary level, can be found in Arbib and Manes (1975) and Goldblatt (1984).

A unique aspect of the course at Maharishi University of Management (previously, Maharishi International University) was the integration of the mathematical content with Maharishi Vedic Science, a systematic understanding of the nature of consciousness and its internal dynamics, formulated by Maharishi Mahesh Yogi. Vedic Science provides not only theoretical principles describing consciousness but also the practical technique to systematically develop higher states of consciousness. All courses at Maharishi University of Management integrate the traditional subject matter with Vedic Science; this allows the student to appreciate the material in a much more intimate way, by seeing how it relates to his own experience of more fundamental levels of the mind. The underlying theme is that the source of each discipline can be identified with the most fundamental and simple state of the student’s own awareness, a totally unified state of awareness described as the experience of pure consciousness or pure intelligence.

Teaching mathematics through this approach has proved to be particularly fruitful in addressing the concern about “relevance” in mathematics education. The abstractness of mathematical knowledge, which has traditionally made mathematics remote from experience and therefore difficult for students to relate to, is found transformed into an asset; because of its abstractness, mathematical knowledge can be related most naturally and profoundly both to the abstract principles of Vedic Science describing the nature and dynamics of intelligence, as well as the student’s direct experience of the totally abstract field of consciousness at the source of his own intelligence—an experience that is at one time completely abstract yet absolutely concrete!

In this article we have many references to principles of Maharishi Vedic Science at relevant places; we hope these will help to illuminate the abstract subject matter. We refer the reader to the author’s 1987 article, *The Samhita of Sets*, reprinted in Part 1 of this volume, for an introduction to the principles of Maharishi Vedic Science in the context of the foundations of mathematics. That exposition contains a section on categories, toposes, and intuitionism, which provides a good
overview of the main themes of the present article, as well as placing
them in the context of the set-theoretic foundation. The presentation
of material on categories in that article is the real introduction to the
present article, and the reader is strongly encouraged to start there. The
remainder of The Samhita of Sets explores the set-theoretic foundation,
which is the natural complement to the categorical approach developed
here. It is hoped that the present article in conjunction with our Sam-
hita article succeeds in providing a reasonably comprehensive picture of
the unified structure of modern mathematics, and that the reader will
share the excitement that the author experienced as he learned of these
extraordinary mathematical developments.

Chapter 1
Abstract Algebra

1.1 Sets With Structure
This chapter is an introduction to the viewpoint of the “working math-
ematician,” the conceptual milieu in which the modern research math-
ematician lives. This is a world inhabited by abstract objects called
mathematical structures or sets with structure. Knowledge of these objects
is organized into abstract theories, each theory describing one par-
ticular type of structure. We shall become familiar with this world
by examining one particular abstract theory: the theory of commuta-
tive rings. Once we become at home with this level of abstraction we
shall journey to the even more abstract realm of category theory in
Chapter 2. This will provide the requisite tools to ascend the summit of
topos theory in Chapter 3, and from there enjoy a vision of the unified
wholeness of mathematical knowledge.

In modern mathematics we find a synthesis of many viewpoints. At
one extreme is the foundational viewpoint of set theory. This sees all of
mathematics structured in terms of sets and the membership relation
between them. The membership relation is the primordial relation of
set theory; in the set-theoretic foundation, all values of relationship are
sequentially unfolded from the membership relation based upon defini-
tions. In the ultimate analysis, sets and the membership relation are all
that there is.
The set-theoretic approach has been extraordinarily successful in providing a unified foundation for modern mathematics. Nevertheless, it does not give full expression to the way the mathematician thinks about mathematics. Certainly, sets and the membership relation are fundamental concepts in all areas of mathematics, but so also are operations, relations and functions. The concept of a function is the fundamental concept expressing the value of mathematical transformation. In the set-theoretic foundation, functions are identified with specific sets: sets of ordered pairs. This works, but it doesn’t do justice to the intuitive concept of a function as a transformation from one set to another. With the growing prominence of the concept of a function in modern mathematics, there naturally arose a second, complementary viewpoint that took the concept of function as the primordial concept: this is the foundational viewpoint of category theory. From the viewpoint of category theory, one can’t even talk about elements of a set. One can only talk about the way functions combine with one another to yield new functions.

Set theory is able to provide a foundation for mathematics because, among other things, it can talk about functions, through the identification of functions with particular sets. For category theory to gain equal dignity, it is necessary to somehow reverse the process and derive the membership relation of set theory from the relationships between functions. This has been achieved through the development of topos theory, which has truly brought fulfillment to the category-theoretic approach to the foundations of mathematics.

The dominating characteristic of topos theory is its value of synthesis. Topos theory provides a striking unification of set theory and category theory by showing how both viewpoints can be naturally translated into one another: they are seen to be just two different ways of viewing the same underlying reality. At the same time, topos theory unifies classical mathematics with intuitionistic mathematics, the school of mathematics founded by the Dutch mathematician L. E. J. Brouwer during the early decades of the 20th century. Brouwer’s intuitionism (Brouwer, 1908) rejected outright the set-theoretic foundation, claiming that the infinitary methods at the heart of set theory were not ultimately meaningful because they could not be effectively carried out by the human intellect. Topos theory has succeeded in integrating
intuitionism into the framework of traditional mathematics by showing how to construct models that provide a classical interpretation of the fundamental intuitionistic theories.

The unification of set theory, category theory, and intuitionism brought about by topos theory has its basis in an extraordinary self-referral structure of knowledge belonging to a topos. This we shall examine in Chapter 3. The role of topos theory in modern mathematics thereby displays a striking parallel to the role of Vedanta in Maharishi Vedic Science.

Maharishi has described the way different parts of the Vedic literature present different intellectual viewpoints on reality; the synthesis of these contrasting viewpoints is presented by Vedanta, which synthesizes all diversity into the transcendental unity of pure consciousness. This unified wholeness of consciousness is described as containing a self-referral structure of knowledge, in which the knower and known are identical. Modern mathematics likewise contains a number of contrasting and even contradictory foundational viewpoints; topos theory provides the grand synthesis of all these viewpoints. This synthesis is achieved on the basis of a self-referral structure of knowledge in which the stages of knowing (values of the knower) are identical to the sets (objects of knowledge). The main goal of this book is to comfortably lead readers to this point of synthesis of mathematical knowledge, so that they may appreciate the unity that has recently been achieved by the discipline of mathematics.

We shall begin our journey in a modest way by examining the abstract-theories viewpoint, the viewpoint of the working mathematician. From this perspective, the discipline of mathematics is organized into a hierarchy of abstract theories, each describing a particular type of structure. At the base of this pyramid of theories is a theory we might call the theory of “sets and functions.”

The theory of sets and functions describes sets of points and transformations between these sets called functions. The concept of set here is somewhat different, however, from that of the set-theoretic foundation. From the viewpoint of the set-theoretic foundation, the elements of a set are themselves sets. Thus each element has its own internal structure and there are natural relationships between the elements of a set; for example, one element could be an element of another element.
From the point of view of the theory of sets and functions, the elements of a set are regarded as structureless, undifferentiated points. The only thing characterizing a set is the number of its elements. Relationships between sets are described in terms of transformations from one set to another called functions.

The concept of function is the most fundamental concept of transformation in mathematics. This concept is simply the following. Suppose $A$ and $B$ are any two sets. A function from $A$ to $B$ is some assignment, to each element of $A$, of a well-defined element of $B$. For example, if $A = \{1, 3, 5, 7\}$, and $B = \{2, 3, 4, 8, 9\}$, then one example of a function from $A$ to $B$ is the transformation that takes $1$ to $2$, $3$ to $8$, $5$ to $3$, and $7$ to $8$. It is not necessary that different elements of $A$ are taken to different elements of $B$; it is also not necessary that every element of $B$ is “hit” by the function. It is required, however, that no element of $A$ goes to two different elements of $B$.

There is a special notation used to describe functions called functional notation. If we introduce the symbol $f$ to designate the function described above, we would write: $f(1) = 2$, $f(3) = 8$, $f(5) = 3$, and $f(7) = 8$. In general, we write $f(a) = b$ if the function takes the element $a$ of $A$ to the element $b$ of $B$. In the table in Figure 1.1 we have listed the values for three other functions, $g$, $h$, and $k$, from $A$ to $B$. Using functional notation, we would write $g(5) = 9$, $g(7) = 4$, $h(3) = 3$, and so on.

Functions can have special properties, which we now describe. One property is the property of being one-to-one or injective. This means that the function takes distinct elements of its source $A$ to distinct elements of its target $B$. For example the functions $g$ and $k$ of the table are one-to-one, but $f$ and $h$ are not; $f$ takes the two elements $3$ and $7$ to the same value $8$, $f(3) = f(7) = 8$, and $h$ takes the two elements $1$ and $3$ to the same value $3$.

A second special property a function can have is the property of being onto or surjective. This means that for every element $b$ in $B$, one can find an element $a$ in $A$ such that the function maps $a$ to $b$; if we call the function $f$, then $f(a) = b$. None of the examples in the table of Figure 1.1 is onto. In fact, no function from the set $\{1, 3, 5, 7\}$ to the set $\{2, 3, 4, 8, 9\}$ can be onto. This is because the set $\{2, 3, 4, 8, 9\}$ contains five elements, whereas the set $\{1, 3, 5, 7\}$ contains only four elements.
An example of a function that is onto is the function $f$ from $\{1, 2, 3\}$ to the set $\{2, 4\}$ defined by $f(1) = 4$, $f(2) = 4$, and $f(3) = 2$.

![Figure 1.1 Examples of functions from the set \{1, 3, 5, 7\} to the set \{2, 3, 4, 8, 9\}.]

A function that is both one-to-one and onto is said to be bijective or an isomorphism of sets. Such a function establishes a one-to-one correspondence between the elements of the two sets, establishing that they have the same number of elements. If there exists such a bijection from a set $A$ to a set $B$, the sets are said to be isomorphic. From the point of view of the theory of sets and functions, isomorphic sets are indistinguishable; the only distinguishing feature of a set is the number of its elements, and isomorphic sets must have the same number of elements. In the example we have been considering, there clearly cannot be a function that is an isomorphism between $A$ and $B$, because they contain a different number of elements: $A$ contains four elements, while $B$ contains five elements. The set $A$ is, however, isomorphic to the set $C = \{6, 7, 8, 9\}$; an example of an isomorphism is the function $f$ defined by $f(1) = 6$, $f(3) = 7$, $f(5) = 8$, and $f(7) = 9$.

In the examples considered above, we have considered sets of natural numbers. From the viewpoint of the theory of sets and functions, these sets are viewed as collections of undifferentiated points, and the numbers only serve to label the points. Thus none of the ordinary arithmetic structure of the number system is considered.

The theory of sets and functions lies at the base of the pyramid of abstract theories; it represents the undifferentiated level of mathematical structure. From this undifferentiated level, the diverse abstract theories of modern mathematics arise by sequentially imposing more and more structural relationships upon the elements of a set. For example,
suppose we begin with the set of integers \( \mathbb{Z} \) considered as a set of undifferentiated points. We can then define an order relation \(<\) between the elements, giving the set an exact sequential structure. This orderly structure satisfies the two axioms for a linear ordering:

**Axiom 1** (trichotomy). For any elements \( a, b \) exactly one of the following three conditions holds: (i) \( a < b \), (ii) \( a = b \), or (iii) \( b < a \).

**Axiom 2** (transitivity). If \( a < b \) and \( b < c \), then \( a < c \).

The set of integers \( \mathbb{Z} \), together with the order relation between integers, constitute a mathematical structure. A mathematical structure is by definition a set of points for which various operations and relations are defined. One extreme case is when there are no operations and relations, in which case one just gets a set of undifferentiated point values. This value of structure (in which there is no structure) is studied by the theory of sets and functions.

If we consider the integers with the order relation, we obtain a different value of structure. We use the notation \((\mathbb{Z}, <)\) to designate this structure; the notation tells us that there are two things being considered: the set of points \( \mathbb{Z} \), and the order relation \(<\) on this set. Because the order relation satisfies the linear-order Axioms 1 and 2 above, the structure \((\mathbb{Z}, <)\) is called a linear ordering. But this is just one example of a linear ordering; there are many other examples of a set \( A \) together with a relation designated \(<\) between elements of \( A \) such that Axioms 1 and 2 are satisfied. Any such structure \((A, <)\) is called a linear ordering. The theory of linear orderings describes all such structures, as well as their relationships to one another.

Introducing an order relation is just one direction we can take in putting structure onto a set of undifferentiated points. A second direction is to introduce an algebraic relation, such as addition or multiplication. Suppose we again start with the set of integers \( \mathbb{Z} \), but this time we introduce the operation of addition rather than the order relation. This gives the set \( \mathbb{Z} \) a different type of orderly structure, a type of algebraic structure called an abelian group. This operation of addition satisfies the following four axioms for an abelian group:
Axiom 1 (commutative law). \( a + b = b + a \)

Axiom 2 (associative law). \( a + (b + c) = (a + b) + c \)

Axiom 3 (identity law). There is a special element, designated 0, such that \( 0 + a = a \).

Axiom 4 (inverse law). For every element \( a \), there exists an element \( -a \), such that \( -a + a = 0 \).

Any structure \((A, +)\), consisting of a set \( A \) and an operation \(+\) on \( A \), such that these four axioms are satisfied is called an abelian group. All these structures are collectively described by the theory of abelian groups.

If we like, we can elaborate the description of an abelian group to specify the zero element 0 and the inverse operation \(-\). An abelian group is then a structure \((A, +, 0, -)\) consisting of:

1. a set \( A \);
2. a binary operation \(+\) on \( A \) (“binary” means that it is applied to two elements, \( a \) and \( b \), to yield one well-defined element, designated \( a + b \));
3. a special element of \( A \) designated 0;
4. a unary operation \(-\) on \( A \) (“unary” means that it is applied to a single element \( a \) of \( A \) to yield a well-defined element of \( A \), designated \(-a\)),

such that the following four axioms are satisfied:

Axiom 1’. \( a + b = b + a \)

Axiom 2’. \( a + (b + c) = (a + b) + c \)

Axiom 3’. \( 0 + a = a \)

Axiom 4’. \( -a + a = 0 \).

The advantage of the second formulation is that the axioms can all be expressed by algebraic equations, without the need for saying “there
exists...” (as is required in the earlier formulation of Axioms 3 and 4). This has advantages for the study of the logical structure of the theory. Also, it makes it possible to extend the algebraic concepts of “abelian group” and “commutative ring” to the more general context of an arbitrary category, as we shall see in Chapter 2.

We have considered now two possible first steps for introducing structure onto the undifferentiated set of integers \( \mathbb{Z} \). This process can naturally be continued further. After addition, it is natural to include a second fundamental operation: multiplication. This gives rise to a structure \((\mathbb{Z}, +, \cdot)\) satisfying in addition to the Axioms 1–4 above the following axioms:

**Axiom 5** (commutative law of multiplication). \( a \cdot b = b \cdot a \)

**Axiom 6** (associative law of multiplication). \( a \cdot (b \cdot c) = (a \cdot b) \cdot c \)

**Axiom 7** (distributive law). \( a \cdot (b + c) = a \cdot b + a \cdot c \)

**Axiom 8** (identity law). There exists a special element, designated 1 (the identity element), such that \( 1 \cdot a = a \).

Notice that Axiom 7 describes the relationship between the operations of addition and multiplication.

The Axioms 1–8 are called the *commutative ring* axioms; any structure \((A, +, \cdot)\) that satisfies these axioms is called a commutative ring. These structures are collectively studied by the theory of commutative rings. In this chapter we shall examine a number of different examples of commutative rings, and we shall introduce the most fundamental concepts of the abstract theory.

If we like, we can absorb the identity element 1 into the definition of a commutative ring just as we did the zero element 0, so that a commutative ring is considered a structure of the form \((A, +, -, \cdot, 0, 1)\). If one takes this approach, the identity law 8 is replaced by the algebraic law:

**Axiom 8’.** \( 1 \cdot a = a \).
We have thus far considered two possible ways of putting structure on the set of integers: either introducing an order relation or introducing the algebraic operations of addition and multiplication. But we can take one further step and introduce both together. This gives rise to a mathematical structure \((\mathbb{Z}, +, \cdot, <)\), satisfying the commutative ring axioms as well as the order axioms; in addition, the following axioms are satisfied, which relate the order relation to the algebraic structure:

**Axiom 9.** If \(a < b\) then \(a + c < b + c\)

**Axiom 10.** If \(a < b\) and \(0 < c\) then \(a \cdot c < b \cdot c\).

Axioms 1–10 collectively characterize a type of structure called an *ordered ring*.

We have thus examined the first few steps through which mathematical structure can be sequentially introduced starting from the undifferentiated level of mathematics, the world of sets and functions. As we shall see, each type of structure is described, in the language of category theory, by a *category*; the simplest type of structure—the world of sets and functions—is described by the *category of sets*. The sequential process we have introduced in the previous pages can be continued on and on, giving rise to an enormous diversity of types of mathematical structure; these include not only a variety of types of algebraic structure but also all conceivable values of continuous, geometric structure, each identified with its own category. The only limit to the range of possibilities that can be mathematically described in this way is the imagination of the mathematician. The range of abstract theories is in principle a field of all possibilities: any type of structure that the mind can conceptualize clearly and articulate in a precise way can form the basis for an abstract theory. For our purposes, however, it is enough just to consider the first few steps to appreciate the underlying theme of sequential development.

This sequential theme has a parallel in Maharishi Vedic Science in the description of the hierarchical structure of natural law sequentially unfolding from the undifferentiated field of pure intelligence, the unified field of natural law. All the diverse expressions of natural law are described as emerging from this undifferentiated field through a
sequential process that closely resembles the process of spontaneous
symmetry breaking in quantum field theories (Hagelin, 1987); this
sequential process is itself the expression of the self-interacting dynam-
ics of the unified field. Maharishi Vedic Science thereby locates, in the
internal dynamics of the unified field, the blueprint for all expressed
values of natural law in creation.

In the context of the hierarchical organization of abstract theories,
the category of sets can be taken to be analogous to the undifferenti-
ated field of pure intelligence. The category of sets is the undifferenti-
atated level of mathematical structure described by the theory of sets and
functions at the basis of the hierarchy of abstract theories. To make this
analogy truly significant, however, it is necessary to locate the blue-
print for the expressed levels of mathematical structure, such as the
category of commutative rings, in the internal dynamics of the category
of sets. We shall see how this can be done using the category-theoretic
concept of triples in Chapter 2. A more complete development of this
theme will be provided by topos theory in Chapter 3, where we shall
see how the set-theoretic foundation can be reconstructed internally in
any topos (of which the category of sets is an example).

In the remaining sections of this chapter we shall focus our atten-
tion on the theory of commutative rings so that we may appreciate the
general structural features of an abstract theory. We shall begin by see-
ing how even a specific theory, such as the theory of rings, embraces
in principle a field of all possibilities, in this case all possible rings.
To gain some feeling for the unboundedness of this range of possibili-
ties, we shall examine a number of examples of commutative rings in
Section 1.2. For a more complete introduction to ring theory, and to
abstract algebra in general, see Hungerford (1980).

1.2 Commutative Rings
We considered in the previous section one example of a commutative
ring: the familiar ring of integers. Two other familiar examples of rings
are the rational numbers \( \mathbb{Q} \) (all possible fractions in reduced form), and
the real numbers \( \mathbb{R} \) (all possible points on a continuous number line).
Both these number systems also satisfy all the axioms for a commuta-
tive ring. There are however many additional examples of rings having
properties very different than the familiar number systems. We shall
consider several such examples in this section to gain some feeling for the range of possibilities embraced by the algebraic concept of a ring.

**Example 1: Modular Systems $\mathbb{Z}_n$.** Modular number systems describe “clock arithmetic.” We will illustrate the general concept by means of an example: the number system $\mathbb{Z}_5$.

Here the underlying set of elements is the five element set {0, 1, 2, 3, 4}. This example is thus fundamentally different from the examples considered above in that the set of elements is a finite set; there are only finitely many distinct elements in a modular number system (in this case exactly five). Addition and multiplication are defined to correspond to “clock arithmetic”: $3 + 2 = 0$, $4 + 3 = 2$, $2 \cdot 4 = 3$, and so on. These operations are described by the addition and multiplication tables given in Figure 1.2.

We can also describe these operations in the following way: To find $a + b$, add $a$ to $b$ in the “usual” way, divide by 5, and take the remainder. To find $a \cdot b$, multiply $a$ times $b$ in the usual way, divide by 5, and take the remainder.

![Addition and Multiplication Tables](image)

The number system $\mathbb{Z}_5$, with the operations of addition and multiplication as defined above, satisfies all the axioms for a commutative ring. The structure $\mathbb{Z}_5$ is thus an example of a commutative ring containing exactly five elements.

One can similarly define a commutative ring $\mathbb{Z}_2$ containing the two elements 0 and 1, a commutative ring $\mathbb{Z}_3$ containing the three elements 0, 1, and 2, and so on. These examples are called *modular number systems*. Unlike the number systems $\mathbb{Z}$ and $\mathbb{Q}$, which contain infinitely many elements, each of the modular systems $\mathbb{Z}_n$ contains only a finite
number of elements. The system $\mathbb{Z}_n$ in fact contains exactly $n$ elements: 0, 1, $\ldots$, $n - 1$.

**Example 2: Trivial Ring.** The next example of a commutative ring we shall consider is the simplest: the trivial ring, containing the single element 0. Addition and multiplication are defined by: $0 + 0 = 0 \cdot 0 = 0$. In the trivial ring, the identity element, 1, is the same as the zero element, 0: $1 = 0$.

In the remaining examples, we shall consider several ways of building up new rings from given rings. These procedures can be applied over and over again, giving rise to an infinite diversity of examples of rings.

**Example 3: Direct Product of Rings.** We consider next a way two rings can be “multiplied” to yield a third ring, called the direct product of the two rings. This operation is based upon the concept of the cartesian product of two sets; we shall first review this concept.

The concept of cartesian product is itself based upon the notion of an ordered pair: if $a$ and $b$ are any two objects, the ordered pair $(a, b)$ consists of two values, $a$ and $b$, in the specific order of $a$ first, followed by $b$. For example, we can form the ordered pair $(3, 5)$ of the two numbers 3 and 5. This ordered pair is different from the ordered pair $(5, 3)$; the number 3 occurs first in the first pair, but the number 5 occurs first in the second pair. Thus, $(3, 5) \neq (5, 3)$. The concept of ordered pair must therefore be distinguished from that of unordered pair in set theory; an unordered pair is simply a two-element set, such as the set $\{a, b\}$. For this latter concept, the order in which one writes the elements is irrelevant: $\{3, 5\} = \{5, 3\}$. Both the notation $\{3, 5\}$ and $\{5, 3\}$, designate the set consisting of the two elements 3 and 5; the order in which we happen to mention them doesn’t matter.

If $A$ and $B$ are any sets, then the cartesian product $A \times B$ is defined to be the set consisting of all ordered pairs $(a, b)$ such that $a$ is an element of $A$ and $b$ is an element of $B$. For example, if $A = \{0, 1\}$ and $B = \{0, 1, 2\}$, then

$$A \times B = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}.$$
In this example we see that the set $A$ contains two elements, the set $B$ contains three elements, and the product set $A \times B$ contains six elements. In general, the number of elements in a cartesian product $A \times B$ is the product of the number of elements of $A$ times the number of elements of $B$.

The cartesian product operation can be used to “multiply” two rings to create a new ring. We shall illustrate this construction by means of an example.

Suppose we start with the two rings $\mathbb{Z}_2$ and $\mathbb{Z}_3$. The ring $\mathbb{Z}_2$ contains the two elements 0 and 1, and $\mathbb{Z}_3$ contains the three elements 0, 1, and 2. The cartesian product of these two sets then is the set

$$S = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}.$$  

We define operations of addition and multiplication for $S$ by the rules:

$$(a, b) + (a', b') = (a + a', b + b')$$
$$(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$$

where the calculations $a + a'$, $a \cdot a'$ are performed in $\mathbb{Z}_2$, and the calculations $b + b'$, $b \cdot b'$ are performed in $\mathbb{Z}_3$. This means simply that one does the arithmetic for the first and second elements of the pairs separately; the arithmetic for the first elements is performed in $\mathbb{Z}_2$ and for the second elements is performed in $\mathbb{Z}_3$. For example:

$$(0, 1) + (1, 2) = (0 + 1, 1 + 2) = (1, 0)$$
$$(1, 2) + (1, 2) = (1 + 1, 2 + 2) = (0, 1)$$
$$(1, 2) \cdot (0, 2) = (1 \cdot 0, 2 \cdot 2) = (0, 1).$$

With these definitions of $+$ and $\cdot$, the mathematical structure $(S, +, \cdot)$ satisfies all the axioms for a commutative ring. The zero element is $(0, 0)$, and the identity is $(1, 1)$. The ring $(S, +, \cdot)$ is called the direct product of the rings $\mathbb{Z}_2$ and $\mathbb{Z}_3$ and is designated by $\mathbb{Z}_2 \times \mathbb{Z}_3$.

More generally, if $R$ and $S$ are any two commutative rings, the same construction can be applied to create the direct product $R \times S$, which will also be a commutative ring.
Example 4: Gaussian Integers. In this example, we begin with the ring of integers \( \mathbb{Z} \) and adjoin a new “imaginary” element \( i \) such that \( i^2 = -1 \). The Gaussian integers \( \mathbb{Z}(i) \) consist of all combinations of the form \( a + bi \) where \( a \) and \( b \) are in \( \mathbb{Z} \), for example, \( 2 + i, 3 - 4i, -2 + 3i, 6, -5i \), and so on. The Gaussian integers are added and multiplied using the usual rules of algebra in conjunction with the relation \( i^2 = -1 \): \( (2 + i) + (3 - 2i) = 5 - i \), \( (2 + i) \cdot (3 - 2i) = 6 + 3i - 4i - 2i^2 = 6 - i + 2 = 8 - i \), and so on. With these definitions of + and ·, the Gaussian integers \( \mathbb{Z}(i) \) satisfy all the axioms for a commutative ring.

Example 5: Polynomial Rings. For any ring \( R \) one can form a new ring \( R[X] \) consisting of all polynomials in the indeterminate \( X \) having coefficients in \( R \). We shall illustrate this construction by means of an example.

Suppose we start with the ring \( R = \mathbb{Z}_5 \). A polynomial in \( X \) with coefficients in \( \mathbb{Z}_5 \) is by definition any expression of the form: \( c_0 + c_1X + c_2X^2 + c_3X^3 + \cdots + c_nX^n \), where the coefficients \( c_0, c_1, c_2, \ldots, c_n \) are all required to be elements of \( \mathbb{Z}_5 \). If \( P \) is such a polynomial, and \( n \) is the largest integer for which the coefficient \( c_n \) is non-zero, then the polynomial \( P \) is said to have degree \( n \). Following are some examples of polynomials in \( \mathbb{Z}_5[X] \) of different degrees:

- degree 0: 1, 2, 3, 4 (any integer constant)
- degree 1: \( X, 3X + 2, 4X, X + 1 \)
- degree 2: \( X^2, 2X^2 + 4, 4X^2 + 2X + 2 \)
- degree 3: \( X^3 + X + 1, 2X^3 + 3X^2 + 4 \)
- degree 13: \( 3X^{13} + X^{11} + 4X^7 + X^2 + 3 \)

Polynomials of degree zero are also called constant polynomials, those of degree one linear polynomials, and those of degree two quadratic polynomials. These polynomials are added and multiplied according to the familiar rules of “high-school” algebra; the only difference is that all arithmetic calculations for the coefficients must be performed in \( \mathbb{Z}_5 \). For example,

\[
(4X^2 + 2X + 2) + (2X^3 + 3X^2 + 4) = 2X^3 + (4 + 3)X^2 + 2X + (2 + 4) = 2X^3 + 2X^2 + 2X + 1
\]
The reader is warned that the arithmetic for the exponents is performed in \( \mathbb{Z} \) and not in \( \mathbb{Z}_5 \). For example, \( X^3 \cdot X^4 = X^7 \) and not \( X^2 \); this is because \( X^7 \) represents \( X \cdot X \cdot X \cdot X \cdot X \cdot X \cdot X \) which is different from \( X \cdot X = X^2 \).

With addition and multiplication defined in this way, the mathematical structure \( \mathbb{Z}_5[X] \) satisfies all the axioms for a commutative ring. The zero element of this ring is the constant polynomial 0, and the identity element is the constant polynomial 1.

More generally, one can start with any commutative ring \( R \) and construct the polynomial ring \( R[X] \) consisting of all polynomials having coefficients in the ring \( R \).

One can also consider polynomial rings having more than one indeterminate. To illustrate, the following two polynomials

\[
X_1^2 + 3X_1X_2 + 2X_2 + 5
\]

\[
X_1^2X_2 + X_2^3X_1^2 + 2X_1 + 7, X_1^2 + 7X_1 + 3
\]

belong to the polynomial ring \( \mathbb{Z}[X_1, X_2] \).

One can consider polynomial rings containing even an infinite number of indeterminates, for example, \( \mathbb{Z}[X_1, X_2, X_3, \ldots] \), for which there is an infinite sequence of indeterminates \( X_1, X_2, X_3, \ldots \). Each element of such a polynomial ring—that is, each particular polynomial—will still, however, contain only a finite number of indeterminates, since a polynomial is required to be a finite symbolic expression. To illustrate, the following polynomials belong to \( \mathbb{Z}[X_1, X_2, X_3, \ldots] \):

\[
2X_1 + 7
\]

\[
4X_1^2X_2 + X_2X_3 + X_1
\]

\[
X_1^3 + X_1^2X_3 + 2X_1 - 6
\]

Example 6: Rings of Functions. We shall consider next a way to construct rings whose elements are functions. We shall illustrate the general construction by means of an example. Let \( S \) be a set consisting of three points: \( S = \{P_1, P_2, P_3\} \), and let \( R = \mathbb{Z}_2 \). We consider the set \( B \) consist-
ing of all possible functions from \( S \) to \( \mathbb{Z}_2 \). \( B \) contains a total of eight functions: \( f_1, f_2, \ldots, f_8 \), defined by the table of values in Figure 1.3: 
\[
\begin{array}{cccccccc}
\hline
 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\
\hline
P_1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
P_2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
P_3 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\hline
\end{array}
\]

\textbf{Figure 1.3} Functions from \( \{P_1, P_2, P_3\} \) to \( \{0, 1\} \).

by simply adding or multiplying their values at each point \( P_1, P_2, P_3 \). For example, \( f_4 + f_7 \) is the function \( g \) defined by:

\[
\begin{align*}
g(P_1) &= f_4(P_1) + f_7(P_1) = 0 + 1 = 1, \\
g(P_2) &= f_4(P_2) + f_7(P_2) = 1 + 1 = 0, \\
g(P_3) &= f_4(P_3) + f_7(P_3) = 1 + 0 = 1.
\end{align*}
\]

This means that \( g = f_6 \), so \( f_4 + f_7 = f_6 \). Similarly \( f_4 \cdot f_7 = f_3 \). These definitions of addition and multiplication give this set of eight functions the structure of a commutative ring. The zero element of the ring is \( f_1 \), and the identity element is \( f_8 \).

This construction can be generalized to any set and any ring: if \( S \) is any set and \( R \) is any ring, then the set of all possible functions from \( S \) to \( R \) has the structure of a ring; addition and multiplication are performed “pointwise” as in the example above. We shall use the notation \( R^S \) to designate this ring of functions.

If \( S \) is a finite set containing \( s \) elements and \( R \) is a finite ring containing \( r \) elements, then the function ring \( R^S \) will contain \( r^s \) elements. In the example above, \( S \) contains three elements, \( R \) contains two elements, and the set of functions \( 2^3 = 8 \) elements.

The constructions described in the above examples can be applied over and over again, yielding an unlimited diversity of examples of rings. For example, we can construct the ring \( (\mathbb{Z}_3 \times \mathbb{Z}_5)[X] \) consisting of all polynomials having coefficients in \( \mathbb{Z}_3 \times \mathbb{Z}_5 \), and then consider the ring of all functions from some set \( S \) to this ring, and so on. The few examples we have given are thus enough to suggest the unbounded
range of possibilities described by a single abstract concept: the concept of a commutative ring. This concept unites an infinity of different examples of mathematical structures: \( \mathbb{Z}, \mathbb{Z}_n, \mathbb{Z}[X], \mathbb{Z}(i), \mathbb{Z}_2 \times \mathbb{Z}_3 \), and so on. All these structures can be described by a common set of algebraic axioms, the commutative ring axioms. In the following section we shall see how, from these axioms, one can sequentially unfold knowledge of universal mathematical principles governing this unbounded range of possibilities. This knowledge will constitute the theory of commutative rings.

1.3 Axiomatic Ring Theory

The last section provided a glimpse of the range of possibilities described by the concept of a commutative ring. We shall consider now how one can develop a mathematical theory that can describe, in a single stroke, this unbounded range of possibilities. This is the theory of commutative rings, which systematically unfolds knowledge of the universal mathematical principles governing all possible commutative rings. This theme of systematic unfoldment of universal knowledge rests on two pillars: the infinitely flexible symbolic language of algebra, and the non-variable criteria of right knowledge provided by logic.

The variables \( x, y, z, \ldots \) used in algebra are infinitely flexible symbolic names. A variable, such as \( x \), can not only speak for all possible integers or rational numbers but it can speak for all possible elements of any mathematical structure. We saw in the last section how an algebraic formula such as \( x + y = y + x \) could be interpreted in the context of any commutative ring: \( \mathbb{Z}, \mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{Z}_2, \mathbb{Z}[X] \), and so on. Algebraic symbolism thus is capable of speaking simultaneously for all possible commutative rings.

In the development of modern mathematics, algebraic symbolism is employed in conjunction with the systematic procedure of gaining knowledge provided by logic. Logic provides the criteria for drawing conclusions in an absolutely reliable way. A logical inference systematically takes one from premises to a conclusion, through a sequence of exact steps that leave no room for error. The logical correctness of an inference means that if the premises are true then the conclusion must be true.
The first systematic development of a mathematical theory based upon logical inference is provided by Euclid’s *Elements* (Euclid, 1956; Moise, 1990). From five geometrical postulates or axioms, expressing the most fundamental properties of points, lines and circles, Euclid systematically derives over three hundred theorems of geometry. The theorems are established sequentially, starting from the axioms: each theorem is logically proved from the axioms together with the previously derived theorems. This means that if the axioms are true, then every theorem, derived sequentially in this way through logical inference, must also be true.

Euclid’s *Elements* presents the theme of sequential unfoldment of knowledge. Starting from a seed expression of knowledge—the axioms—more and more elaborated expressions of knowledge—the theorems—are seen to unfold sequentially. The mechanics of this process of sequential unfoldment of knowledge is provided by the principles of logical inference.

This theme of sequential unfoldment of knowledge has a parallel in Vedic Science in Maharishi’s *Apaurusheya Bhashya* (uncreated commentary) of Rik Veda. According to the *Apaurusheya Bhashya*, the *Richas*, or verses, of the Rik Veda provide a sequential elaboration and commentary on the expression of total knowledge contained in the first syllable of Rik Veda, AK. The syllable AK is the direct expression of the self-interacting dynamics of the unified field, the Samhita; it is the expression of total knowledge in its most compact form. The rest of the Rik Veda, and in fact all of the Vedic literature, then provides a sequence of “packages of knowledge” each elaborating and commenting on the previous package. The mechanics of transformation that sequentially unfold the different expressions of knowledge are themselves the expression of the self-interacting dynamics of the Samhita.

The axiomatic structure of Euclid’s *Elements* parallels this theme of sequential elaboration of knowledge. The axioms contain the total knowledge of the theory in its most compact expression. The theorems sequentially elaborate this knowledge, each emerging from the previously derived theorems together with the axioms. The mechanics of transformation in this case are provided by the principles of logical inference, the non-variable “laws of thought.” These principles are a direct expression of the internal dynamics of intelligence, as are the
mechanics of transformation that sequentially unfold the expressions of knowledge of the Veda. This is perhaps one reason why mathematical knowledge enjoys, to such a great degree, the qualities of stability and non-variability associated with Vedic knowledge.

If one brings together the theme of sequential unfoldment of knowledge found in Euclid’s *Elements* with the infinitely flexible symbolic language of algebra, one is led to the field of abstract algebra. We shall illustrate the approach of abstract algebra by means of an example of an abstract algebraic theory: the theory of commutative rings.

In abstract algebra one begins with axioms that are expressed in the symbolic language of algebra, by algebraic formulas. For the theory of commutative rings, these axioms are simply the commutative ring axioms introduced in Section 1.1. From these axioms, one sequentially derives the theorems of the theory by means of the principles of logical inference. Each theorem has a proof, presenting the steps of logical inference through which the theorem is validated on the basis of the axioms and the previously established theorems. To give a feeling for how this works, we present the proofs of two theorems from the theory of commutative rings.

**Theorem 1 (Cancellation Law of Addition).** If \( a + b = a + c \), then \( b = c \).

**Proof.** Suppose \( a + b = a + c \). Add \(-a\) to both sides of the equation: \(-a + (a + b) = -a + (a + c)\). Expanding the left side: \(-a + (a + b) = (-a + a) + b = 0 + b = b\), where we have used in succession Axioms 2, 4, and 3. Following the same steps, the right side evaluates to \( c \); that is, \(-a + (a + c) = c\). Hence \( b = c \). This completes the proof of Theorem 1.

**Theorem 2.** \( a \cdot 0 = 0 \).

**Proof.** \( 0 + 0 = 0 \) (by Axiom 3). Multiply both sides of the equation by \( a \): \( a \cdot (0 + 0) = a \cdot 0 \). Evaluating the left side: \( a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \) (by Axiom 7). And the right side becomes: \( a \cdot 0 = 0 + a \cdot 0 = a \cdot 0 + 0 \) (by Axioms 3 and 1). Thus, \( a \cdot 0 + a \cdot 0 = a \cdot 0 + 0 \). Hence \( a \cdot 0 = 0 \) (by Theorem 1). This completes the proof of Theorem 2.
The above two proofs make implicit use of the laws of equality:

**Reflexive law.** $a = a.$

**Symmetric law.** If $a = b,$ then $b = a.$

**Transitive law.** If $a = b$ and $b = c,$ then $a = c.$

They also use the **substitution rule:**

**Substitution rule.** If $a = b,$ then we can substitute $b$ for any occurrence of $a$ in any symbolic formula.

These four fundamental principles governing equations have their basis in the meaning of the equals sign itself. An equation $r = s$ expresses the fact that the symbolic expressions on both sides of the equals sign name the same mathematical object. For example, the equation $1 + 1 = 2$ expresses the fact that the symbolic expressions “$1 + 1$” and “$2$” designate the same object, namely the number two. The three laws of equality as well as the substitution rule are direct consequences of the meaning of the equals sign, and are therefore valid for all mathematical theories.

Following is an elaborated version of the proof of Theorem 1, making explicit all the applications of the laws of equality and the substitution rule.

**Theorem 1.** If $a + b = a + c,$ then $b = c.$

**Proof:**

<table>
<thead>
<tr>
<th>Statements</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Suppose $a + b = a + c$</td>
<td>Assumption</td>
</tr>
<tr>
<td>2. $-a + (a + b) = -a + (a + b)$</td>
<td>Reflexive law</td>
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<tr>
<td>3. $-a + (a + b) = -a + (a + c)$</td>
<td>Substitution rule: substituting $a + c$ for $a + b$ in the right hand side of 2.</td>
</tr>
<tr>
<td>4. $-a + (a + b) = (-a + a) + b$</td>
<td>Axiom 2</td>
</tr>
<tr>
<td>5. $-a + a = 0$</td>
<td>Axiom 4</td>
</tr>
</tbody>
</table>
6. \(-a + (a + b) = 0 + b\)  
   Substitution rule: substituting 0 for \(-a + a\) in 4

7. \(0 + b = b\)  
   Axiom 3

8. \(-a + (a + b) = b\)  
   Transitivity rule (from 6, 7)

9. \(-a + (a + c) = c\)  
   Same argument as \(-a + (a + b) = b\)

10. \(b = c\)  
    Substitution rule: substituting \(b\) for \(-a + (a + b)\) and \(c\) for \(-a + (a + c)\) in 3

Each version of the above proofs has the purpose of demonstrating how the theorems follow logically from the axioms: if the axioms are true, then the theorems must also be true. This means that if \((\mathbb{R}, +, \cdot)\) is any mathematical structure that satisfies the commutative ring axioms, it must also satisfy Theorems 1 and 2. It is not necessary to verify the theorems separately for each new commutative ring one encounters. One proves the theorems once and then knows they must be true for all possible commutative rings, even commutative rings that may not yet have been discovered.

In this we find the significance of the “abstract” approach of abstract algebra. We intentionally leave the algebraic symbolism uninterpreted: we do not tie the symbols down to any fixed interpretation. For example, we do not restrict \(x, y, z,\) and so on, to designate only the familiar rational numbers. By leaving the symbols uninterpreted, abstract algebra functions on the level of all possibilities: it unfolds knowledge that is simultaneously valid for all possible commutative rings. By operating on the abstract level of all possibilities, the knowledge gained is most general and powerful.

Knowledge gained on the abstract level furthermore requires least effort. It is a striking fact that it is often much simpler to derive a general principle abstractly than it is to concretely verify the principle for a specific mathematical structure (for example, a specific commutative ring). When the mathematician operates on the abstract level of all possibilities, his awareness is freed from the rigid boundaries of any specific mathematical structure; on this level of pure knowledge, the abstract values of organizing power embodied in the axioms of the theory can be most effortlessly unfolded. Mathematical proofs structured at the
abstract level are invariably most elegant and transparent, in addition to being most general and powerful; mathematical activity at the abstract level is consequently most fulfilling.

Modern mathematics has thus discovered the secret of “doing less and accomplishing more” by acting from the most abstract, unified, and comprehensive level, the level of “all possibilities.” Action from this level is most powerful, effortless, illuminating, and fulfilling. This theme of skill in action finds its most perfect expression in the structure of the state of enlightenment, as described by Maharishi Vedic Science. In this most developed state of human consciousness, the awareness is firmly established in the experience of pure intelligence. This experience is, at the same time, completely abstract yet absolutely concrete. It is the direct experience of the ultimate, unified field of natural law at the unmanifest source of creation; this is the level of natural law in which the total organizing power of nature is lively. When awareness is grounded in this experience of pure intelligence, then all the laws of nature are spontaneously enlisted to bring fulfillment to every desire. In the state of enlightenment, one “does nothing and accomplishes everything”: The spontaneity of action in this state gives all activity an effortless, joyful quality; at the same time, the level of achievement is maximized, because one is functioning in harmony with the totality of natural law.

1.4 Ring Homomorphisms
In the last section we introduced one aspect of the theory of rings: the way in which one sequentially unfolds knowledge from the axioms. The body of knowledge one obtains in this way is called the elementary theory of rings. The theorems of the elementary theory of rings are statements expressed in terms of the algebraic operations of addition and multiplication; because the theorems are logically derived from the axioms, they are simultaneously true for all possible rings.

The general theory of rings in modern mathematics has a much richer structure than just the elementary theory discussed so far. The general theory not only describes the common elementary properties of rings, but further describes the complex interrelationships between the numerous diverse examples of rings, thereby bringing to light the individuality of these different examples. In ring theory, the main tool used to study these relationships between rings is the concept of a coherent
type of transformation from one ring to another called a ring homomorphism.

A ring homomorphism is a certain kind of function: it is a function that respects the integrity of the algebraic structure of the rings involved. It is a special case of the more general concept of a morphism, or structure-preserving transformation, of an abstract theory. Each abstract theory has its own morphisms; in the case of rings, these are called ring homomorphisms. Category theory provides an abstract language for the study of morphisms in general. Our study of ring homomorphisms in this and the following sections will provide motivation for the more abstract category-theoretic viewpoint introduced in Chapter 2. We shall begin with the definition of a ring homomorphism, and shall then consider a number of examples. The precise definition is the following.

**Definition.** A ring homomorphism is a function \( f \) from a commutative ring \( A \) to a commutative ring \( B \) that has the following three properties:

1. \( f(a + b) = f(a) + f(b) \).
2. \( f(a \cdot b) = f(a) \cdot f(b) \).
3. \( f(1) = 1' \), where 1 is the identity of \( A \), and 1’ is the identity of \( B \).

Note that \( a + b \) and \( a \cdot b \) are calculated in \( A \), while \( f(a) + f(b) \) and \( f(a) \cdot f(b) \) are calculated in \( B \).

Condition (1) says that if we first add \( a + b \) in \( A \), and then apply \( f \), the result is the same as first applying \( f \) to \( a \) and \( b \) separately, and adding the results in \( B \). Equivalently, if \( a + b = c \), then \( f(a) + f(b) = f(c) \). Condition (2) expresses the analogous property for multiplication. Condition (3) says that \( f \) takes the identity element of \( A \) to the identity element of \( B \).

These three conditions just express the fact that the transformation \( f \) and the operations of addition and multiplication all fit together in a coherent way; it is the expression of coherence of a mathematical transformation.

Two additional properties that every homomorphism satisfies are the following:
(4) \( f(0) = 0' \), where 0 is the zero element of \( A \) and 0' the zero element of \( B \), and

(5) \( f(-a) = -f(a) \).

The properties (4) and (5) are a consequence of the properties (1)–(3). We can see this as follows. Since \( a + 0 = a \) for any element \( a \) in \( A \), it follows that \( f(a) = f(a + 0) = f(a) + f(0) \) by property (1). But \( f(a) = f(a) + 0' \), since \( b + 0' = b \) for every element \( b \) in \( B \). Hence \( f(a) + f(0) = f(a) + 0' \), and it follows that \( f(0) = 0' \) from the cancellation law (Theorem 1). This establishes (4).

To prove (5), we begin with the relation \( a + (-a) = 0 \). It follows that \( f(a + (-a)) = f(0) = 0' \), by property (4), which we just derived. By property (1), on the other hand, we have \( f(a + (-a)) = f(a) + f(-a) \), hence \( f(a) + f(-a) = 0' \). Next, \( f(a) + f(-a) = 0' \), since \( b + (-b) = 0' \) for every element \( b \) in \( B \). It follows that \( f(a) + f(-a) = f(a) + (-f(a)) \), and again, by the cancellation law, that \( f(-a) = -f(a) \). This establishes (5).

In the remainder of this section, we shall consider a number of examples of homomorphisms to illustrate this fundamental concept.

**Example 1: Identity Homomorphisms.** For any set \( S \), the identity function \( 1_S \) is defined by \( 1_S(a) = a \) for every element \( a \) in \( S \). The identity function \( 1_S \) is the transformation from \( S \) to itself that does nothing; it leaves every element of \( S \) unchanged.

For any commutative ring \( A \), the identity function \( 1_A \) from \( A \) to itself is always a homomorphism, called the identity homomorphism. Identity homomorphisms express the unmanifest value of transformation in the theory of commutative rings.

**Example 2: Subrings.** Consider the function \( f \) from the integers \( \mathbb{Z} \) to the rationals \( \mathbb{Q} \) that takes each integer to itself, regarded as an element of \( \mathbb{Q} \): \( f(0) = 0, f(1) = 1, f(2) = 2 \), and so on. Because we add and multiply integers the same way whether we consider them as elements of \( \mathbb{Z} \) or elements of \( \mathbb{Q} \), it follows that the function \( f \) satisfies conditions (1) and (2) for a ring homomorphism. Further, the integer 1 has the property that \( 1 \cdot a = a \) for every rational number \( a \), so the identity element 1 of \( \mathbb{Z} \) is at the same time the identity element of \( \mathbb{Q} \), showing that condition (3) is also satisfied. The function \( f \) is thus a homomorphism.
from $Z$ to $Q$. This homomorphism $f$ shows how the ring $Z$ is included as a subring of the ring $Q$.

A similar example is provided by the homomorphism $g$ from $Z$ to $Z[X]$ that takes each integer to itself regarded as an element of $Z[X]$, that is, a constant polynomial. This shows that $Z$ is included as a subring of $Z[X]$.

More generally, for any ring $R$, we can consider the homomorphism from $R$ to $R[X]$ that takes each element of $R$ to itself, regarded as a constant polynomial.

**Example 3: Zero Homomorphisms.** Let $R$ be any ring, and let $f$ be the unique function from $R$ to the trivial ring: $f(a) = 0$ for every element $a$ in $R$. The function $f$ is easily seen to satisfy the conditions for a ring homomorphism. For example, $f(1) = 0$, which is the identity element of the trivial ring, so condition (3) is satisfied.

**Example 4: Diagonal Homomorphisms.** Consider the function $f$ from $Z_2$ to $Z_2 \times Z_2$ defined by $f(0) = (0, 0), f(1) = (1, 1)$. It is easily checked that $f$ is a homomorphism. For example, $f(1 + 1) = f(0) = (0, 0), f(1) + f(1) = (1, 1) + (1, 1) = (0, 0)$, so $f(1 + 1) = f(1) + f(1)$, and so on. The homomorphism $f$ is called the diagonal homomorphism from $Z_2$ to $Z_2 \times Z_2$.

More generally, for any commutative ring $R$, one has a diagonal homomorphism $f$ from $R$ to $R \times R$ defined by $f(a) = (a, a)$ for every element $a$ of $R$.

**Example 5: Isomorphisms.** Let $f$ be the function from $Z_6$ to $Z_2 \times Z_3$ defined as follows:

- $f(0) = (0, 0) = 0'$ (the zero element of $Z_2 \times Z_3$)
- $f(1) = (1, 1) = 1'$ (the identity element of $Z_2 \times Z_3$)
- $f(2) = 1' + 1' = (0, 2)$
- $f(3) = 1' + 1' + 1' = (1, 0)$
- $f(4) = 1' + 1' + 1' + 1' = (0, 1)$
- $f(5) = 1' + 1' + 1' + 1' + 1' = (1, 2)$

The function $f$ can be easily seen to satisfy the defining conditions for a ring homomorphism. At the same time, the function $f$ is a one-to-one
correspondence between $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$. This means that the operations of addition and multiplication in $\mathbb{Z}_6$ mirror exactly the operations of addition and multiplication in $\mathbb{Z}_2 \times \mathbb{Z}_3$. The two rings, $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$, consequently have identical algebraic properties—they are structurally indistinguishable as far as the theory of commutative rings is concerned. The two rings are said to be isomorphic, and the homomorphism $f$ is called an isomorphism.

In general, a homomorphism $f$ from a commutative ring $A$ to a commutative ring $B$ that is a one-to-one correspondence is called an isomorphism, and the commutative rings $A$ and $B$ are said to be isomorphic. Isomorphic rings always have identical structural properties.

Example 6: Projections of a Direct Product. Consider the function $\pi_1$ from $\mathbb{Z}_2 \times \mathbb{Z}_3$ to $\mathbb{Z}_2$ defined by $\pi_1((a, b)) = a$, that is:

$$
\begin{align*}
\pi_1((0, 0)) &= \pi_1((0, 1)) = \pi_1((0, 2)) = 0, \\
\pi_1((1, 0)) &= \pi_1((1, 1)) = \pi_1((1, 2)) = 1.
\end{align*}
$$

It is easily verified that $\pi_1$ is a homomorphism from $\mathbb{Z}_2 \times \mathbb{Z}_3$ to $\mathbb{Z}_2$. The homomorphism $\pi_1$ is called the projection of $\mathbb{Z}_2 \times \mathbb{Z}_3$ onto $\mathbb{Z}_2$.

One can similarly define the projection $\pi_2$ of $\mathbb{Z}_2 \times \mathbb{Z}_3$ onto $\mathbb{Z}_3$ by $\pi_2((a, b)) = b$:

$$
\begin{align*}
\pi_2((0, 0)) &= \pi_2((1, 0)) = 0, \\
\pi_2((0, 1)) &= \pi_2((1, 1)) = 1, \\
\pi_2((0, 2)) &= \pi_2((1, 2)) = 2.
\end{align*}
$$

More generally, for any commutative rings $A$ and $B$, one can define a projection $\pi_1$ from $A \times B$ to $A$ and a projection $\pi_2$ from $A \times B$ to $B$ by the formulas: $\pi_1((a, b)) = a$ and $\pi_2((a, b)) = b$ for all $(a, b)$ in $A \times B$; once again, $\pi_1$ and $\pi_2$ are ring homomorphisms.

Example 7: Projection of $\mathbb{Z}$ onto $\mathbb{Z}_2$. Consider the function $f$ from $\mathbb{Z}$ to $\mathbb{Z}_2$ defined as follows: $f(a) = 0$ whenever $a$ is even, and $f(a) = 1$ whenever $a$ is odd. From the properties: even + even = odd + odd = even, even + odd = odd, even $\cdot$ even = even $\cdot$ odd = even, odd $\cdot$ odd = odd, we can see that $f$ is a ring homomorphism.
Similarly, consider the function $f$ from $\mathbb{Z}$ to $\mathbb{Z}_3$ defined as follows:

\begin{align*}
f(a) &= 0 \text{ whenever } a \text{ is divisible by } 3; \\
f(a) &= 1 \text{ whenever } a \text{ leaves remainder } 1 \text{ when divided by } 3; \\
f(a) &= 2 \text{ whenever } a \text{ leaves remainder } 2 \text{ when divided by } 3.
\end{align*}

This means

\[
f(a) = \begin{cases} 
0, & \text{if } a \in \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}; \\
1, & \text{if } a \in \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}; \\
2, & \text{if } a \in \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}.
\end{cases}
\]

The function $f$, as defined above, is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_3$.

More generally, for any positive integer $n$ one can define a homomorphism $f$ from $\mathbb{Z}$ to $\mathbb{Z}_n$ by the rule: For any integer $a$, $f(a)$ is the remainder obtained when $a$ is divided by $n$ (the remainder is always one of the numbers $0, 1, 2, \ldots, n-1$).

**Example 8: Evaluation of Polynomials.** Let $f$ be the function from $\mathbb{Z}[X]$ to $\mathbb{Z}$ that takes each polynomial in $\mathbb{Z}[X]$ to its constant term; for example, $f(X^2 + 2X + 3) = 3, f(X^3 - X^2 - 7) = -7, f(2X^2 + X) = 0$, and so on. Then $f$ is easily verified to be a homomorphism from $\mathbb{Z}[X]$ to $\mathbb{Z}$. This follows from the facts that: (1) when one adds two polynomials the constant terms simply add, and (2) when one multiplies two polynomials the constant terms simply multiply.

We can also describe the above homomorphism in the following way: If $P$ is any polynomial in $\mathbb{Z}[X], f(P)$ is obtained by substituting $0$ for $X$ in $P$, for example, $f(X^2 + 2X + 3) = 0^2 + 2 \cdot 0 + 3 = 3$, and so on.

More generally, if $n$ is any integer, we can define a homomorphism $f$ from $\mathbb{Z}[X]$ to $\mathbb{Z}$ by substituting the value $n$ for $X$ in every polynomial in $\mathbb{Z}[X]$. For example, taking $n = 3$, we obtain a homomorphism $f$ such that $f(2X + 7) = 2 \cdot 3 + 7 = 13, f(X^2 - 3X + 4) = 3^2 - 3 \cdot 3 + 4 = 4, f(X) = 3, f(X^3 + 1) = 3^3 + 1 = 28$, and so on.

We can see why the requirements (1–3) for a homomorphism are satisfied in the following way. The rules for adding and multiplying polynomials in $\mathbb{Z}[X]$ are directly derived from the ring axioms, treating $X$ just as if it were the name for any element of the ring $\mathbb{Z}$. As a
consequence, if one substitutes for \( X \) the name of a particular element of \( \mathbb{Z} \), the algebraic relationships between polynomials give rise to valid relationships between the elements of \( \mathbb{Z} \).

For example, if \( P = 2X - 1 \) and \( Q = X^2 + 3X \), then \( P + Q = (2X - 1) + (X^2 + 3X) = X^2 + 5X - 1 \), where the sum was calculated using the ring axioms and treating \( X \) like any ring element. Because the axioms are valid for all actual elements of the ring \( \mathbb{Z} \), we can substitute 3 for \( X \) to obtain the valid formula:

\[
(1.1) \quad (2 \cdot 3 - 1) + (3^2 + 3 \cdot 3) = 3^2 + 5 \cdot 3 - 1.
\]

Now the way we defined the function \( f \) above was simply to substitute 3 for \( X \) in every polynomial, so:

\[
\begin{align*}
f(P) &= 2 \cdot 3 - 1, \\
f(Q) &= 3^2 + 3 \cdot 3, \\
f(P+Q) &= 3^2 + 5 \cdot 3 - 1.
\end{align*}
\]

The homomorphism requirement (1) says \( f(P + Q) = f(P) + f(Q) \); if we substitute the definitions for the three terms, we just obtain the formula (1.1) derived above.

This situation generalizes to arbitrary coefficient rings \( R \). In the polynomial ring \( R[X] \) one can define a homomorphism by substituting for \( X \) any element of \( R \). For example, consider the polynomial ring \( \mathbb{Z}_3[X] \). We can define a homomorphism \( f \) from \( \mathbb{Z}_3[X] \) to \( \mathbb{Z}_3 \) by substituting the value 2 for \( X \) in every polynomial: \( f(X^2 + 2X + 1) = 2^2 + 2 \cdot 2 + 1 = 1 + 1 + 1 = 0, f(X^3 + 2) = 2^3 + 2 + 2 = 1 \), and so on.

**Example 9: Rings of Functions.** Consider the ring of functions \( R^S \) consisting of all possible functions from a set \( S \) to a commutative ring \( R \). Each point \( P \) in \( S \) gives rise to a homomorphism from \( R^S \) to \( R \), which simply evaluates every function at that point. For example, in Example 6 of Section 1.2 the point \( P_3 \) gives rise to the following homomorphism \( g \):

\[
\begin{align*}
g(f_1) &= f_1(P_3) = 0, \\
g(f_2) &= f_2(P_3) = 1,
\end{align*}
\]
The defining properties (1)–(3) in this case are a direct consequence of the “pointwise” definition of addition and multiplication in the ring $R^s$.

**Example 10: Universal property of $\mathbb{Z}$ and $\mathbb{Z}[X]$.** The rings $\mathbb{Z}$ and $\mathbb{Z}[X]$ have a very special place in the family of rings, as we shall now describe.

The special property of $\mathbb{Z}$ is the following: For any ring $R$, there always exists a unique homomorphism from $\mathbb{Z}$ to $R$. This means:

1. for any ring $R$ there will always be some homomorphism from $\mathbb{Z}$ to $R$, and
2. for any ring $R$ there will never be more than one homomorphism from $\mathbb{Z}$ to $R$.

Condition (1) is the *existence* condition, and (2) is the *uniqueness* condition. Let us see why each of these conditions is satisfied.

We first consider existence. Suppose that $R$ is any ring; we wish to find a homomorphism $f$ from $\mathbb{Z}$ to $R$. We know that $f$ must have the properties $f(0) = 0'$, and $f(1) = 1'$, where $0'$ and $1'$ designate the zero element and identity element of $R$. Now every non-zero integer $n$ can be expressed uniquely in one of the two forms: $n = 1 + 1 + \cdots + 1$, or $n = -(1 + 1 + \cdots + 1)$. For example, $5 = 1 + 1 + 1 + 1 + 1$, $-3 = -(1 + 1 + 1)$, and so on. We define the function $f$ as follows:

- $f(0) = 0$,
- If $n$ is positive and $n = 1 + 1 + \cdots + 1$, then $f(n) = 1' + 1' + \cdots + 1'$ ($n$ terms),
- If $n$ is negative and $n = -(1 + 1 + \cdots + 1)$, then $f(n) = -(1' + 1' + \cdots + 1')$ ($n$ terms).

For example, $f(3) = f(1 + 1 + 1) = 1' + 1' + 1'$, $f(-2) = -(1' + 1')$, and so on. It is not difficult to verify that the function $f$ defined in this way satisfies all the requirements for a ring homomorphism. For example, $f(3 + (-2)) = f(1) = 1'$, and $f(3) + f(-2) = (1' + 1' + 1') +(-(1' + 1')) = 1'$—so that $f(3 + (-2)) = f(3) + f(-2)$. Note that the last equality is a
consequence of the ring axioms applied to the ring $R$. (It is in fact a consequence of the four abelian group axioms governing addition.) This establishes the existence part of the assertion.

We now consider uniqueness. To establish uniqueness we must show that there can be at most one homomorphism from $\mathbb{Z}$ to $R$. We can see this as follows.

Suppose $f$ and $g$ are homomorphisms from $\mathbb{Z}$ to $R$. We will show that $f = g$. To begin, we know that

\[
\begin{align*}
f(0) &= 0' = g(0) \quad \text{(condition 4)}, \\
f(1) &= 1' = g(1) \quad \text{(condition 3), and} \\
f(-a) &= -f(a) \text{ and } g(-a) = -g(a) \quad \text{(condition 5)}.
\end{align*}
\]

It follows from condition (1) then that

\[
\begin{align*}
f(2) &= f(1 + 1) \\
&= f(1) + f(1) \\
&= 1' + 1' \\
&= g(1) + g(1) \\
&= g(1 + 1) \\
&= g(2),
\end{align*}
\]

\[
\begin{align*}
f(3) &= f(1 + 1 + 1) \\
&= f(1) + f(1) + f(1) \\
&= 1' + 1' + 1' \\
&= g(1) + g(1) + g(1) \\
&= g(1 + 1 + 1) \\
&= g(3),
\end{align*}
\]

\[
\begin{align*}
f(-1) &= -f(1) = -1' = -g(1) = g(-1),
\end{align*}
\]

\[
\begin{align*}
f(-2) &= f(-1 + (-1)) \\
&= f(-1) + f(-1) \\
&= -1' + (-1') \\
&= g(-1) + g(-1) \\
&= g(-1 + (-1)) \\
&= g(-2),
\end{align*}
\]
and so on. In this way one establishes that $f = g$, and that, therefore, there is at most one homomorphism from $\mathbb{Z}$ to $R$.

We have already seen examples of homomorphisms from $\mathbb{Z}$ to various rings $R$. When $R$ is the modular system $\mathbb{Z}_n$, for instance, the unique homomorphism is the one described in Example 7. When $R$ is the ring $\mathbb{Z}[X]$ of polynomials with integer coefficients, the unique homomorphism $f$ is defined by assigning to each $a$ in $R$ the constant polynomial $a$.

We next consider a special property of the polynomial ring $\mathbb{Z}[X]$: For any ring $R$ and any element $a$ in $R$, there always exists a unique homomorphism $f$ from $\mathbb{Z}[X]$ to $R$ such that $f(X) = a$; that is, $f$ takes the polynomial $X$ to $a$.

We establish the existence of such a homomorphism $f$ as follows. Suppose $P$ is any element of $\mathbb{Z}[X]$, that is, $P$ is a polynomial in $X$ with integer coefficients. Let $P = c_0 + c_1 X + c_2 X^2 + \cdots + c_n X^n$. Let $g$ be the unique homomorphism from $\mathbb{Z}$ to $R$ found in the first part of this example. We define a function $f$ from $\mathbb{Z}[X]$ to $R$ by the rule:

$$f(P) = g(c_0) + g(c_1) \cdot a + g(c_2) \cdot a^2 + \cdots + g(c_n) \cdot a^n.$$ 

Using the rules for adding and multiplying polynomials, it can be verified that $f$ is a homomorphism.

Uniqueness can be seen as follows. Suppose $f$ is any homomorphism from $\mathbb{Z}[X]$ to $R$ such that $f(X) = a$. If we restrict $f$ to the constant polynomials, we obtain a homomorphism $g$ from $\mathbb{Z}$ to $R$, which must therefore be the unique homomorphism from $\mathbb{Z}$ to $R$. If now $P$ is any polynomial in $\mathbb{Z}[X]$, and $P = c_0 + c_1 X + c_2 X^2 + \cdots + c_n X^n$, then

$$f(P) = f(c_0 + c_1 X + c_2 X^2 + \cdots + c_n X^n)$$
$$= f(c_0 + c_1 \cdot X + c_2 \cdot X \cdot X + \cdots + c_n \cdot X \cdot X \cdots \cdot X) = f(c_0) + f(c_1 \cdot X) + f(c_2 \cdot X \cdot X) + \cdots + f(c_n \cdot X \cdot X \cdots \cdot X)$$
$$= f(c_0) + f(c_1) \cdot f(X) + f(c_2) \cdot f(X) \cdot f(X) + \cdots + f(c_n) \cdot f(X) \cdots \cdots \cdot f(X)$$
$$= g(c_0) + g(c_1) \cdot a + g(c_2) \cdot a \cdot a + \cdots + g(c_n) \cdot a \cdot a \cdots \cdots \cdot a$$
$$= g(c_0) + g(c_1) \cdot a + g(c_2) \cdot a^2 + \cdots + g(c_n) \cdot a^n.$$
The value of the function $f$ on any polynomial $P$ is in this way completely determined. This establishes uniqueness.

We shall consider one example. Let $R$ be the ring $\mathbb{Z}(i)$ of Gaussian integers, and let $a = i$. We illustrate the behavior of the homomorphism $f$ from $\mathbb{Z}[X]$ to $\mathbb{Z}(i)$ that takes $X$ to $i$ in two cases:

$$f(2X^2 + 3) = 2i^2 + 3 = 2 \cdot (-1) + 3 = 1,$$
$$f(X^3 + X) = i^3 + i = -i + i = 0.$$

In this example we see that the homomorphism is not one-to-one: many different polynomials, such as $X^2 + 1$ and $X^3 + X$, are mapped to the same element 0.

The ring $\mathbb{Z}[X]$ is called the free ring generated by the element $X$. Here, “freedom” means that the element $X$ is unconstrained by any binding relationships within the ring $\mathbb{Z}[X]$. As a consequence of this freedom, the element $X$ is the lively embodiment of all possibilities: if $a$ is any element of any commutative ring $R$, then there exists a ring homomorphism $\mathbb{Z}[X] \rightarrow R$ that transforms the element $X$ to $a$. We shall discuss free rings further in Section 1.6.

From these examples, we see the range of transformations available in the theory of commutative rings, which establish the relationships between the diverse examples of commutative rings. In the next section we shall look more deeply into the mechanics of transformation expressed in a ring homomorphism. We shall locate the basis for all these diverse values of transformation in the internal dynamics of a ring. This will provide a mathematical expression of the phenomenon called Akshara in Maharishi Vedic Science: the description of the self-interacting dynamics of the unified field in terms of the collapse of wholeness to a point.

### 1.5 Homomorphism Theorem

In the last section we introduced the concept of a ring homomorphism, a coherent transformation from one ring to another. We shall see in this section how every such value of transformation can be described in terms of a self-referral value of transformation whereby a ring collapses within itself to yield a new ring structure.

Suppose we have a homomorphism $f$ from a ring $R$ to a ring $R'$. The homomorphism $f$ is a function from $R$ to $R'$, but what does $f$ actually
transform $R$ into? The real value of transformation is from $R$ to a subring $S$ of $R'$, consisting of all those elements of $R'$ that are “hit” by the function. The subring $S$ is called the image of the homomorphism $f$.

For example, if we let $f$ be the unique homomorphism from $\mathbb{Z}$ to $\mathbb{Z}[X]$, then the image of $f$ is the subring of $\mathbb{Z}[X]$ consisting of all constant polynomials.

In general, if $f$ is a homomorphism from $R$ to $S$, then in the process of transformation distinct points of $R$ can sometimes collapse to the same point of $S$. For example, consider the homomorphism $f$ from $\mathbb{Z}$ to $\mathbb{Z}_3$ considered in Example 7 of Section 1.4. Let $A_0$, $A_1$, and $A_2$ be the sets:

$$A_0 = \{ \ldots, -6, -3, 0, 3, 6, \ldots \}$$
$$A_1 = \{ \ldots, -5, -2, 1, 4, 7, \ldots \}$$
$$A_2 = \{ \ldots, -4, -1, 2, 5, 8, \ldots \}$$

Then every integer in the set $A_0$ is mapped to the single point 0 in $\mathbb{Z}_3$; every integer in the set $A_1$ is mapped to the single point 1 in $\mathbb{Z}_3$, and every integer in the set $A_2$ is mapped to the single point 2 in $\mathbb{Z}_3$. The homomorphism $f$ has the effect of partitioning the original ring $R$ into disjoint equivalence classes, where each equivalence class collapses to a single point of the image. In our example, there are three equivalence classes: $A_0$, $A_1$, $A_2$. $A_0$ collapses to 0, $A_1$ collapses to 1, and $A_2$ collapses to 2.

Now suppose we form the set $T$ of all the equivalence classes. Since there is precisely one equivalence class for each point in the image $S$, we have a one-to-one correspondence between the elements of $T$ and the elements of $S$. In our example, $T$ is the three element set $\{A_0, A_1, A_2\}$ and the one-to-one correspondence between $T$ and $S$ is simply: $A_0 \leftrightarrow 0, A_1 \leftrightarrow 1, A_2 \leftrightarrow 2$.

Let $g$ designate the function from $T$ to $S$ that represents this one-to-one correspondence. In our example, $g(A_0) = 0, g(A_1) = 1, g(A_2) = 2$.

Let $h$ be the function from $R$ to $T$ that takes each element of $R$ to its corresponding equivalence class. In our example, $h(6) = A_0, h(4) = A_1, h(5) = A_2, h(11) = A_2$, and so on.

The homomorphism $f$ can be computed by first applying the function $h$, and then applying the function $g$. In our example,
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\[ f(6) = g(b(6)) = g(A_0) = 0, \quad f(4) = g(b(4)) = g(A_1) = 1, \text{ and so on}. \]

We write \( g \circ h \) to designate the result of performing the two functions \( h \) and \( g \) in succession; \( g \circ h \) is called the composition of the functions \( g \) and \( h \). Thus \( f = g \circ h \).

The equivalence class that collapses to the zero element is called the kernel of the homomorphism. In our example, the kernel \( A_0 \) consists of all multiples of the integer 3.

We see in this example how all the different equivalence classes can be obtained from the kernel by simply adding the different possible elements of \( \mathbb{Z} \). For example, if we add 7 to each of the elements in \( A_0 \), we obtain the equivalence class \( A_1 \).

This situation holds in general: for any ring homomorphism \( f \) from \( R \) to \( R' \), all the equivalence classes can be obtained from the kernel \( I \) by simply adding the different possible elements of \( R \) to \( I \). Thus the knowledge of the kernel gives knowledge of all the equivalence classes.

Now the set \( T \) of equivalence classes can always be made into a commutative ring in the following way: (i) to add two equivalence classes, simply add their elements in all possible combinations; and (ii) to multiply two equivalence classes, multiply their elements in all possible combinations.

Thus, in our example, to add \( A_1 = \{ \ldots, -5, -2, 1, 4, 7, \ldots \} \) to \( A_2 = \{ \ldots, -4, -1, 2, 5, 8, \ldots \} \), we simply add their elements in all possible combinations: \(-5 + (-4) = -9\), \(-5 + 8 = 3\), \(1 + 5 = 6\), and so on. We end up with precisely the set \( A_0 = \{ \ldots, -6, -3, 0, 3, 6, \ldots \} \), so \( A_1 + A_2 = A_0 \). In this way we can define addition and multiplication tables for \( T \).

These operations of addition and multiplication satisfy all the commutative ring axioms; the mathematical structure \((T, +, \cdot)\) is thus a commutative ring. It is called the quotient ring of \( \mathbb{Z} \) mod \( I \), and is designated \( \mathbb{Z}/I \). The function \( h \) from \( R \) to \( T \) described above is easily seen to be a homomorphism from \( R \) to the quotient ring \( R/I \) and is called the canonical projection of \( R \) onto \( R/I \). Furthermore, the function \( g \) from \( T \) to \( S \) is easily seen to be an isomorphism of commutative rings. We thus find that the original homomorphism \( f \) can be “factored” in the form \( f = g \circ h \), where \( h \) is the projection of \( R \) onto \( R/I \), and \( g \) is an isomorphism from \( R/I \) onto \( S \). This result is called the homomorphism theorem.
Homomorphism Theorem. Let $R$, $R'$ be commutative rings and let $f$ be a homomorphism from $R$ to $R'$. Let $I$ be the kernel of $f$ and let $S$ be the image of $f$. Then $f$ can be expressed in the form $f = g \circ h$, where $h$ is the projection of $R$ onto the quotient ring $R/I$, and where $g$ is an isomorphism from $R/I$ onto $S$.

We make two observations. First, since the algebraic structure of the quotient ring $R/I$ is completely determined by the knowledge of the subset $I$, and since the image $S$ is isomorphic to the quotient $R/I$, this means that the algebraic structure of the image $S$ is completely determined by the knowledge of the kernel of $f$, the set of elements of $R$ that collapse to 0. For example, if $f$ is any homomorphism from $\mathbb{Z}$ to a ring $R$, and if the kernel of $f$ consists of all multiples of 3, then the image of $f$ must have the algebraic structure of $\mathbb{Z}/3$; there is no other possibility!

Secondly, an isomorphism expresses the relationship of structural equivalence, non-changing structural value. This means that the whole dynamics of transformation is found in the projection $h$ from $R$ onto the quotient ring $R/I$. This projection is an internal transformation within $R$, in which each equivalence class collapses to a point value.

The homomorphism theorem thus analyzes all values of transformation between rings in terms of the internal mechanics of transformation of a ring; the most essential feature of this internal dynamics is the way in which subsets of the original ring collapse to points of the quotient ring. This mechanics of transformation has a parallel in Maharishi Vedic Science in the phenomenon of Akshara.

Maharishi has described the way in which the internal dynamics of the Samhita can be analyzed in terms of the collapse of infinity to a point value within it, a phenomenon called Akshara. The field of pure intelligence, the Samhita, has the structure of an unbounded ocean of consciousness, which expresses the value of infinity. On the other hand, the intelligent nature of this ocean of consciousness makes the discriminative value of the intellect lively within it, which then identifies a point value within that continuum, a point that expresses the extreme value of nothingness. The simultaneity of these two complementary values lively within the structure of intelligence gives rise to the intellectually conceived “collapse” of infinity to the point value within it, and this is the primordial expression of the self-interacting dynamics of consciousness.
Maharishi has explained how the first syllable of Rik Veda, AK, gives expression to this phenomenon. The open sound A expresses the value of infinity, and the stop K expresses the value of a point. The syllable AK thus expresses the collapse of infinity, A, to the point, K. The term *Akshara*, meaning “collapse (kshara) of A,” is thus used to describe this phenomenon.

In the context of ring theory, the construction of a quotient ring is a mathematical expression of the phenomenon of *Akshara*; the construction of the quotient ring is an internal transformation of a ring in which the different equivalence classes each collapse to a single point value in the quotient ring. In Maharishi Vedic Science, all values of transformation in nature are seen to have their ultimate basis in the self-referral value of transformation in the phenomenon of *Akshara*; ring theory likewise analyzes all values of transformation between rings in terms of the internal dynamics of a ring as expressed in the quotient-ring construction, a mathematical expression of the phenomenon of *Akshara*.

**Example.** Consider the homomorphism $f$ from $\mathbb{Z}[X]$ to $\mathbb{Z}(i)$ described in Section 1.4. The image of $f$ contains all the Gaussian integers, $\mathbb{Z}(i)$ (since, for any Gaussian integer $a + bi$, the polynomial $a + bX$ maps to it). The ring of Gaussian integers is thus isomorphic to the quotient ring $\mathbb{Z}[X]/I$, where $I$ is the kernel of $f$. We note that the polynomial $X^2 + 1$ is contained in the kernel of $f$: $f(X^2 + 1) = i^2 + 1 = -1 + 1 = 0$. It can be shown that the kernel of $f$ consists of all multiples of the polynomial $X^2 + 1$, that is, all polynomials of the form $(X^2 + 1) \cdot P(X)$, where $P(X)$ can be any polynomial. We see then the ring of Gaussian integers $\mathbb{Z}(i)$ arises from the polynomial ring $\mathbb{Z}[X]$ through the mechanics of *Akshara*, whereby the polynomial $X^2 + 1$ collapses to 0. Through this process of collapse, the indeterminate $X$ gets transformed to a value $i$, such that $i^2 = -1$.

In the ring of integers, $\mathbb{Z}$, the equation $X^2 + 1 = 0$ has no solution: it is an “impossibility”. To actualize this impossibility, it is only necessary to go to the field of all possibilities, the free ring $\mathbb{Z}[X]$, in which all possibilities are lively, and then utilize the mechanics of *Akshara* to bring out any desired possibility. This theme will be elaborated in the next section, in which we shall see how all possible commutative rings
can be created from free commutative rings, through the mechanics of *Akshara*.

**Ideals.** Suppose $R$ is any commutative ring. We can ask, what are the distinguishing properties of subsets of $R$ that are kernels of homomorphisms? It is not difficult to verify that the kernel $I$ of a homomorphism always has the following two properties:

1. If $a, b$ are elements of $I$, then the difference $a - b$ is also an element of $I$. (This is equivalent to the condition that $I$ is a group under the addition operation, $+$, of $R$.)

2. If $a$ is an element of $I$ and $r$ is any element of $R$, then $r \cdot a$ is an element of $I$.

Condition (2) follows from the property of homomorphisms that $f(a \cdot r) = f(a) \cdot f(r)$, so if $f(a) = 0$, then $f(a \cdot r) = 0 \cdot f(r) = 0$ by Theorem 2. Thus if $a$ is any element of the kernel of $f$, $a \cdot r$ likewise must be an element of the kernel of $f$.

Condition (1) follows from the property of homomorphisms that $f(a + b) = f(a) + f(b)$, together with properties of addition described by Axioms 1–4.

Any subset $I$ of a commutative ring $R$ that satisfies properties (1) and (2) above is called an *ideal* in $R$. An ideal $I$ always partitions a commutative ring into disjoint equivalence classes, obtained by adding all the different elements of $R$ to $I$. The set of equivalence classes always has the structure of a commutative ring: the classes are added or multiplied by adding or multiplying their elements in all possible combinations. This ring is called the *quotient ring* $R/I$. We write $R/I = \{ r + I : r \in R \}$.

The function $h$ that takes each element of $R$ to its equivalence class (that is, $h(r) = r + I$ for each $r \in R$) is a homomorphism from $R$ to $R/I$, called the *canonical projection* onto the quotient ring. The kernel of this homomorphism is the ideal $I$ itself.

The subsets of a commutative ring that are kernels of homomorphisms are thus precisely its ideals. In this way the whole range of possibilities of transformation for a commutative ring can be identified internally, within the ring, in terms of its own algebraic structure.
Each ideal presents a unique channel of transformation that collapses precisely that ideal to zero.

1.6 Free Rings
We have seen that each level of mathematical structure has its own range of transformations. The undifferentiated level, the field of sets, has all possibilities available: all possible functions. At differentiated levels of structure, the range of possibilities becomes restricted: only those transformations are allowed that preserve the structural relationships belonging to that level. For example, in the category of commutative rings, only those functions are permitted that are ring homomorphisms, functions that preserve the algebraic relations.

Nevertheless, there is a way in which the full range of possibilities belonging to the undifferentiated level, the field of sets, can be actualized in the differentiated field, for example, the world of commutative rings. This actualization of “all possibilities” is achieved by free structures. We shall examine in this section the concept of a free structure in the context of the theory of commutative rings.

The intuitive concept of a free ring is the following. We begin with some set $S$. From the elements of $S$, we “generate” a commutative ring. We must do this in such a way that there are no binding relationships among the generators (the elements of $S$); this is the meaning of “freedom” in this context. We shall consider different possible cases to see how this works.

1. $S$ is the empty set: $S = \emptyset = \{\}$. This is the simplest case, in which there are no generators. We wish to determine the smallest possible ring $R$ in which the only equational relationships that hold true are those that are required by the axioms of commutative rings; this will be called the free commutative ring on no generators. To begin, we know that $R$ must contain a zero element $0$ and an identity element $1$ (in order to satisfy the commutative ring axioms). The commutative ring $R$ must also contain $1 + 1$, $1 + 1 + 1$, $-1$, $-(1 + 1)$, $(1 + 1) \cdot (1 + 1 + 1)$, and so on. Since the basic principle governing free commutative rings is that there must be no equational relationships other than those that are logical consequences of the commutative ring axioms, we expect that the relation $(1 + 1) \cdot (1 + 1 + 1) = 1 + 1 +$
1 + 1 + 1 + 1 will be satisfied (because it follows from the axioms: It is a theorem of the theory of commutative rings) but not the relation 1 + 1 + 1 = 0 (as it is only true for certain commutative rings and not for others). This is the meaning of “no binding relationships.” It follows that the elements 0, 1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1, . . . , –1, –(1 + 1), . . . are all distinct (for example, 1 + 1 = 1 + 1 + 1 + 1 + 1 would be a “binding” relationship).

Consider now the function \( f \) from \( \mathbb{Z} \) to \( R \) defined by: \( f(0) = 0, f(1) = 1, f(2) = 1 + 1, f(3) = 1 + 1 + 1, f(-1) = -1, f(-2) = -(1 + 1), f(-3) = -(1 + 1 + 1) \), and so on. (Note that the 1’s on the right hand side of the equations designate the multiplicative identity of \( R \), not the integer one). The function \( f \) establishes a one-to-one function from \( \mathbb{Z} \) into \( R \). It is not difficult to verify that \( f \) must be a ring homomorphism. For example, \( f(2) \cdot f(3) = (1 + 1) \cdot (1 + 1 + 1) \), and \( f(6) = 1 + 1 + 1 + 1 + 1 + 1 = (1 + 1) \cdot (1 + 1 + 1) \), since this relationship is a consequence of the commutative ring axioms. Moreover, we can see, intuitively at least, that in order for \( R \) to be the “smallest” ring in which there are no binding relationships, the homorphism \( f \) must in fact be onto. For instance, we notice that the direct product \( \mathbb{Z} \times \mathbb{Z} \) is also free of binding relationships, but is not the “smallest possible” ring with this property. We thus conclude that the free commutative ring with no generators is isomorphic to the ring of integers \( \mathbb{Z} \).

The free commutative ring \( \mathbb{Z} \) has the following universal property: for any ring \( R' \), there exists a unique homomorphism \( f \) from \( \mathbb{Z} \) to \( R' \) (as we saw in Section 1.4). Let \( 1' \) be the identity of \( R' \). There are two different cases:

Case 1: The elements 0', 1', 1' + 1', 1' + 1' + 1', . . . are all distinct. Then the kernel of \( f \) is \{0\}, and \( f \) is an isomorphism from \( \mathbb{Z} \) to a subring \( S \) of \( R' \). For instance, when \( R' = \mathbb{Z} \times \mathbb{Z} \), the subring \( S \) consists of all elements of \( R' \) of the form \((n, n)\), where \( n \) belongs to \( \mathbb{Z} \); \( S \) is an isomorphic copy of \( \mathbb{Z} \) in \( R' \). All the examples in this case make precise the intuition that the free ring on no generators is the “smallest possible” ring without binding relationships: in mathematical language, it means precisely that an isomorphic copy of \( \mathbb{Z} \) can be found in any ring in which there are no binding relationships.

Case 2: Some finite sum \( 1' + 1' + \cdots + 1' = 0' \). If the least such sum contains \( n \) 1’s, then the image of \( f \) is a subring of \( R' \) isomorphic to \( \mathbb{Z}/n \). For
example, if $0', 1', 1' + 1'$ are all distinct, but $1' + 1' + 1' = 0'$, then the three elements $0', 1', 1' + 1'$ form a subring of $R'$ isomorphic to $\mathbb{Z}_3$, and this subring is the image of $f: f(0) = 0', f(1) = 1', f(2) = 1' + 1', f(3) = 0', f(4) = 1', f(-1) = 1' + 1'$, and so on. The kernel of the homomorphism $f$ in this case consists of all multiples of the integer $n$.

(2) $S$ contains a single element: $S = \{a\}$.

As in case (1), the free commutative ring $R$ must contain the integers $\mathbb{Z}$. In addition, $R$ must contain $a$ and all elements generated from $\mathbb{Z}$ and $a$ using the operations of addition and multiplication: $2a + 5$, $3a^2 - 4a + 7$, and so on. As in case (1), the only equational relations permitted are those that follow from the commutative ring axioms. Thus $(a + 1) \cdot (a - 1) = a^2 - 1$, because this is a theorem of the theory of commutative rings, but the relation $a^2 - 1 = 0$ is not satisfied, because this does not follow from the commutative ring axioms. (If it did, the square of any element of any commutative ring would be the identity element!) It follows that any two distinct polynomials in $a$ with coefficients in $\mathbb{Z}$ must represent distinct elements of $R$. The free ring $R$ thus turns out to be isomorphic to the polynomial ring $\mathbb{Z}[X_a]$. We have designated the indeterminate by $X_a$ to indicate that it represents the generator $a$.

This free commutative ring has the following universal property: If $R'$ is any commutative ring, and $r$ is any element of $R'$, then there exists a unique homomorphism $f$ from $R$ to $R'$ that takes the generator $X_a$ to $r$: $f(X_a) = r$. We described the homomorphism $f$ in Section 1.4.

(3) $S$ contains two elements: $S = \{a, b\}$.

In this case the free ring $R$ turns out to be the polynomial ring in two indeterminates having integer coefficients, $\mathbb{Z}[X_a, X_b]$. This free ring $R$ has the following universal property: If $R'$ is any ring, and $r, s$ are any elements of $R'$, then there exists a unique homomorphism $f$ from $R$ to $R'$ that takes $X_a$ to $r$ and $X_b$ to $s$: $f(X_a) = r$ and $f(X_b) = s$.

(4) $S$ is any set.

Suppose now $S$ is any set. The free ring $R$ generated by $S$ turns out to be the polynomial ring having coefficients in $\mathbb{Z}$ and having one indeterminate $X_a$ for each element $a$ of $S$. The universal property of the free ring $R$ is then the following. Suppose $R'$ is any commutative ring and $g$
is any function from the set \( S \) to \( R' \). Then there exists a unique homomorphism \( f \) from \( R \) to \( R' \) such that \( f(X_a) = g(a) \) for every element \( a \) in \( S \).

This means that any function applied to the set of generators extends to a homomorphism of the free commutative ring. In this way the free commutative ring gives expression, in the world of commutative rings, to the full range of possibilities of transformation belonging to the undifferentiated field, the world of sets. Free rings are the “representatives” of the undifferentiated, unified field of sets in the expressed field of structure described by the theory of commutative rings. We shall develop this theme further in Chapter 2, when we systematically explore transformations between different mathematical theories.

The universal property of free rings can be used to show how any ring can be actualized as a quotient ring of a free ring. This is done as follows.

Suppose \( R' \) is any commutative ring, and let \( S \) designate the underlying set of elements of \( R' \), so that \( R' = (S, +, \cdot) \). Let \( R \) be the free commutative ring generated by the set \( S \). Consider the function \( g \) from \( S \) to \( R' \) that takes each element \( a \) of \( S \) to itself (regarded as an element of \( R' \)). Because of the universal property of the free commutative ring \( R \), there exists a homomorphism \( f \) from \( R \) to \( R' \) such that \( f(X_a) = g(a) \) for every element \( a \) in \( S \), that is, each generator \( X_a \) of \( R \) is taken to the corresponding element \( a \) of \( S \), regarded as an element of \( R' \). The image of the homomorphism \( f \) is clearly all of \( R' \).

By the homomorphism theorem, it follows that the commutative ring \( R' \) is isomorphic to the quotient \( R/I \), where \( I \) is the kernel of the homomorphism \( f \). Thus, any commutative ring \( R' \) can be actualized (up to isomorphism) as a quotient ring of a free commutative ring \( R \). In this way every possible commutative ring structure can be derived from a free commutative ring through the quotient-ring construction. This provides a further expression of the way free rings are the lively embodiment of “all possibilities.” It also shows how all possible values of ring structure can be created through the internal mechanics of the quotient-ring construction, which as we have analyzed is a mathematical expression of the phenomenon of \( \text{Akshara} \), the phenomenon Maharishi Vedic Science identifies at the source of all expressed values of structure in creation.
Chapter 2
Category Theory

2.1 Definition and Examples
In Chapter 1 we examined an example of an abstract theory: the theory of commutative rings. This provided an introduction to the viewpoint of the “working mathematician.” The theory of rings is just one example of an abstract theory. Modern mathematics contains numerous abstract theories, each describing a different type of structure. In addition to rings, one can study groups, fields, vector spaces, topological spaces, topological vector spaces, . . . . The list goes on and on. Each of these diverse types of structure is characterized by a specific collection of axioms, and each thereby gives rise to its own abstract theory. The range of these theories embraces in principle every exact, orderly type of structure the mind can conceptualize and describe in precise, symbolic language. Modern mathematics in this way embraces the field of all possibilities.

Each abstract theory has its own unique flavor and structure of knowledge. Nevertheless, there are certain features shared by all abstract theories. The isolation of several of these features has given rise to a theory of abstract theories called category theory, founded by Eilenberg and Mac Lane in 1945 (Eilenberg & Mac Lane, 1945).

Category theory has its roots in the observation that all abstract theories share the following features:

1. There is a certain collection of mathematical structures described by the theory. These are simply the structures that satisfy the axioms of the theory. In the case of the theory of rings described in Chapter 1, these are all possible commutative rings.

2. There are certain transformations between structures that are studied in the theory. These are functions from one structure to another that preserve the basic structural relationships of the theory. In the case of the theory of commutative rings, these transformations are the ring homomorphisms. In general, these transformations are called the morphisms of the theory.

3. Morphisms can always be combined in sequence to yield new morphisms: If $f$ is a morphism from a structure $A$ to a structure $B$, and $g$ is a morphism from $B$ to a structure $C$, then there will
always exist a third morphism—the composite function $g \circ f$—from $A$ to $C$.

(4) The composition operation $\circ$ satisfies the associative law:

$$(f \circ g) \circ h = f \circ (g \circ h).$$

(5) For every structure $A$, the identity function $1_A$ from $A$ to itself is always an example of a morphism of the theory. The identity morphism $1_A$ satisfies the identity laws:

$$f \circ 1_A = f,$$

for any morphism $f$ from $A$ to a structure $B$, and

$$1_A \circ g = g,$$

for any morphism $g$ from a structure $C$ to $A$.

Based on these simple observations, Eilenberg and Mac Lane introduced the concept of a new type of abstract structure called a category, whereby a single category would represent the structure of an entire abstract theory.

We shall begin with the definition of a category and then will consider several examples.

**Definition.** A category consists of objects: $A$, $B$, $C$, . . . and arrows: $f$, $g$, $h$, . . . . Each arrow has a source and a target (which are objects). If the arrow $f$ has source $A$ and target $B$ we write $f : A \to B$. The arrows have a composition operation $\circ$: If $f : A \to B$ and $g : B \to C$, then there is always defined a composition arrow $h = g \circ f : A \to C$. The composition of arrows must satisfy the following axioms:

**Axiom 1** (associative law). $(f \circ g) \circ h = f \circ (g \circ h)$ whenever the compositions are defined.

**Axiom 2** (identity law). For every object $A$ there is an identity arrow $1_A : A \to A$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ whenever the products are defined.

Every abstract theory gives rise to a category, whose objects are all possible structures satisfying the axioms of the theory, and whose arrows are all possible morphisms of the theory. The theory of rings, for example, gives rise to the category whose objects are all possible commutative rings and whose arrows are all possible ring homomorphisms. This category is called the category of commutative rings.
Category theory is itself an abstract theory. It is developed axiomatically from the above axioms. The theorems of category theory apply to all possible categories; they apply in particular to the categories corresponding to the different abstract theories of modern mathematics. In this way category theory provides a theory of abstract theories. In later sections we shall see how the language of category theory provides a unified theory of abstract mathematical theories. In the remainder of this section we shall consider several examples of categories that we shall require for the further development of category theory and topos theory.

We have observed that every abstract theory gives rise to a category. Two such categories that will be of particular interest to us are the category of sets and the category of commutative rings. In most of our examples, the morphisms of the category will be functions of some kind, but there are other ways of constructing categories as well. Several such constructions are described in Examples 5–8 below.

**Example 1: Category of Sets.** The theory of sets and functions discussed in Section 1.1 describes the category of sets (\textbf{Set}), whose objects are all possible sets and whose arrows \( f : A \to B \) are all possible functions \( f \) from \( A \) to \( B \).

**Example 2: Category of Commutative Rings.** The theory of commutative rings describes the category of commutative rings (\textbf{CR}), whose objects are all possible commutative rings and whose arrows \( f : A \to B \) are all possible ring homomorphisms. These are the functions that preserve the ring structure.

**Example 3: Category of Groups.** A group is a mathematical structure \((G, \ast)\), having a single binary operation \( \ast \), which satisfies the following three axioms:

**Axiom 1** (associative law). \( a \ast (b \ast c) = (a \ast b) \ast c \).

**Axiom 2** (identity law). There is an element \( e \) (the identity element) such that \( e \ast a = a \ast e = a \) for every \( a \) in \( G \).

**Axiom 3** (inverse law). For each \( a \) in \( G \), there exists an element \( a^{-1} \) (the inverse of \( a \)) such that \( a^{-1} \ast a = e = a \ast a^{-1} \).
Group theory describes the category of groups \((\text{Grp})\), whose objects are all possible groups and whose arrows \(f : A \to B\) are all possible group homomorphisms, that is, all possible functions \(f\) from \(A\) to \(B\) that have the property \(f(a \ast b) = f(a) \ast f(b)\). These are the functions that preserve the group structure.

Example 4: Category of Topological Spaces. Point-set topology describes the category of topological spaces \((\text{Top})\), whose objects are all possible topological spaces and whose arrows \(f : A \to B\) are all possible continuous functions from \(A\) to \(B\). (These are functions that preserve the topological structure.)

In the above examples, the composition rule of arrows is the composition rule for functions: If \(f : A \to B\) and \(g : B \to C\) then \(h = g \circ f : A \to C\) is defined by \(h(a) = g(f(a))\) for all \(a\) in \(A\), so \(g \circ f\) is simply the result of first performing \(f\) and then performing \(g\). The identity arrow \(1_A\) is the identity function for \(A\), the function defined by \(1_A(a) = a\) for all \(a\) in \(A\).

A category is, however, simply defined as a type of abstract structure. It is not necessary that the arrows of a category represent functions nor that the objects represent structures satisfying the axioms of an abstract theory. Following are several examples in which the arrows do not represent functions.

Example 5: Partial Orderings. A partial ordering is a mathematical structure \((S, \leq)\) that satisfies the following three axioms:

**Axiom 1** (reflexive law). \(a \leq a\).

**Axiom 2** (antisymmetric law). If \(a \leq b\) and \(b \leq a\), then \(a = b\).

**Axiom 3** (transitive law). If \(a \leq b\) and \(b \leq c\), then \(a \leq c\).

Every linear ordering \(<\) (discussed in Section 1.1) gives rise to a partial ordering if we interpret \(\leq\) as “less than or equal to”; that is, we define \(a \leq b\) to be satisfied if either \(a < b\) or \(a = b\). There are, however, many additional examples of partial orderings. One very important example is the collection of subsets of a given set, with the subset relation: if \(A\) is any set, let \(S = P(A)\) (the set of all possible subsets
of $A$, and define the ordering on $S$ by declaring that, for any elements $B, C$ of $S, B \leq C$ if and only if $B \subseteq C$ (recall that $B \subseteq C$ means that every element of $B$ is also an element of $C$). It is easily checked that the structure $(S, \leq) = (P(A), \subseteq)$ satisfies the three axioms for a partial ordering.

Suppose now $(S, \leq)$ is any partial ordering. We can construct a category $K$ describing this partial ordering as follows:

1. The objects of $K$ are all the elements of $S$.
2. For every pair $a, b$ of elements of $S$, we introduce a single arrow with source $a$ and target $b$ just when $a \leq b$; otherwise, we introduce no arrow with source $a$ and target $b$.
3. If $f : a \rightarrow b$ and $g : b \rightarrow c$, then we define $g \circ f$ to be the unique arrow having source $a$ and target $c$. (If $a \leq b$ and $b \leq c$, then $a \leq c$ by the transitivity law, so there will exist an arrow $h : a \rightarrow c$.)

It is easily verified that $K$ satisfies the axioms for a category. (In particular, the existence of the identity arrow $1_a : a \rightarrow a$ is a consequence of the reflexive law: $a \leq a$.) Categories that arise in this way from partial orderings play an important role in topos theory, as we shall see later.

Example 6: Monoids. A monoid is a mathematical structure $(M, \cdot, e)$ that satisfies the following two axioms:

Axiom 1 (associative law). $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
Axiom 2 (identity law). $e \cdot a = a \cdot e = a$.

Every group is a monoid, but there are other examples as well. Two monoids that are not groups are $(\mathbb{N}, +, 0)$ and $(\mathbb{N}, \cdot, 1)$, where $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \ldots\}$, ‘$+$’ is the operation of addition, and ‘$\cdot$’ is the operation of multiplication.

Suppose now $(M, \cdot, e)$ is any monoid. We construct a category $K$ representing the structure of this monoid as follows:

1. There is a single object $A$ in $K$.
2. For every element $a$ of $M$, introduce an arrow $f_a : A \rightarrow A$.
3. Define the composition of arrows by: $f_a \circ f_b = f_{a \cdot b}$, whenever $c = a \cdot b$. 

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The axioms for a category are clearly satisfied: the associative law follows from the associative law for \((M, \cdot, e)\), and the identity arrow is \(f_e : a \to a\), where \(e\) is the identity element of the monoid.

We notice that for this example of a category, we have not collected together all structures of a particular type (such as groups, rings, sets, monoids); instead, the structure of the category arises from viewing the structure of a single mathematical object—a monoid—through “categorical eyes,” identifying the object itself as the only object in the category, and defining arrows from the monoid’s operation. This category is another example in which arrows are not a special kind of function.

**Example 7: Discrete Categories.** Let \(S\) be any set. One can define a category \(K\) whose objects are the elements of \(S\) and whose arrows are the identity arrows only. A category of this form is called a *discrete category*.

**Example 8: The Categories \(0, 1, 2, 3\).** The special categories \(0, 1, 2,\) and \(3\) are defined as follows.

The category \(0\) is the empty category, having no objects and no arrows. The category \(1\) is the category having a single object \(M\) and a single arrow \(1_M : M \to M\). The category \(2\) is the category having two objects \(N_1, N_2\) and a single arrow \(n_{12} : N_1 \to N_2\), in addition to the identity arrows \(1_{N_1}\) and \(1_{N_2}\). The category \(3\) is the category having three objects \(L_1, L_2, L_3\) and three arrows \(l_{12} : L_1 \to L_2, l_{23} : L_2 \to L_3, l_{13} : L_1 \to L_3\), with \(l_{13} = l_{23} \circ l_{12}\), in addition to the identity arrows \(1_{L_1}, 1_{L_2}, 1_{L_3}\).

Of course, we can define a category \(n\) of this kind for each natural number \(n\). It is worthwhile to observe that the category \(n\), consisting of \(n\) objects together with arrows defined according to the pattern given above, is structurally the same as the partial order category defined on the set \(\{1, 2, 3, \ldots, n\}\), with arrows defined in accord with the natural relation \(\leq\). The two categories are therefore said to be *isomorphic*. We will discuss the concept of isomorphism between categories shortly.

We have given here just a handful of examples. Category theory applies to any collection of objects and arrows that satisfies the category theory axioms. We shall encounter many other examples as we continue with our development of category theory and topos theory.
It should be noticed that the language of category theory does not make reference to the internal structure of the objects. If $A$ is an object in a category, we cannot talk about the elements of $A$. We can only talk about the arrows between objects and the way these compose with one another to yield new arrows. The vision of category theory is thus focused not on the internal structure of the objects, but rather on the “gaps” between the objects. It is in these gaps that one locates the transformations between the objects, the morphisms of an abstract theory. The vision of category theory is all in terms of these values of transformation and their sequential relationships to one another. The objects themselves serve just to label the starting point and ending point of the arrows, so that one knows when two arrows can be combined sequentially to yield a third. Category theory is thus the language of the “gaps.”

In different abstract theories, the objects have different types of internal structure, each of which requires its own special language to describe. The language of category theory is formulated on a more abstract level—the level of arrows, the level of the “gaps.” This level is common to all mathematical theories. Category theory thus provides a universal mathematical language, capable of speaking for all mathematical theories simultaneously. On this basis category theory has been able to articulate the most basic structural concepts of mathematics in a language that is valid for all mathematical theories. This conceptual framework has provided a totally new insight into the fundamental constructions of mathematics. We shall describe this new formulation of the basic mathematical constructions and relationships in the following two sections.

Maharishi Vedic Science has its own theme of analysis of the universal mechanics of transformation belonging to the “gaps.” In Maharishi’s analysis of the sequential emergence of the Rik Veda from Akshara, the dynamics of transformation are located in the gaps between the sequentially emerging expressions of knowledge. We shall see later how universal constructions in category theory give mathematical expression to the basic characteristics of the “gap” described by Maharishi Vedic Science.
2.2 Monics and Epics

In this and the following section we shall see how the most fundamental structural concepts and constructions of modern mathematics can be formulated in the universal language of category theory. In this way, familiar mathematical constructions will be appreciated in a totally new light in terms of arrows rather than elements.

*Isomorphisms.* An *isomorphism* expresses the concept of structural equivalence. In the category of sets, an isomorphism is a function \( f: A \rightarrow B \) that establishes a one-to-one correspondence between the sets \( A \) and \( B \). In the category of groups, an isomorphism is a group homomorphism that establishes a one-to-one correspondence between two groups. In the category of commutative rings, an isomorphism is a ring homomorphism that establishes a one-to-one correspondence between two rings. In the category of topological spaces, an isomorphism (usually called a *homeomorphism*) is a continuous function \( f: A \rightarrow B \) that establishes a one-to-one correspondence between the topological spaces \( A \) and \( B \) and that has the additional property that the inverse function \( g: B \rightarrow A \) also is continuous.

We notice in all these cases that if an arrow \( f: A \rightarrow B \) is an isomorphism, then there always exists an “opposite” arrow \( g: B \rightarrow A \) with the characteristic that, when we execute \( f \) and \( g \) in succession, in either order, we obtain an identity arrow: \( g \circ f = 1_A \) and \( f \circ g = 1_B \). This property can be used to formulate the concept of an isomorphism in a completely general way in the universal language of category theory.

*Definition.* An arrow \( f: A \rightarrow B \) in a category \( K \) is called an *isomorphism* if there exists an arrow \( g: B \rightarrow A \) in \( K \) such that \( f \circ g = 1_B \) and \( g \circ f = 1_A \). The arrow \( g \) is called the *inverse* of \( f \). The objects \( A \) and \( B \) are said to be *isomorphic*.

Isomorphic objects in any category always have identical structural properties. The simple abstract definition of an isomorphism given above in terms of arrows provides a unified description of the concept of structural equivalence that is applicable simultaneously to all possible abstract theories.
Monics. The concept of a monic arrow generalizes the idea of a function that is one-to-one. We recall that a function \( f : A \to B \) is said to be one-to-one or \textit{injective} if it always takes distinct elements of \( A \) to distinct elements of \( B \). This description of a one-to-one function is in terms of elements. We can find, however, an equivalent description in terms of arrows as follows:

Consider as an example the function \( f : A \to B \) where \( A = \{1, 2, 3\} \), \( B = \{2, 4, 6, 8\} \), and where the values of \( f \) are defined by \( f(1) = 4 \), \( f(2) = 8 \), and \( f(3) = 8 \). Since \( f(2) = f(3) \), the function \( f \) is not one-to-one (\( f \) takes distinct elements 2, 3 to the same value 8). We can now find a set \( C \) and two different functions \( g, h : C \to A \) with the property that \( f \circ g = f \circ h \). For example, we can take \( C = \{0\} \), with \( g \) and \( h \) defined by \( g(0) = 2 \) and \( h(0) = 3 \). This situation generalizes to any function \( f : S \to T \) that is not one-to-one: if \( f(a) = f(b) \), then if we take \( C = \{0\} \) and define the functions \( g, h : C \to S \) by \( g(0) = a \) and \( h(0) = b \) then \( f \circ g = f \circ h \). If, however, \( f \) is one-to-one, then no such pair of functions could exist, for if \( c \in C \) is an element at which such functions \( g, h \) disagree—that is, \( g(c) \neq h(c) \)—then the one-to-one property of \( f \) ensures that \( f(g(c)) \neq f(h(c)) \), and so \( f \circ g \neq f \circ h \). These observations lead to the following definition of a monic arrow.

\textit{Definition.} An arrow \( f : A \to B \) in a category \( K \) is \textit{monic} if it has the following property: whenever \( C \) is an object of \( K \) and \( g, h : C \to A \) are arrows in \( K \) satisfying \( f \circ g = f \circ h \), then we must have \( g = h \).

The definition of a monic arrow generalizes the concept of a one-to-one function to the universal language of category theory, applicable to all possible categories.

Epics. The concept of an epic arrow generalizes the concept of a function that is onto. We recall that a function \( f : B \to A \) is said to be onto or \textit{surjective} if every element \( y \) in \( A \) can be expressed as \( y = f(x) \) for some \( x \) in \( B \); that is, every element of \( A \) is “hit” by the function. This description of an onto function is in terms of elements. We can find, however, an equivalent description in terms of arrows as follows.
Consider the function \( f : B \rightarrow A \), where now \( B = \{1, 2, 3\} \), \( A = \{2, 4, 6, 8\} \) and where the values of \( f \) are defined by \( f(1) = 4, f(2) = 8 \), and \( f(3) = 8 \). Since there is no \( x \) in \( B \) such that \( f(x) = 2 \), the function \( f \) is not onto. We can now find a set \( C \) and two different functions \( g, h : A \rightarrow C \) with the property that \( g \circ f = h \circ f \).

For example, we can take \( C = \{0, 1\} \), with \( g \) and \( h \) defined by \( g(2) = g(6) = g(8) = 0 \), \( h(2) = 1 \), \( h(4) = h(6) = h(8) = 0 \). The trick is to define \( g \) and \( h \) in such a way that they agree on output values of \( f \), but are defined so that they disagree on at least one element of \( A \) that is \textit{not} in the range of \( f \). This technique generalizes to any function \( f : S \rightarrow T \) that is not onto: if we take \( C = \{0, 1\} \), then we can, in the same way, find a pair of functions \( g, h : A \rightarrow C \) such that \( g \circ f = h \circ f \). Certainly this technique will not work if \( f \) is onto. In fact, it can be shown that whenever \( f \) is onto and \( g, h : A \rightarrow C \) (for \textit{any} set \( C \)) are distinct, then \( g \circ f \) and \( h \circ f \) must also be distinct. This observation leads to the following definition of an epic arrow.

\textit{Definition.} An arrow \( f : B \rightarrow A \) in a category \( K \) is \textit{epic} if it has the following property: whenever \( g, h : A \rightarrow C \) are arrows in \( K \) satisfying \( g \circ f = h \circ f \), we must have \( g = h \).

Note that the definitions of monics and epics have identical form, except that the direction of all arrows is reversed. This reflects a deep symmetry of category theory: symmetry under reversal of arrows. The axioms of category theory are unchanged if one reverses the direction of all arrows. It follows that, if \( S \) is any theorem of category theory, then the statement one obtains by reversing all the arrows in \( S \) must also be a theorem. This principle is called the \textit{principle of duality}.

For every concept of category theory there is a \textit{dual concept}, obtained by reversing the direction of all arrows. Thus epics and monics are dual concepts. The concept of an isomorphism is self-dual; if we reverse all the arrows we obtain the same definition. As we continue we shall see many examples of dual concepts. The principle of duality is a profound expression of perfect balance in the structure of category theory.

\textit{Subobjects.} Every abstract theory has its own concept of \textit{subobjects}; set theory has the concept of a \textit{subset} of a set; group theory has the concept
of a subgroup of a group; ring theory has the concept of a subring of a ring; and so on. The concepts of subset, subgroup, subring, and so on, are ordinarily defined in terms of elements. For example, \( S \) is a subset of \( A \) if every element of \( S \) is an element of \( A \). We can find, however, an equivalent description of a subset in terms of arrows as follows. We can think of the subset \( S \) as being described by an isomorphic copy \( B \) of \( S \), together with a monic arrow \( f : B \to A \) showing how the set \( B \) is embedded as a subset of \( A \). For example, suppose \( A = \{1, 2, 3, 4\} \) and \( S = \{2, 3\} \). \( S \) is isomorphic to any two-element set \( B = \{a_1, a_2\} \) (remember that isomorphism in the category of sets is the same as one-to-one correspondence). We can therefore capture the notion that “\( S \) is a subset of \( A \)” abstractly by defining an embedding function \( f : B \to A \) with \( f(a_1) = 2 \) and \( f(a_2) = 3 \). In this way, \( S \), as a subset of \( A \), has been represented by a monic arrow \( f : B \to A \).

When we represent subsets by monic arrows we must, however, take into account that different arrows can represent the same subset. In the example considered above, if \( C = \{b_1, b_2\} \) is any other two-element set, then the function \( g : C \to A \) defined by \( g(b_1) = 2 \), \( g(b_2) = 3 \) describes the same subset \( S \subseteq A \) as \( f : B \to A \). However, we also observe that the sets \( B \) and \( C \) must be close relatives of each other; in particular, there is an isomorphism \( h : B \to C \) such that \( g \circ h = f \). Here, \( h \) is defined by \( h(a_1) = b_1 \) and \( h(a_2) = b_2 \). Moreover, this latter property shows that the functions \( f \) and \( g \) are themselves closely related: they “differ” only by the isomorphism \( h \). For this reason, we can think of \( f \) and \( g \) as being essentially the same.

This situation generalizes to arbitrary sets. Whenever two monics represent the same subset of a set there will always exist such a connecting isomorphism; conversely, whenever two monics represent different subsets, no such connecting isomorphism will exist. This observation solves the problem of characterizing the concept of subobject in terms of the language of arrows.

**Definition.** A subobject of \( A \) in a category \( K \) is a monic arrow \( f : B \to A \) in \( K \), for some object \( B \) in \( K \). Two subobjects \( f : B \to A \) and \( g : C \to A \) in \( K \) are equivalent (that is, they represent the same subobject) if and only if there exists an isomorphism \( h : B \to C \) in \( K \) such that \( g \circ h = f \).
(To be more precise, we must define a subobject of $A$ to be an *equivalence class* of monic arrows with target $A$, where $f$ is *equivalent* to $g$ whenever $f = g \circ h$ for some isomorphism $h$. When we speak of a “subobject $f$” we shall always mean the equivalence class of $f$; the “subobject $f$” and “subobject $g$” will be the same whenever $f = g \circ h$.)

In category theory one often uses *commutative diagrams* to display relationships among arrows. Commutativity means that, whatever path one follows through the diagram from one object to another, the resulting arrow is the same. The relationship $f = g \circ h$ in the above definition can be expressed diagrammatically by asserting that the diagram (2.1) commutes:

\[
\begin{array}{ccc}
B & \xrightarrow{h} & C \\
\downarrow{f} & & \downarrow{g} \\
A & & 
\end{array}
\]

As another example, diagram (2.2) asserts that $f \circ p = g \circ q = r$:

\[
\begin{array}{ccc}
D & \xrightarrow{q} & C \\
\downarrow{p} & & \downarrow{g} \\
B & \xrightarrow{f} & A \\
\end{array}
\]

We shall see many examples of diagrams in this chapter. Diagrams provide an effective way to visually present the mathematical relationships among morphisms and are an integral part of the symbolic language of category theory.

*Subobject Category $P(A)$. If $A$ is an object in a category, we can define an inclusion relation $\subset$ between subobjects of $A$ in the following way.*

*Definition.* If $f : B \to A$ and $g : C \to A$ are each subobjects of $A$ in a category $K$, then $f \subset g$ if there exists an arrow $h : B \to C$ such that $f = g \circ h$; that is, diagram (2.1) commutes.

It is easily verified that the inclusion relation $\subset$ between subobjects of $A$ satisfies the three axioms for a partial ordering. We can thus form a
category \( P(A) \), whose objects are the subobjects of \( A \), and whose arrows describe the inclusion relation \( \subseteq \).

**Quotient Objects.** The concept of subobject, as all concepts in category theory, has a dual concept, obtained by simply reversing the direction of all arrows in the definition. This dual concept is the concept of a **quotient object**.

**Definition.** A quotient object of \( A \) in a category \( K \) is described by an epic arrow \( f: A \to B \). Two epics \( f: A \to B \) and \( g: A \to C \) in \( K \) are equivalent (they describe the same quotient object) if and only if there exists an isomorphism \( h: C \to B \) in \( K \) such that \( h \circ g = f \); that is, such that diagram (2.3) commutes:

![Diagram](image)

Let us see how this definition applies to the category of sets. Let \( f \) be a surjective function from a set \( A \) to a set \( B \). For each element \( b \in B \) we can consider the subset \( f^{-1}(b) \subseteq A \) consisting of all elements \( x \in A \) such that \( f(x) = b \). Since \( f \) is surjective, each of these subsets will be non-empty. Also, they will clearly be disjoint; that is, no two will have any elements in common. These subsets will thus partition the set \( A \) into disjoint **equivalence classes**, with each equivalence class corresponding to a unique element of \( B \). The set \( B \) is thus isomorphic to the set of equivalence classes (where each equivalence class is collapsed to a point value).

We now further observe that if one has a second surjection \( g: A \to C \) describing the same partition of \( A \) into equivalence classes, then there must be a connecting isomorphism \( h: C \to B \) such that \( h \circ g = f \). Furthermore, if \( g: A \to C \) is any surjection for which there exists such a connecting isomorphism then the two epics \( f, g \) must define the same partition of \( A \) into equivalence classes. The concept of quotient object, in the category of sets, thus corresponds to the concept of a set of equivalence classes.

In the category of groups, the concept of quotient object corresponds to a **quotient group**, and in the category of commutative rings, a **quotient**
We have seen in this section how certain fundamental structural concepts and relationships of abstract mathematical theories—isomorphisms, monic maps, epic maps, subobjects, quotient objects—which are ordinarily described in the individual language of each theory in terms of relationships among elements, can be described in the universal language of category theory in terms of the relationships among arrows. This description, in the universal language of the “gaps,” not only unifies these mathematical concepts of the diverse abstract theories, but further brings to light the underlying symmetry in the structure of knowledge—the principle of duality, which is hidden from view when one’s vision is focused on the elements of the mathematical structures rather than the gaps. In the coming sections we shall see how the commentary on the gaps provided by category theory has brought about a profound integration and unification of the diverse theories of modern mathematics, creating a connected wholeness of mathematical knowledge.

2.3 Universal Constructions

In this section we shall describe the basic structural concept of category theory, the concept of universality. There are a number of different equivalent ways of formulating this concept. We shall develop in this section one of these, the concept of a universal cone. We shall consider first several of the most important special cases of universal cones, and then we shall consider the general formulation of this concept.

*Terminal Objects.* The simplest expression of universality is the concept of a terminal object; this expresses the concept of a point value in the language of arrows.

*Definition.* An object $A$ in a category $K$ is a *terminal object* if for every object $B$ in $K$ there exists exactly one arrow from $B$ to $A$.

In the category of sets, any set $A$ containing a single element $p$ is a terminal object: for any set $B$ there is a unique function from $B$ to $A$, namely the function $f$ that takes every element of $x$ in $B$ to $p$, that is $f(x) = p$. 

*ring.* (In both cases, the equivalence classes are called the *cosets* of the quotient object.)
In the category of groups, the terminal object is the identity group, consisting of a single element \( e \) such that \( e \cdot e = e \). For every group \( G \), the function \( h: G \to \{e\} \) defined by \( h(g) = e \), for every \( g \) in \( G \), is the unique group homomorphism from \( G \) to the identity group.

In the category of commutative rings, the terminal object is the trivial ring consisting of the single element 0, with \( 0 + 0 = 0 \cdot 0 = 0 \). As in the case of groups, the function \( h: R \to \{0\} \) defined by \( h(r) = 0 \), for each \( r \) in \( R \), is the unique ring homomorphism from the ring \( R \) to the trivial ring.

In the category of topological spaces, any space consisting of a single point is a terminal object. In this case, it turns out that any function from a space \( X \) to a single point \( p, f: X \to \{p\} \) (treating \( \{p\} \) as a topological space), is continuous, and, as in the case of sets, must be unique.

We see from these examples that a terminal object is not necessarily unique. For example, in \( \text{Set} \) any set containing a single point will be a terminal object. What is always true, however, is that a terminal object will be unique up to isomorphism; that is, any two terminal objects will necessarily be isomorphic. This we can see as follows:

Suppose \( A \) and \( B \) are each terminal objects in a category \( K \). Then, in \( K \), there will exist a unique arrow \( f: A \to B \) (because \( B \) is a terminal object), and likewise there will exist a unique arrow \( g: B \to A \) (because \( A \) is a terminal object). The composition \( f \circ g: B \to B \) must then equal the identity arrow 1\(_B\) (because there is only one arrow from \( B \) to \( B \)), and likewise \( g \circ f \) must equal 1\(_A\). This means that \( f \) is an isomorphism, so that \( A \) and \( B \) are isomorphic.

As we will see, terminal objects in a category arise from universal constructions—each is a vertex of a universal cone. In general, universal constructions in category theory will only determine objects up to isomorphism, as we observed in the previous paragraph. When we think of the meaning of isomorphism in the context of abstract theories this makes sense; isomorphic objects have identical structural properties.

The concept of a terminal object expresses the concept of a point value \( A \) in the language of arrows. Since the language of arrows cannot speak directly about elements, we must express the concept in terms of arrows having target \( A \), arrows that “collapse” an arbitrary object \( B \) to the point value \( A \). This suggests the theme of Akshara in Maharishi Vedic Science, the collapse of fullness, \( A \), to the point value, \( K \). This is the pri-
consciousness-based education and mathematics

mordial expression of the dynamics of consciousness in Vedic Science, as we discussed in Section 1.5. As we explore the abstract description of the field of mathematical transformation in category theory, we shall find a number of mathematical expressions of the fundamental themes of analysis of transformation in Maharishi Vedic Science.

**Products.** Suppose we wish to characterize the cartesian product $A \times B$ of two sets $A$, $B$ in terms of arrows. Recall that $A \times B$ is the set consisting of all ordered pairs $(a, b)$ with $a$ in $A$ and $b$ in $B$. We observe first that there are two important arrows associated with $A \times B$, which project $A \times B$ onto $A$ and onto $B$ respectively; these are $\pi_1 : A \times B \to A$ defined by $\pi_1((x, y)) = x$ and $\pi_2 : A \times B \to B$ defined by $\pi_2((x, y)) = y$.

These two projection operators define the “cone” (2.4).

![Diagram](2.4)

There will be of course many possible cones, obtained by using other objects $X$ in place of $A \times B$, and other functions $f: X \to A$, $g: X \to B$ in place of $\pi_1$ and $\pi_2$. But, the cone with vertex $A \times B$ and projections $\pi_1$ and $\pi_2$ can be seen to have the following universal property: For any cone (2.5)

![Diagram](2.5)

that is, any pair of arrows $f: X \to A$ and $g: X \to B$, there exists a unique arrow $h: X \to A \times B$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$; that is, the “two-cone” diagram (2.6) commutes.

![Diagram](2.6)
The function $h : X \to A \times B$ is simply defined by $h(x) = (f(x), g(x))$.

This universal property of direct products allows us to generalize the concept of direct product to an arbitrary category.

**Definition.** In any category, the *product* of two objects $A, B$ consists of an object $A \times B$ together with two arrows, $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$, having the following universal property: for any object $X$ and arrows $f : X \to A$, $g : X \to B$, there exists a unique arrow $h : X \to A \times B$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$; that is, the two-cone diagram (2.6) commutes.

We use the notation $(f, g)$ to designate the arrow $h$, that is $h = (f, g)$. Thus, if $f : X \to A$ and $g : X \to B$, then $(f, g) : X \to A \times B$ is the unique arrow from $X$ to $A \times B$ making diagram (2.6) commute. In the category of sets, $(f, g)$ is the function $x \to (f(x), g(x))$.

We should mention here that there are unusual categories in which products of objects cannot always be defined; in such categories, one cannot find the unique arrow $h$ as in (2.6) in certain cases. When a category $K$ has the characteristic that products always exist, we will say that $K$ has products.

The cone (2.4) is called a *universal cone*. Its “universal” property is that every cone $C$ can be unfolded from the universal cone by means of a unique arrow $h$ to the vertex $A \times B$ of the cone: once $h$ is known, the projection maps $f$ and $g$ of $C$ are uniquely determined: $f = \pi_1 \circ h$ and $g = \pi_2 \circ h$.

You may recognize a similarity in feeling between the definitions of a product and a terminal object. The concept of a product in a category is in fact a special case of a terminal object. This is seen as follows. Suppose $A$ and $B$ are fixed objects in a category, and consider all possible cones (2.5).

These cones can be taken to be the objects of a new category, a *category of cones*. The objects of this category are cones of the type in (2.5). The arrows of this category are defined to be commutative two-cone diagrams of the form (2.7).
Diagram (2.7) represents an arrow from the cone (2.5) to the cone (2.8).

\[
\begin{array}{c}
\text{Diagram (2.7)} \\
X \xrightarrow{h} X' \xleftarrow{g} B \\
A \xrightarrow{f'} \xleftarrow{g'} A' \xrightarrow{f} B
\end{array}
\]

The definition of product simply says then that the product cone (2.4) is a terminal object in this category of cones; from any cone (2.5) there exists a unique arrow in the category of cones to the universal cone (2.4).

Suppose now we have a pair of arrows \( f : A \to C, \ g : B \to D \). We can define a product arrow \( f \times g : A \times B \to C \times D \) in the following way.

Let \( \pi_1, \pi_2 \) be the projections from \( A \times B \) to \( A \) and \( B \) respectively: \( \pi_1 : A \times B \to A, \pi_2 : A \times B \to B \). Consider the composite arrows \( f \circ \pi_1 : A \times B \to C, \ g \circ \pi_2 : A \times B \to D \). By the universal property of the product \( C \times D \), these two arrows determine a unique arrow \( (f \circ \pi_1, g \circ \pi_2) : A \times B \to C \times D \):

\[
\begin{array}{c}
A \times B \xrightarrow{f \times g} C \times D \\
A \xrightarrow{f} C \xleftarrow{\pi'_1} \xrightarrow{f \circ \pi_1} C \times D \\
B \xrightarrow{g} D \xleftarrow{\pi'_2} \xrightarrow{g \circ \pi_2} C \times D
\end{array}
\]

We use the notation \( f \times g \) to designate this arrow: \( f \times g = (f \circ \pi_1, g \circ \pi_2) \). In this way, every pair of arrows \( f : A \to C, \ g : B \to D \) determines a unique arrow \( f \times g : A \times B \to C \times D \). In the category of sets, \( f \times g \) is the function \( (a, b) \to (f(a), g(b)) \).

**Multiple Products.** The definition of product in a category generalizes to products of more than two objects. Thus, the *triple product* \( A \times B \times C \) is
characterized by a universal cone in which there are three projections, 
\( \pi_1 : A \times B \times C \rightarrow A \), \( \pi_2 : A \times B \times C \rightarrow B \), \( \pi_3 : A \times B \times C \rightarrow C \).  

![Diagram](image)

The universal property is the following. For any object \( X \) and arrows \( f : X \rightarrow A \), \( g : X \rightarrow B \), \( h : X \rightarrow C \), there is a unique arrow \( k : X \rightarrow A \times B \times C \) such that \( \pi_1 \circ k = f \), \( \pi_2 \circ k = g \), \( \pi_3 \circ k = h \). The universal cone (2.10) is a terminal object in the category of cones of the form (2.11).

\![Diagram](image)

**Application of Products: Internal Algebraic Structures.** The concepts of product and terminal object make it possible to internalize the definitions of algebraic structures such as monoids, groups, rings, so that one can talk about monoid objects, group objects, ring objects, and so on, within a wide range of categories. This is done in the following way.

We consider first the concept of a monoid; recall that a monoid is a set with structure, \((S, \cdot, e)\), where \( S \) is a set, \( \cdot \) is a binary operation on \( S \), and \( e \) is an element of \( S \) (the identity element) such that the following algebraic laws are satisfied:

\[
\begin{align*}
(1) & \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c \\
(2) & \quad e \cdot a = a \cdot e = a
\end{align*}
\]

Now the binary operation \( \cdot \) is a rule that assigns to each pair \((a, b)\) of elements of \( S \) a single element \( c \) of \( S \), which we designate \( a \cdot b \). This means that the operation \( \cdot \) is just a function \( f : S \times S \rightarrow S \), from the cartesian product \( S \times S \) to \( S \), where we simply designate the value \( f((a, b)) \) by the notation \( a \cdot b \). This characterization of a binary operation \( \cdot \) allows us to abstract this concept and apply it within virtually any category \( K \) (specifically, within any category that has products).
Definition. An internal binary operation on an object $A$ of $K$ is an arrow $f : A \times A \to A$.

Furthermore, we can express, in the language of arrows, what it means for such an operation to satisfy a given equational relationship. For example, the associative law $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ is expressed by the commutativity of diagram (2.12).

If we examine what this diagram says in terms of elements in the category of sets, we see that it indeed yields the associative law, as in (2.13) below. Proceeding to the left in the diagram, starting at $(a, b, c)$, leads to the expression $a \cdot (b \cdot c)$; proceeding to the right, starting again at $(a, b, c)$ leads to $(a \cdot b) \cdot c$. Commutativity of the diagram tells us that these two expressions must be equal.

The identity law $e \cdot a = a \cdot e = a$ can likewise be formulated in the language of arrows in the following way. Let 1 designate a set containing a single point $p$, $1 = \{p\}$, and let $g$ be the function from 1 to $S$ defined by $g(p) = e$, that is, $g$ takes the single element of 1 to the identity element $e$ of $S$. The identity law $e \cdot a = a \cdot e = a$ is then expressed by the commutativity of diagram (2.14):
Diagram (2.14) has the following meaning in terms of elements as shown in (2.15): In the left diagram, for example, applying the projection \( \pi_2 \) to \((p, a)\) yields \(a\), whereas applying \(g \times 1_A\) to \((p, a)\) yields \((e, a)\), and then applying the product function \(f\) to \((e, a)\) produces \(e \cdot a\). By commutativity of the diagram, it follows that \(e \cdot a = a\). Similar logic applies to the right diagram, leading to the equation \(a \cdot e = a\).

\[
\begin{array}{ccc}
  (p, a) & \xrightarrow{g \times 1_A} & (e, a) \\
  \downarrow & \searrow f & \downarrow \nearrow e \cdot a = a \\
  (a, e) & \xrightarrow{1_A \times g} & (a, p) \\
  \downarrow & \searrow f & \downarrow \nearrow a \cdot e = a \\
  & \swarrow \pi_1 & \\
  & & \ \end{array}
\]

The set 1 as we have been using it here, consisting of a single point, can be represented in the language of arrows as a terminal object in the category of sets. This allows us to generalize the identity law to an arbitrary category; the identity element \(e\) simply gets replaced by an arrow \(g : 1 \to A\), where 1 is a terminal object of the category.

We are thus led to the following definition of an internal monoid in a category.

**Definition.** Let \(K\) be a category that has products and a terminal object 1. An internal monoid in \(K\) is a triple \((A, f, g)\) where: (i) \(A\) is an object of \(K\), (ii) \(f\) is an arrow \(f : A \times A \to A\), and (iii) \(g\) is an arrow \(g : 1 \to A\), such that diagrams (2.12) and (2.14) commute.

In a similar way, one can define an internal group, internal ring, and so on. We shall see later how it is possible to even internalize the whole development of set theory in special categories called toposes.

**Pullbacks.** We shall describe now another fundamental type of universal cone called a pullback diagram.

\[
\begin{array}{ccc}
  X & \xrightarrow{g} & C \\
  \downarrow & \searrow k & \downarrow \nearrow B \\
  A & \xrightarrow{h} & C \\
  \downarrow & \swarrow f & \downarrow \nearrow C \\
  & \swarrow g & \\
  & & \ \end{array}
\]

(2.16)
Instead of considering cones of the form (2.5), we consider cones of the form (2.16), where we have added two fixed arrows $h : A \to C$, $k : B \to C$. This means that in the present context, a cone consists of a pair of arrows $f : X \to A$, $g : X \to B$ such that $h \circ f = k \circ g$—for this discussion, we will call such a cone a cone based on $h$ and $k$. These cones form the category of all cones based on $h$ and $k$. An arrow in this category, from a cone with vertex $X$ to a cone with vertex $X'$, is a commutative two-cone diagram of the form:

$$\begin{array}{c}
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet \\
\end{array}
\end{array}$$

A terminal object $T$ given by $p : D \to A$, $q : D \to B$ in this category of cones is called a pullback diagram; see (2.18).

$$\begin{array}{c}
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet \\
\end{array}
\end{array}$$

Being a terminal object in this category means that for every cone $C$ based on $h$ and $k$, there is a unique arrow in the category from $C$ to $T$. In particular, this means that a pullback diagram has the following universal property: For any cone $C$ of the form (2.16) there is a unique arrow $r : X \to D$ such that the two-cone diagram (2.19) commutes.

$$\begin{array}{c}
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet \\
\end{array}
\end{array}$$

In $\textbf{Set}$, the pullback object $D$ is the subset of the cartesian product $A \times B$ consisting of all ordered pairs $(a, b)$ such that $h(a) = k(b)$, and $p$ and $q$ are just the two projections, acting on such pairs in the usual way: $\pi_1((a, b)) = a$ and $\pi_2((a, b)) = b$. 

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Let us verify that this cone has the required universal property. Suppose we are given any cone of the form (2.16). That is, we are given a set $X$ and a pair of functions $f : X \to A$ and $g : X \to B$ such that $b \circ f = k \circ g$. Consider the function $(f, g) : X \to A \times B$, which maps each $x \in X$ to $(f(x), g(x))$. It follows from the relation $b \circ f = k \circ g$ that, for every $x \in X$, the point $(f(x), g(x))$ is in $D$. If we set $r(x) = (f(x), g(x))$, then it is clear that the two-cone diagram (2.19) commutes.

It remains to verify that $r$ is unique. Suppose now $r' : X \to D$ is any arrow making diagram (2.19) commute, with $r$ replaced by $r'$. Let $x$ be any element of $X$ and suppose $r'(x) = (a, b)$. It follows from the commutativity of diagram (2.19) that $f(x) = a$ and $g(x) = b$. Hence $r'(x) = (f(x), g(x)) = r(x)$, establishing uniqueness of $r$.

The pullback diagram is generally drawn as a square rather than a triangle:

\[
\begin{array}{ccc}
D & \xrightarrow{q} & B \\
\downarrow{p} & & \downarrow{k} \\
A & \rightarrow & C \\
\end{array}
\]  

(2.20)

The arrow $p$ is called the pullback of $k$ along $h$, and the arrow $q$ is called the pullback of $h$ along $k$.

If $h : A \to C$ is a given arrow, then the pullback of $h$ along itself is called the kernel pair of $h$. In Set, the kernel pair object $Y$ is the subset of $A \times A$ consisting of all pairs $(a, b)$ such that $h(a) = h(b)$.

Pullback diagrams can be used to internalize the concept of the intersection of two subsets of a set. We recall a subobject of $A$ is described by a monic arrow $f : B \to A$. Suppose we are given two subobjects, $f : B \to A$ and $g : C \to A$. If we complete the pullback square of diagram (2.21), the diagonal arrow $r = f \circ p : D \to A$ will be a monic arrow, which we define to be the intersection of the two subobjects $f$ and $g$.

\[
\begin{array}{ccc}
D & \xrightarrow{q} & C \\
\downarrow{p} & & \downarrow{g} \\
B & \xrightarrow{f} & A \\
\end{array}
\]  

(2.21)
Tracing through the diagram in the case in which all the arrows are inclusion maps in the category of sets shows why this definition makes sense. (Note that \( h: X \to Y \) is an inclusion map if \( h(x) = x \) for all \( x \) in \( X \)—this is another way of saying that \( X \subseteq Y \).)

**Equalizers.** Suppose we are given two functions, \( h, k \), from a set \( A \) to a set \( B \). The **equalizer** of \( h \) and \( k \) is defined to be the subset \( C \) of \( A \) consisting of all elements \( a \) such that \( h(a) = k(a) \). This concept can be described in the language of category theory by the universal cone:

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow{f} & & \downarrow{g} \\
C & \xrightarrow{f} & A & \xrightarrow{k} & B
\end{array}
\]

(2.22)

Diagram (2.22) says simply that \( h \circ f = k \circ f = g \). Since the arrow \( g \) is determined by the other arrows, it is redundant, and the diagram is generally drawn in the linear form:

\[
\begin{array}{ccc}
C & \xrightarrow{f} & A & \xleftarrow{k} & B
\end{array}
\]

(2.23)

The universal property of the equalizer diagram is the following: If \( s: X \to A \) is any arrow such that \( h \circ s = k \circ s \), then there exists a unique arrow \( r: X \to C \) such that \( f \circ r = s \), that is, diagram (2.24) commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{r} & C & \xrightarrow{f} & A & \xleftarrow{k} & B \\
\downarrow{s} & & \downarrow{} \\
\end{array}
\]

(2.24)

The arrow \( f: C \to A \) is called the **equalizer** of \( h \) and \( k \). It is always a monic arrow, and in \( \text{Set} \) it represents the subset \( C \) of \( A \) described above.

**Universal Cones.** After looking at these examples, we are ready to discuss the general concept of a universal cone. We begin with some commutative diagram. In the case of terminal objects, this beginning point
is simply the empty diagram. In the case of direct products, we begin with a *discrete diagram*—one that has no arrows:

\[
\begin{array}{cc}
A & B \\
\end{array}
\]

This is also the starting point for equalizers. In the case of pullbacks, the starting point is a diagram:

\[
A \rightarrow C \leftarrow B
\]

In principle, we can start with any diagram. Let us call the given diagram \( D \). A *cone over \( D \)* is then an object \( X \) together with arrows to all objects in the diagram \( D \) such that the new diagram commutes; we shall designate such a cone symbolically by a double arrow: \( X \Rightarrow D \).

A *universal cone* is a cone \( Y \Rightarrow D \) with the following universal property: for every cone \( X \Rightarrow D \) there exists a unique arrow \( h : X \rightarrow Y \) such that the two-cone diagram (2.25) commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow & & \downarrow \\
D & & D
\end{array}
\]

This property of the universal cone simply says that the universal cone is a terminal object in the category of cones over \( D \). The vertex \( Y \) of the universal cone is called the *limit of the diagram \( D \)*. The universal cone is also called a *limiting cone*.

**Universal Cocones.** The dual concept of a universal cone is called a *universal cocone*. Its definition is obtained by simply reversing the direction of all arrows in the definition of a universal cone. Some examples follow.

<table>
<thead>
<tr>
<th><strong>Universal Cone</strong></th>
<th><strong>Universal Cocone</strong></th>
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<tr>
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An *initial object* $A$ has the property that for every object $B$ there exists exactly one arrow $A \to B$ from $A$ to $B$. In **Set**, the null set is the initial object. In **Grp**, the identity group $\{e\}$ is the initial object. (In **Grp**, the initial object and terminal object are the same.) In **CR**, the ring of integers $\mathbb{Z}$ is the initial object. In **Top**, the empty space is the initial object.

The coproduct $A \coprod B$ of two objects $A, B$ is characterized by the universal cocone:

$$
\begin{array}{ccc}
A & \rightarrow & X \\
i_1 & & \downarrow h \\
\downarrow f & & \downarrow \text{commutes} \\
A \coprod B & \leftarrow & B \\
i_2 & & \\
\end{array}
$$

The universal property (sometimes called a *co-universal property* in this new context of cocones) is the following: for any object $X$ and arrows $f : A \to X$ and $g : B \to X$, there exists a unique arrow $h : A \coprod B \to X$ such that the diagram (2.27) commutes.

In **Set**, the coproduct $A \coprod B$ is the *disjoint union* of $A$ and $B$, that is, a union of disjoint copies of $A$ and $B$. More precisely, we define $A \coprod B$ as follows. Let $p : A \to A_1$ and $q : B \to B_1$ be isomorphisms such that $A_1$ and $B_1$ are disjoint. Set $A \coprod B = A_1 \cup B_1$. Let $r : A_1 \to A_1 \cup B_1$ and $s : B_1 \to A_1 \cup B_1$ be the *inclusion functions*; that is, $r(x) = x$ for all $x \in A_1$, and $s(x) = x$ for all $x \in B_1$. Set $i_1 = r \circ p$ and $i_2 = s \circ q$. These functions describe how $A$ and $B$ respectively are embedded in their coproduct $A \coprod B$. We outline the proof that the resulting diagram (2.26) has the required universal property: Suppose $f : A \to X$ and $g : B \to X$ are functions, as in (2.27). We define a function $h : A \coprod B \to X$ by letting $h(p(a)) = f(a)$ for all $a \in A$ and $h(q(b)) = g(b)$ for all $b \in B$; note that every element of $A_1$ has the form $p(a)$ for some $a \in A$, and likewise every element of $B_1$ has the form $q(b)$ for some $b \in B$. It is easy to check that this definition makes the diagram (2.27) commute. The final step is to verify that $h$ is the *only* function that makes (2.27) commute. To this end, suppose $k : A \coprod B \to X$ is a function for which (2.27)
does commute (with $h$ replaced by $k$); we show $k = h$. But now, by
commutativity of the diagrams, we must have:

$$h(p(a)) = f(a) = k(i_1(a)) = k(p(a)), \text{ for all } a \in A, \text{ and}$$
$$h(q(b)) = g(b) = k(i_2(b)) = k(q(b)), \text{ for all } b \in B.$$ 

Since $p$ and $q$ are isomorphisms, it follows that $h = k$, as required. This
completes the discussion on the definition of coproducts in $\textbf{Set}$.

In $\textbf{CR}$, the coproduct $A \amalg B$ is known in ring theory as the tensor product of the rings $A$ and $B$. In the special case when both $A$ and $B$ are the polynomial ring $\mathbb{Z}[X]$, the coproduct $\mathbb{Z}[X] \amalg \mathbb{Z}[X] = \mathbb{Z}[X_1, X_2]$, the polynomial ring in two indeterminates. In this case, $i_1 : \mathbb{Z}[X] \to \mathbb{Z}[X_1, X_2]$ is the homomorphism that takes $X$ to $X_1$, and $i_2 : \mathbb{Z}[X] \to \mathbb{Z}[X_1, X_2]$ is the homomorphism that takes $X$ to $X_2$. Universal-
ity of the coproduct diagram (2.26) is an easy consequence of the prop-
erties of the free rings $\mathbb{Z}[X]$ and $\mathbb{Z}[X_1, X_2]$ described in Section 1.6.

In $\textbf{Top}$, the coproduct $A \amalg B$ is the disjoint union of the topological
spaces $A$ and $B$. As in $\textbf{Set}$, one maps $A$ and $B$ respectively to disjoint
copies $A_1$, $B_1$, and defines topologies on each so that $A$ is isomorphic to
$A_1$ and $B$ is isomorphic to $B_1$ (relative to the category $\textbf{Top}$). The defini-
tion and proofs in this case are similar to those for $\textbf{Set}$. Finally, in $\textbf{Grp}$,
the coproduct $A \amalg B$ is the free product of $A$ and $B$.

In this section we have examined the most fundamental examples of
universal constructions in category theory, as well as the general con-
cept, the concept of a universal cone, and the dual concept, that of a
universal cocone. Universal constructions lie at the heart of category
theory. We shall conclude this section by noting some parallels to the
description of the nature of the “gap” in Maharishi Vedic Science.

Vedic Science locates the basic mechanics of transformation of natural law in the gaps between the sequentially emerg-
ing expressions of the Rik Veda. In $\textit{Nyaya}$, the Vitarka Samgraha enumerates four fundamental qualities of the gap:

1. pradhvamsabhava: annihilation, or convergence to the point;
2. atyantabhava: emptiness or nothingness;
3. anyonyabhava: infinite dynamism;
4. pragabhava: creation, or expansion from the point.
In the transformation of each syllable into the succeeding syllable, these characteristics of the gap are expressed sequentially. The preceding syllable is first annihilated, the expression of pradhvamsabhava. This leaves a value of a vacuum or nothingness, atyantabhava. This vacuum contains within it, however, infinite dynamism, the unmanifest liveliness of “all possibilities,” anyonyabhava. This infinite dynamism is the expression of the unmanifest, self-interacting dynamics of the Samhita. Through these dynamics, the succeeding syllable is created, the expression of pragabhava.

Universal constructions in category theory provide a somewhat parallel analysis of the sequential mechanics of transformation. The fundamental theme is that all transformations of a given class can be expressed as a sequence of two transformations, passing through an intermediate point value in which all possible values of transformation are lively. This “point of all possibilities” is the vertex of the universal cone or cocone. Being an object of a category, it is the expression of a “point value” in category theory; we have observed how the objects of a category simply label the start and end of arrows, and have no internal structure, thereby expressing the “emptiness” of a point. The liveliness of “all possibilities” in this point is expressed in the fact that all possible cones or cocones can be unfolded from this point by applying a suitable arrow. The value of convergence to a point is expressed in the structure of a universal cocone, in which a diagram, an extended structural object of category theory, collapses to a single object, a point value. The value of expansion from a point is similarly expressed in the structure of a universal cone.

The deeper connections between category theory and Maharishi Vedic Science will require the study of the arrows of category theory, called functors; these we shall introduce in the next section.

2.4 Functors
We have seen how every abstract mathematical theory describes a category. The objects of the category are all the different possible mathematical structures that satisfy the axioms of the theory. The arrows of the category are all possible functions from one structure to another that preserve the integrity of the defining structural relationships. For example, the theory of commutative rings describes the category of
commutative rings, whose objects are all possible commutative rings, and whose arrows are all possible ring homomorphisms.

Now category theory has itself the structure of an abstract mathematical theory, so it itself describes a category. The objects of this category are all the different structures that satisfy the axioms of category theory. That is, the objects of this category are all the diverse categories themselves; it is the category of all categories. The arrows of this category are all the transformations from one category to another that preserve the integrity of the defining structural relationships of category theory. These transformations are called functors.

Definition. Let $K, K'$ be categories. A functor $F$ from $K$ to $K'$ is a rule that assigns to each object $A$ in $K$ an object $F(A)$ in $K'$, and that assigns to each arrow $f: A \to B$ in $K$ an arrow $F(f): F(A) \to F(B)$ between the corresponding objects in $K'$, such that

(i) $F(g \circ f) = F(g) \circ F(f)$, and
(ii) $F(1_A) = 1_{F(A)}$.

Condition (i) says that $F$ preserves the composition operation of arrows. Equivalently, $F$ takes commutative diagrams in $K$ to commutative diagrams in $K'$: if $h = g \circ f$ in $K$, then $F(h) = F(g) \circ F(f)$ in $K'$, as in Diagram 2.28.

Condition (ii) says that $F$ takes identity arrows in $K$ to identity arrows in $K'$.

Note that conditions (i) and (ii) are analogous to the requirements that a function $f$ (from one ring to another) must satisfy to be a ring homomorphism:

(i) (preservation of the operations) $f(a \cdot b) = f(a) \cdot f(b), f(a + b) = f(a) + f(b)$;
(ii) (preservation of identity) $f(1) = 1'$.
Functors are just the homomorphisms between categories. We saw in Chapter 1 the central role played by homomorphisms in the theory of commutative rings. For the development of every abstract theory, the study of the morphisms of the theory plays a central role. Category theory is no exception; the study of functors lies at the heart of the development of category theory.

Functors provide a natural language to describe transformations between mathematical theories. For example, in algebraic topology one studies functors from the category of topological spaces to algebraic categories, such as the category of groups or the category of commutative rings. The category of topological spaces is the category described by the abstract theory called point-set topology, which is the most abstract foundation for geometry. The functors studied in algebraic topology describe transformations from this theory to the fundamental algebraic theories, such as group theory and ring theory. On this basis, a profound unification and integration of the continuous perspective of geometry with the discrete perspective of algebra has been achieved.

This is just one example of the power of the abstract viewpoint of category theory, which can describe transformations between mathematical theories in general. We have discussed above the way in which category theory is the universal language of the “gap” in mathematics. In his exposition of Vedic Science, Maharishi has brought out the significance of different levels of gaps in the structure of the Veda; there are gaps between syllables, gaps between Padas, gaps between Richas, gaps between Suktas, and gaps between Mandalas. Each level of gap has its own mechanics of transformation. In modern mathematics, category theory provides a commentary on two levels of gaps. The first is the gap between the different structures described by a particular theory: the gap between two groups, or two rings, and so on. The dynamics of transformation belonging to these gaps is described in terms of the arrows of the different theories. The second level of gaps is that between the different theories themselves. The dynamic structure of these great gaps is described in terms of functors, mathematical transformations from one category to another. It is this level of commentary on the gaps that has been able to integrate all the diverse abstract theories of modern mathematics into the unified, holistic structure of the category of all categories.
In the remainder of this section we shall consider a number of fundamental examples of functors. In the next section we shall discuss the most fundamental and important relationship between functors, the relationship of adjointness. This will provide a new insight into the relationship between the category of sets and the categories of structures described by abstract algebraic theories. This chapter will conclude with a discussion of a foundational approach based upon the self-referral application of the category-theoretic viewpoint to describe the structure of category theory itself, the theory of the category of all categories.

Example 1: Identity Functors. Every category $K$ has an identity functor $I: K \rightarrow K$ that simply takes each object $A$ in $K$ to itself, and each arrow $f$ in $K$ to itself: $I(A) = A$, $I(f) = f$.

Example 2: Forgetful Functors. We saw in Chapter 1 that the abstract theories of modern mathematics are arranged in a hierarchy grounded in the category of sets. The category of sets represents the undifferentiated level of mathematical structure; the other categories arise by sequentially imposing more and more structural relationships. The objects in each of these categories are sets for which specific structural relationships are defined. For example, a group is a set for which a single operation is defined (group multiplication), which satisfies the axioms of group theory; a commutative ring is a set for which two operations are defined, addition and multiplication, which satisfy the commutative ring axioms; and so on. A forgetful functor starts with a category with structure and simply forgets some, or all, of the structural relationships. For example, there is a forgetful functor from $\text{Grp}$ to $\text{Set}$, which simply forgets the group multiplication operation, leaving a set without structure. There is also a forgetful functor from $\text{CR}$ to $\text{Set}$, which forgets both addition and multiplication.

Example 3: Inclusion Functors. It often happens that the objects of one category are included in a larger category. For example, every abelian group is certainly a group, so the category of abelian groups is contained within the category of groups. This gives rise to an inclusion
functor \( J : \text{Ab} \to \text{Grp} \), which takes every abelian group to the same group, regarded as an element of the category of groups.

Example 4: Hom-Functors. Let \( K \) be any category and let \( A, B \) be any objects in \( K \). The hom-set \( K(A, B) \) is defined to be the set consisting of all possible arrows from \( A \) to \( B \). The hom-set \( K(A, B) \) is thus always an object in the category of sets.

Suppose now \( A \) is any fixed object in \( K \). The object \( A \) gives rise to a functor \( K(A, -) \) from \( K \) to \( \text{Set} \), called a hom-functor, which takes each object \( B \) of \( K \) to the hom-set \( K(A, B) \); that is, for each object \( B \) in \( K \),

\[
K(A, -)(B) = K(A, B).
\]

Moreover, if \( f : B \to C \) is any arrow of \( K \), then the hom-functor \( K(A, -) \) takes \( f \) to the function \( r \) from \( K(A, B) \) to \( K(A, C) \), defined as follows: if \( g : A \to B \) belongs to \( K(A, B) \), then \( r(g) \) is defined to be the \( K(A, C) \) arrow \( f \circ g : A \to C \).

In the special case when \( K \) is the category of sets, the hom-functor \( K(A, -) \) is a functor from \( \text{Set} \) to \( \text{Set} \); that is, it is a functor from the category of sets to itself. This functor is called an exponential functor and is designated \( -^A \). The exponential notation \( B^A \) is used to designate the set of all functions from \( A \) to \( B \), that is, the hom-set \( \text{Set}(A, B) \). Therefore,

\[
\text{Set}(A, B) = B^A \text{ for each } B \text{ in } \text{Set}.
\]

Example 5: Product Functors. Let \( K \) be a category for which products \( A \times B \) always exist. Let \( A \) be a fixed object in \( K \). Then the product functor \( - \times A \) from \( K \) to \( K \) takes each object \( B \) in \( K \) to the object \( B \times A \); that is,

\[
( - \times A)(B) = B \times A.
\]

Moreover, for any arrow \( f : B \to C \) of \( K \), the product functor \( - \times A \) takes \( f \) to the arrow \( f \times 1_A : B \times A \to C \times A \).

Example 7: Free-Algebra Functors. The generation of free mathematical structures is always described by a functor from the category of sets to the appropriate algebraic category. The free-commutative-ring functor \( F : \text{Set} \to \text{CR} \) is defined as follows.

(i) If \( S \) is any set, \( F(S) \) is the free commutative ring generated by \( S \).

(ii) Suppose \( f \) is an arrow in \( \text{Set} \); that is, \( f \) is a function, \( f : S \to V \). Then \( F(f) \) is the ring homomorphism \( F(f) : F(S) \to F(V) \), defined as follows: Let \( g \) be the function from \( V \) to \( F(V) \) that embeds the set \( V \) of generators in the free ring \( F(V) \); that is, \( g(a) = X_a \), where
$X_a$ is the element of $F(V)$ corresponding to the element $a$ of $V$ (see Section 1.6). Then $g \circ f: S \rightarrow F(V)$ is a function from the set $S$ to the commutative ring $F(V)$. By the universal property of the free commutative ring $F(S)$, the function $g \circ f$ extends uniquely to a homomorphism $h: F(S) \rightarrow F(V)$. We set $F(f) = h$.

**Example 8: Diagrams.** Every commutative diagram in a category $K$ defines a functor from a suitable index category $J$ to $K$. For example, a commutative triangle in $K$ is simply a functor from the category 3 to $K$. The triangle of diagram (2.29) corresponds to the functor $F$ from 3 to $K$ defined by: $F(L_1) = A$, $F(L_2) = B$, $F(L_3) = C$, $F(l_{12}) = f$, $F(l_{23}) = g$, $F(l_{13}) = h$.

![Diagram 2.29](Image)

**Example 9: Diagonal Functors.** We considered categories of cones in the previous section. More generally, one can consider categories of diagrams of any specified form. For example, let $K$ be a category, and consider all commutative diagrams in $K$ of the form displayed in diagram (2.30); that is, all commutative triangles in $K$.

![Diagram 2.30](Image)

These are the objects of a category of diagrams; the arrows of this category from the diagram (2.30) to the diagram (2.31) are defined to be triples of arrows $(k_1, k_2, k_3)$ in $K$ from the objects of the first diagram to the corresponding objects of the second such that the composite diagram (2.32) commutes.
Let us call the resulting category of diagrams $K'$.

We can define now the diagonal functor $F$ from $K$ to $K'$ as follows. If $A$ is an object of $K$, then $F(A)$ is the diagram (2.33) all of whose objects are $A$ and all of whose arrows are the identity arrow $1_A$.

If $f : A \rightarrow B$ is an arrow of $K$, then we define $F(f)$ to be the triple $(f, f, f) : F(A) \rightarrow F(B)$.

A diagonal functor can be similarly defined for any category of diagrams over $K$.

**Example 10: Inverse-Image Functors.** Suppose $f$ is a function from a set $A$ to a set $B$. If $C$ is any subset of $B$, the inverse image of $C$, designated $f^{-1}(C)$, is defined to be the set of all points $a$ in $A$ such that $f(a)$ is in $C$. Thus, for each subset $C$ of $B$, $f^{-1}(C)$ is a subset of $A$. In this way we obtain a function $f^{-1}$ from the power set $P(B)$ to the power set $P(A)$.

Now suppose $C, D$ are each subsets of $B$, and $C \subseteq D$ (that is, $C$ is a subset of $D$). Then it is immediate that $f^{-1}(C) \subseteq f^{-1}(D)$. This means that the function $f^{-1} : P(B) \rightarrow P(A)$ preserves the partial ordering $\subseteq$ defined
by the inclusion relation. It follows that $f^{-1}$ is in fact a functor from the category $P(B)$ to the category $P(A)$ (see Section 2.1).

More generally, let $K$ be any category having pullbacks and let $f: A \to B$ be an arrow of $K$. Then the “pullback along $f$” defines a functor $f^{-1}$ from $P(B)$ to $P(A)$, where $P(A)$ and $P(B)$ are the categories of subobjects of $A$ and $B$, respectively. Specifically, if $g : C \to B$ is a subobject of $B$, then $f^{-1}(g) : D \to A$ is defined to be the pullback of $g$ along $f$.

2.5 Adjoint Functors

One of the striking features of category theory is the way in which functors of interest tend to occur in perfectly balanced pairs representing complementary values of transformation; these are called adjoint functors. The concept of adjoint functors was introduced by Daniel Kan (1958) and has been central to virtually all major developments in category theory since then. In this section we shall define the concept of adjoint functors and examine a number of examples. See Mac Lane (1971) for a more complete and detailed discussion.

Suppose $K, K'$ are categories. A pair of adjoint functors consists of two functors, $F : K \to K'$ and $G : K' \to K$, for which the following situation holds: For every object $A$ in $K$ and every object $B$ in $K'$, the relationship between $F(A)$ and $B$ in $K'$ directly mirrors the relationship between $A$ and $G(B)$ in $K$.

Specifically, what is required is that for every object $A$ in $K$ and every object $B$ in $K'$ there is an isomorphism of hom-sets $\theta_{A,B} : K(A, G(B)) \to K'(F(A), B)$. This means that there is a rule $\theta_{A,B}$ that associates to each arrow $f : A \to G(B)$ in $K$ a corresponding arrow $\theta_{A,B}(f) : F(A) \to B$ in $K'$ such that the correspondence $f \mapsto \theta_{A,B}(f)$ is a one-to-one correspondence between the set $K(A, G(B))$ of all arrows from $A$ to $G(B)$ in $K$ and the set $K'(F(A), B)$ of all arrows from $F(A)$ to $B$ in $K'$.

There are, in addition, the following coherence requirements:

(i) For every $g : B \to C$ in $K'$, we have $\theta_{A,B,G(C)}(G(g) \circ f) = g \circ \theta_{A,B}(f)$.

(ii) For every $h : D \to A$ in $K$, we have $\theta_{D,B}(f \circ h) = \theta_{A,B}(f) \circ F(h)$.

The coherence conditions (i) and (ii) ensure that the different one-to-one correspondences between hom-sets that one obtains as one varies the objects $A$ in $K$ and $B$ in $K'$ all fit together in a coherent way. This
ensures that the pairing \( f \mapsto \theta_{A,B}(f) \) is natural, and not just an arbitrarily defined one-to-one correspondence.

When the above situation holds, \( F \) is said to be a left adjoint of \( G \), and \( G \) a right adjoint of \( F \). The triple \((F, G, \theta)\) is called an adjoint situation or an adjunction. If \( g = \theta_{A,B}(f) \), then \( g \) is called the left adjoint of \( f \), and \( f \) is called the right adjoint of \( g \); we write \( g = \text{lad}(f) \) and \( f = \text{rad}(g) \).

Following are several examples of adjoint situations.

Example 1: Product Functor - \( \times A \) and the Exponential Functor \(-^A\). Let \( A \) be any set, and consider the product functor \(- \times A : \text{Set} \to \text{Set}\), which takes any set \( B \) to the cartesian product \( B \times A \). The right adjoint of this functor is the exponential functor \(-^A : \text{Set} \to \text{Set}\), which takes any set \( C \) to the set \( C^A \) consisting of all possible functions from \( A \) to \( C \). The adjunction relation between these two functors means the following: For any sets \( B, C \) there exists a one-to-one correspondence between the arrows \( B \to C^A \) and the arrows \( B \times A \to C \). Since the arrows in the category of sets are all possible functions, the adjunction must establish a one-to-one correspondence between functions \( f : B \to C^A \) and functions \( g : B \times A \to C \). This one-to-one correspondence is defined as follows:

Suppose \( f \) is a function from \( B \) to \( C^A \). Then we define \( g \) to be the function from \( B \times A \) to \( C \) that takes any element \((b, a)\) of \( B \times A \) to \( f(b)(a) \); that is, \( g((b, a)) = f(b)(a) \). (Since \( f(b) \) is an element of \( C^A \), \( f(b) \) is a function from \( A \) to \( C \), so if we evaluate this function \( f(b) \) at an element \( a \) of \( A \), we obtain an element of \( C \).)

Example 2: Inclusion Functors and Reflection Functors. (This example is for group theorists only!) Let \( J \) be the inclusion functor \( J : \text{Ab} \to \text{Grp} \). Then the left adjoint to \( J \) is the factor-commutator functor \( \text{Grp} \to \text{Ab} \) that takes any group \( G \) to the quotient group \( G/N \), where \( N \) is the commutator subgroup of \( G \). (Such \( N \) always exists and has the property that \( G/N \) is the largest abelian quotient group of \( G \).) In this example, the adjunction is based on the following universal property of \( G/N \): If \( G \) is any group and \( A \) is any abelian group, then every homomorphism \( f : G \to A \) (in \( \text{Grp} \)) factors uniquely through \( G/N \); that is, there is a unique homomorphism \( g : G/N \to A \) (in \( \text{Ab} \)) such that \( f = g \circ p \), where \( p \) is the projection homomorphism \( p : G \to G/N \).
In general, a left adjoint to an inclusion functor is called a *reflection functor*; it provides a “mirror” that reflects a category within a subcategory.

**Example 3: Forgetful Functors and Free-Algebra Functors.** Let $F$ be the forgetful functor $F : \text{CR} \rightarrow \text{Set}$. Then the left adjoint of $F$ is the free commutative-ring functor $T : \text{Set} \rightarrow \text{CR}$, which takes any set $S$ to the free commutative ring generated by $S$.

The adjunction relation between the forgetful functor $F$ and the free commutative ring functor $T$ signifies the following: Let $A$ be any set and let $R$ be any commutative ring. Then for every function $f$ from $A$ to the underlying set $F(R)$ of $R$, there corresponds a unique ring homomorphism $g$ from the free commutative ring $T(S)$ to $R$.

Now the category of sets is the undifferentiated field, in which the full range of transformations is available; the arrows are all possible functions. The category of commutative rings is a differentiated level of structure for which the available channels of transformations are more restricted; the arrows are required to be functions that preserve addition and multiplication, that is, ring homomorphisms. The free commutative ring functor provides a way of taking the total range of possibilities for transformation for a set $A$ in the undifferentiated field ($\text{Set}$), and “lifting” all these possibilities to the differentiated level ($\text{CR}$), so that they all can be actualized as ring homomorphisms. The free commutative ring functor in this way gives expression to the “all possibilities” quality of the undifferentiated field in the boundaries of the differentiated field, by creating a “universal” commutative ring structure generated from a given set.

**Example 4: Diagonal Functors and Limits, Colimits.** Let $K$ be a category and let $K'$ be the category of diagrams over $K$ of some fixed type. Let $F$ be the diagonal functor from $K$ to $K'$. Then the left adjoint of $K$ takes each diagram to its colimit, and the right adjoint of $K$ takes each diagram to its limit. We shall illustrate this adjunction by means of an example.

Let $K'$ be the category of all diagrams of the form shown in (2.35),

$$
\begin{array}{ccc}
C & \xrightarrow{h} & A & \xleftarrow{k} & B \\
\end{array}
$$

(2.35)
where $A$, $B$, $C$ range over all objects of $K$. The diagonal functor $F$ takes each object $S$ of $K$ to the diagram (2.36)

$$S \xrightarrow{1_S} S \xleftarrow{1_S} S$$ (2.36)

The right adjoint $G$ is the functor from $K'$ to $K$ that takes each diagram (2.35) to its limit, that is, the vertex object $D$ of the universal cone over diagram (2.35). In this example, $D$ is the pullback object. The functor $G$ acts on arrows as follows:

Let $g$ be an arrow from the diagram (2.37)

$$\begin{array}{c}
C' \xrightarrow{h'} A' \xleftarrow{k'} B' \\
\begin{array}{c}
\downarrow g_1 \\
\downarrow g_2 \\
\downarrow g_3 \\
C \xrightarrow{h} A \xleftarrow{k} B
\end{array}
\end{array}$$ (2.37)

to the diagram (2.35). This means $g$ is a triple of arrows of $K$, $g = (g_1, g_2, g_3)$, such that the diagram (2.38) commutes.

$$\begin{array}{c}
C' \xrightarrow{h'} A' \xleftarrow{k'} B' \\
\begin{array}{c}
\downarrow g_1 \\
\downarrow g_2 \\
\downarrow g_3 \\
C \xrightarrow{h} A \xleftarrow{k} B
\end{array}
\end{array}$$ (2.38)

Let $D$ be the limit of diagram (2.35) and let $D'$ be the limit of diagram (2.37). Let diagram (2.39) be the universal cone (pullback diagram) over diagram (2.35).

$$\begin{array}{c}
C' \xrightarrow{r} D' \xleftarrow{s} B' \\
\begin{array}{c}
\downarrow h' \\
\downarrow g_1 \\
\downarrow g_2 \\
C \xrightarrow{h} A \xleftarrow{k} B
\end{array}
\end{array}$$ (2.39)

Putting together diagram (2.38) and (2.39) we obtain the cone (2.40).

$$\begin{array}{c}
C' \xrightarrow{g_1 \circ r} D' \xleftarrow{g_2 \circ s} B' \\
\begin{array}{c}
\downarrow h' \\
\downarrow g_1 \\
\downarrow g_2 \\
C \xrightarrow{h} A \xleftarrow{k} B
\end{array}
\end{array}$$ (2.40)

By the universal property of the pullback diagram over (2.35), there is determined a unique arrow $f$ from $D'$ to $D$. We set $G(g) = f$. This completes the description of the functor $G$.

It remains to verify that $G$ is the right adjoint to $F$. For this, we must establish a one-to-one correspondence between arrows from (2.36) to
(2.35) in \( K' \), and arrows from \( S \) to \( D \) in \( K \). But an arrow from (2.36) to (2.35) is simply a diagram of the form (2.41)

\[
\begin{array}{c}
S \\
\downarrow g_1 \\
C \\
\downarrow h \\
A \\
\downarrow k \\
B \\
\end{array}
\begin{array}{c}
1_S \\
\downarrow g_2 \\
S \\
\downarrow g_3 \\
S \\
\end{array}
\]

and such a diagram collapses to the cone (2.42)

\[
\begin{array}{c}
S \\
\downarrow g_1 \\
C \\
\downarrow h \\
A \\
\downarrow k \\
B \\
\end{array}
\begin{array}{c}
1_S \\
\downarrow g_2 \\
S \\
\downarrow g_3 \\
S \\
\end{array}
\]

because of the identity arrows \( 1_S \).

The adjunction therefore must establish a one-to-one correspondence between cones over (2.35) having vertex \( S \) and arrows from \( S \) to \( D \). But every such cone with vertex \( S \) corresponds to a unique arrow \( S \to D \) by virtue of the universal property of the pullback diagram. This correspondence defines the adjunction between \( F \) and \( G \).

**Example 5: Inverse Image Functors and Existential and Universal Quantification Functors.** Let \( K \) be a category, and let \( f : A \to B \) be an arrow in \( K \). Then we have seen that \( f \) gives rise to an inverse-image functor \( f^{-1} : P(B) \to P(A) \). The left adjoint of \( f^{-1} \) is called *existential quantification along* \( f \) and is designated \( \exists_f \). The right adjoint of \( f^{-1} \) is called *universal quantification along* \( f \) and is designated \( \forall_f \).

Let us see how these functors are defined in the category of sets. Suppose \( f \) is a function from \( A \) to \( B \). Let \( S \) be a subset of \( A \). Then:

(i) \( \exists_f S \) is the subset \( T \) of \( B \) consisting of all points that are images of points in \( S \), that is, all points \( r \) of \( B \) with the property that there exists at least one point \( u \) in \( S \) such that \( f(u) = r \). This subset of \( B \) is called the *direct image* of \( S \).

(ii) \( \forall_f S \) is the subset \( R \) of \( B \) consisting of all points that are images of only points in \( S \), that is, all points \( r \) of \( B \) with the property that \( f^{-1}(r) \) is a subset of \( S \).

To verify the adjunction relation, we recall that the arrows of \( P(A) \) and \( P(B) \) correspond to the inclusion relation between subsets: If
$C \subseteq D$ then there exists exactly one arrow from $C$ to $D$; otherwise, there is no arrow from $C$ to $D$. The adjunction relation between $f^{-1}$ and $\exists f S$ thus says: If $S$ is any subset of $A$ and $U$ is any subset of $B$, then $\forall f.S \subseteq U$ if and only if $S \subseteq f^{-1}(U)$. The adjunction relation between $f^{-1}$ and $\forall f$ says: If $S$ is any subset of $A$ and $U$ is any subset of $B$, then $U \subseteq \exists f.S$ if and only if $f^{-1}(U) \subseteq S$.

The existential quantification functor $\exists f$ can be used to give a categorical construction of the image of an arrow. We recall that the image of a function $f$ from a set $A$ to a set $B$ is defined to be the subset of $B$ consisting of all elements $b \in B$ that are “hit” by the arrow; that is, all elements $b$ for which there exists an element $a \in A$ such that $f(a) = b$. It is easily verified that the image of the function $f$ is the subset of $B$ obtained by applying the functor $\exists f$ to the subset $A \subseteq A$. This property can be used to generalize the concept of the image of a function to an arbitrary category; the image of an arrow $f: A \to B$ is defined to be $\exists f 1_A$, where $1_A: A \to A$ is the identity subobject of $A$.

The functors $\exists f$ and $\forall f$ play an important role in the development of categorical logic. They make it possible to represent the logical quantifiers $\forall$ (for all) and $\exists$ (there exists) by functors, and thereby interpret the symbolic language of mathematical logic internally, within a category. This technique will be essential for our development of topos theory in Chapter 3.

A striking feature of certain adjoint situations that tend to occur frequently in practice is that one of the functors is very simple and “silent,” doing almost nothing, whereas its adjoint is extremely dynamic and complex. For example, the forgetful functor does nothing but “forget” structural relationships; its adjoint, the free-algebra functor, dynamically generates an algebraic structure from any set. Again, an inclusion functor simply expresses the way one category is included within a larger category; it doesn’t change anything. Its adjoint, a reflection functor, can transform structures in a way that is extremely complex. Diagonal functors transform objects into diagrams built entirely from identity arrows, expressing the value of non-change. Their adjoints express the dynamic transformation of diagrams into their limits and colimits. Product functors, in the case of categories associated with abstract theories, express the combination of structures in an essentially trivial way. The product ring
\( A \times B \), for example, combines the rings \( A \) and \( B \) in such a way that the two do not interact with one another; the two components are simply added and multiplied independently. The adjoint functor is a hom-functor that takes each ring \( B \) to the hom-set \( \mathbf{CR}(A, B) \) expressing the total dynamism of all values of transformation from the ring \( A \) to the ring \( B \).

In all these examples, the adjunction expresses the unity of two complementary values of transformation. In one part of the adjunction, the value of non-change or silence predominates, while in the other, the value of dynamic transformation of structure is dominant. The adjunction, which serves to unify these two, is itself formulated in terms of the isomorphism of hom-sets in two categories, thereby expressing a theme of perfect balance, in which values of transformation in two categories are seen to directly mirror one another.

We find here the expression of two deep themes of Maharishi Vedic Science: the unification of absolute silence and infinite dynamism in the structure of the Samhita, and the role of perfect balance among the different fundamental impulses of natural law in maintaining that integrity of that unified wholeness of nature.

2.6 Natural Transformations
We have thus far considered two basic levels of transformation in category theory: transformations between the objects of a category, described by arrows, and transformations between categories, described by functors. In this section we shall consider a third level of transformation; these are transformations between functors, called natural transformations. We shall see that every adjunction has associated with it two natural transformations, the unit and counit of the adjunction. From these we shall derive a type of structure called a triple, which contains, in many important cases, complete knowledge of the adjunction. The application of this construction to the forgetful functor will provide a striking mathematical parallel to deep principles of Maharishi Vedic Science, which locate the blueprint for the expressed field of natural law in the internal, self-referral dynamics of the Samhita.

We shall begin with the definition of the concept of a natural transformation.
**Definition.** Let $K, K'$ be categories, and let $F, G$ be functors from $K$ to $K'$. A **natural transformation** $\tau : F \to G$ is a rule that assigns to every object $A$ of $K$ an arrow $\tau(A) : F(A) \to G(A)$ in $K'$, such that the following “coherence” condition is satisfied: for every arrow $f : A \to B$ in $K$, $\tau(B) \circ F(f) = G(f) \circ \tau(A)$; that is, diagram (2.43) commutes.

\[
\begin{array}{cccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\downarrow{\tau(A)} & & \downarrow{\tau(B)} \\
G(A) & \xrightarrow{G(f)} & G(B)
\end{array}
\]  

(2.43)

The “coherence” condition guarantees the arrows $\tau(A)$ are not defined arbitrarily, but are defined in a “natural” way, so they all act in a coherent, perfectly correlated way. This coherence condition can be motivated as follows. Suppose we take any commutative triangle (2.44) in $K$.

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\text{ } & \text{ } & \text{ } \\
B & \text{ } & \text{ }
\end{array}
\]  

(2.44)

The functors $F$ and $G$ take this triangle to two triangles in $K'$, as indicated in diagram (2.45).

\[
\begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
A & \xrightarrow{h} & C \\
\text{ } & \text{ } & \text{ } \\
B & \xrightarrow{f} & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ }
\end{array}
\]  

(2.45)

The natural transformation $\tau$ provides arrows between the corresponding vertices of these two triangles: $\tau(A) : F(A) \to G(A)$, $\tau(B) : F(B) \to G(B)$, $\tau(C) : F(C) \to G(C)$. The coherence condition is just the requirement that the diagram (2.46) obtained by joining the two triangles with these three arrows commutes.
The concept of natural transformation can be used to construct categories whose objects are functors.

**Definition.** Let $J$, $K$ be categories. The functor category $K^J$ is the category whose objects are all possible functors from $J$ to $K$, and whose arrows are all possible natural transformations between these functors.

We have actually already encountered examples of functor categories, in the guise of categories of diagrams. We saw in Section 2.4 how every diagram in a category $K$ is just a functor from a suitable index category $J$ to $K$. The category of all diagrams of a given form is simply the category $K^J$ of all functors from $J$ to $K$. The arrows of this category of diagrams are just the natural transformations between these functors.

In terms of $J$, we can describe the diagonal functor $D$ as follows: $D$ is the functor from $K$ to the functor category $K^J$ that takes each element $A$ of $K$ to the functor $F : J \to K$ defined by: (i) $F(C) = A$ for every object $C$ in $J$, and (ii) $F(f) = 1_A$ for every arrow $f$ in $J$. That is, the functor $D(A) = F$ simply collapses the index category $J$ to the point $A$ of $K$. The diagonal functor in this way is seen to give mathematical expression to the phenomenon of *Akshara*.

**Units and Counits.** There are two special natural transformations associated with an adjoint situation called the unit and counit of the adjunction. Suppose we have an adjunction $(F, G, \theta)$, with $F : K \to K'$ and $G : K' \to K$. Let $A$ be any object of $K$, and consider the identity arrow $1_{F(A)} : F(A) \to F(A)$ in $K'$. The right adjoint of this arrow will be an arrow from $A$ to $GF(A)$ in $K$, $\text{rad}(1_{F(A)}) : A \to GF(A)$. We designate the arrow $\text{rad}(1_{F(A)})$ by $\eta(A)$; namely, $\eta(A) = \text{rad}(1_{F(A)}) : A \to GF(A)$. Thus, for each object $A$ of $K$, there is determined an arrow...
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\( \eta(A) : A \to GF(A) \). It is not difficult to verify (using the coherence conditions (i) and (ii) for an adjunction) that \( \eta \) is a natural transformation from the identity functor \( I_K \) to the composite functor \( G \circ F \); that is, \( \eta : I_K \to G \circ F \). The natural transformation \( \eta \) is called the \textit{unit} of the adjunction.

Similarly, let \( B \) be any object of \( K' \), and consider the identity arrow \( 1_{G(B)} : G(B) \to G(B) \) in \( K \). The left adjoint of this arrow will be an arrow from \( FG(B) \) to \( B \) in \( K' \), \( \text{lad}(1_{G(B)}) : FG(B) \to B \). We designate the arrow \( \text{lad}(1_{G(B)}) \) by \( \epsilon(B) \), that is, \( \epsilon(B) = \text{lad}(1_{G(B)}) : FG(B) \to B \). It can likewise verified that \( \epsilon(B) \) is a natural transformation from the functor \( F \circ G \) to the identity functor \( i_{K'} \), that is, \( \epsilon : F \circ G \to I_{K'} \). The natural transformation \( \epsilon \) is called the \textit{counit} of the adjunction.

The unit \( \eta \) and counit \( \epsilon \) play an important role in an adjoint situation. We shall consider several examples.

\textit{Example 1: Product Functors and Exponential Functors.} Let \( F \) be the cartesian product functor \(- \times A\), and let \( G \) be the exponential functor \(-^A\). The unit \( \eta(B) : B \to (B \times A)^A \) is the function that takes each element \( b \) of \( B \) to the function \( f \) from \( A \) to \( B \times A \) defined by \( f(a) = (b, a) \). The counit \( \epsilon(B) : (B^A) \times A \to B \) is the \textit{evaluation arrow} defined by \( \epsilon(B) : (g, a) \to g(a) \) for \( g \) in \( B^A \) and \( a \) in \( A \). (Recall \( B^A \) is the set of functions from \( A \) to \( B \).)

\textit{Example 2: Forgetful Functors and Free Algebra Functors.} Let \( G \) be the forgetful functor \( G : \text{CR} \to \text{Set} \), and let \( F : \text{Set} \to \text{CR} \) be the free commutative ring functor. Then the unit \( \eta(S) : S \to FG(S) \) is the \textit{embedding of generators} function \( a \to X_a \), which shows how each element \( a \) of \( S \) is embedded in the (underlying set of the) free commutative ring generated by \( S \). The counit \( \epsilon(A) : FG(A) \to A \) is the representation of an arbitrary commutative ring \( A \) as a quotient of the free commutative ring generated by the underlying set of \( A \) (see Section 1.6).

\textit{Example 3: Diagonal Functors and Limits.} Let \( K' \) be a category of diagrams over \( K \), let \( F \) be the diagonal functor \( F : K \to K' \), and let \( G : K' \to K \) be the functor that takes each diagram to its limit. Then the unit \( \eta(A) : A \to GF(A) \) is simply the identity arrow \( 1_A \), that is, \( \eta \) is the identity natural transformation, \( \eta = I_K \). The counit \( \epsilon(D) : FG(D) \to D \) is the
universal cone over \( D \), presented as an arrow from a constant diagram, representing the vertex of the cone, to the diagram \( D \).

**Triples.** Suppose we have an adjunction \((F, G, \theta)\), as above. Let \( A \) be an object of \( K \). Then \( F(A) \) is an object of \( K' \), and the counit \( \epsilon(F(A)) \) is an arrow from \( FGF(A) \) to \( F(A) \) in \( K' \); that is, \( \epsilon(F(A)) : FGF(A) \to F(A) \). If we now apply the functor \( G \) to this arrow, we obtain an arrow \( \mu(A) = G(\epsilon(F(A))) : GFGF(A) \to GF(A) \) in \( K \). Let us call the composite functor \( T \); that is, \( G \circ F = T \). \( T \) is then an endofunctor from \( K \) to \( K \). (An endofunctor is a functor from a category to itself.) For every \( A \) in \( K \), \( \mu(A) \) is an arrow from \( T \circ T(A) \) to \( T(A) \). It is straightforward to verify that \( \mu \) is a natural transformation from \( T \circ T \) to \( T \).

From the adjunction \((F, G, \theta)\) we thus obtain the following three transformations within \( K \):

(i) the endofunctor \( T : K \to K \),
(ii) the natural transformation \( \eta : I_K \to T \),
(iii) the natural transformation \( \mu : T \circ T \to T \).

It is straightforward to verify that the following diagrams commute. These diagrams define a type of structure called a *triple*.

**Definition.** A *triple* in a category \( K \) consists of a triple \((T, \eta, \mu)\), where \( T : K \to K \) is an endofunctor of \( K \), and \( \eta : I_K \to T \), \( \mu : T \circ T \to T \) are natural transformations, such that the diagrams (2.47) and (2.48) commute, where in the diagrams, \( I \) denotes \( I_K \).

\[
\begin{array}{ccc}
T^3 & \xrightarrow{T\mu} & T^2 \\
\mu^T \downarrow & & \downarrow \mu \\
T^2 & \xrightarrow{\mu} & T
\end{array}
\]

and

\[
\begin{array}{ccc}
IT & \xleftarrow{\eta^T} & T^2 \\
\| & & \downarrow \mu \\
T = & & T
\end{array}
\]

\[
\begin{array}{ccc}
IT & \xleftarrow{T\eta} & T^2 \\
\| & & \downarrow \mu \\
TI & \xrightarrow{\mu} & T
\end{array}
\]
You should observe that the diagrams characterizing a triple are identical in form to the diagrams characterizing an internal monoid in a category. The only difference is that the direct product operation \( A \times B \) is replaced by the composition operation \( A \circ B \), and the terminal object 1 (the identity for the direct product operation \( \times \)), is replaced by the identity functor \( I_k \) (the identity for the composition operation \( \circ \)). This means that a triple is simply a “monoid in a category of endofunctors.”

Thus, from an adjunction \((F, G, \theta)\), with \( F : K \to K' \) and \( G : K' \to K \), one obtains a triple \((T, \eta, \mu)\) in \( K \). This triple is defined completely in terms of transformations within \( K \).

Now it is a remarkable fact that these transformations, within \( K \), can contain the complete “memory” of the second category \( K' \) as well as the two functors \( F \) and \( G \). A deep theorem of category theory, Beck’s theorem, gives precise conditions for the existence of this memory. Those conditions are satisfied, in particular, for forgetful functors for algebraic theories.

What this means is the following. Suppose we start with some category \( K \) that happens to be the category of models for some algebraic theory; \( K \) might be \( \text{Grp} \), or \( \text{CR} \), or \( \text{Ab} \), and so forth. Consider the forgetful functor \( G : K \to \text{Set} \). The forgetful functor \( G \), together with its adjoint, the free-algebra functor \( F \), will induce a triple \((T, \eta, \mu)\) in \( \text{Set} \), consisting of transformations within \( \text{Set} \). From this triple one can reconstruct, within \( \text{Set} \), the entire structure of the category \( K \), as well as the two functors \( F \) and \( G \) connecting the two categories. The triple thereby contains the complete memory of the category \( K \), the differentiated field, in the category of sets, the undifferentiated field. Thus, although the forgetful functor \( G \) simply “forgets,” the dynamism it induces (in conjunction with its adjoint) in the undifferentiated field contains the complete memory of the “forgotten” differentiated structure of \( K \). In short, the triple “remembers” what the forgetful functor forgets.

The memory is reconstructed in the following way: For any triple \((T, \eta, \mu)\) in a category \( K \) one can construct a category \( K' \) and a pair of adjoint functors \( F : K \to K' \), \( G : K' \to K \) such that the triple \( T \) is precisely the triple induced by the adjunction \((F, G, \theta)\). This is done as fol-
The category $K'$ is defined to be the category of $T$-algebras, where a $T$-algebra is a pair $(A, h)$ consisting of an object $A$ of $K$ and an arrow $h : TA \to A$ such that the two diagrams of (2.49) commute.

These diagrams describe, in categorical language, the concept of a monoid action on a set. The object $A$ is called the underlying object of the algebra, and the arrow $h : TA \to A$ is called the structure map of the algebra. The functor $G : K' \to K$ is then the forgetful functor defined by $G((A, h)) = A$. The functor $F : K \to K'$ is defined by $F(A) = (Tx, \mu(x))$; this functor takes each object $A$ of $K$ to the free $T$-algebra generated by $A$.

Now suppose we are given any adjunction $(F, G, \theta)$, with $F : K \to K'$ and $G : K' \to K$, and suppose the conditions of Beck’s theorem are satisfied. Then if we form the triple $(T, \eta, \mu)$ as above, Beck’s theorem tells us that the category $K'$ can be actualized as the category of $T$-algebras in $K$, with $F$ the free $T$-algebra functor and $G$ the forgetful functor.

The mathematical study of the forgetful functor in category theory provides an interesting mathematical commentary on the theme of “forgetting” at the basis of the Transcendental Meditations programs. In the TM-Sidhi program, the mechanics of forgetting is utilized to systematically “forget” the boundaries of a thought and thereby experience the unbounded wholeness of the Samhita, the undifferentiated field of pure consciousness. The effect is to enliven in awareness the internal, self-interacting dynamics of the Samhita, which contains the blueprint of creation.

In a parallel way, Beck’s theorem locates the blueprint for the expressed level of mathematical reality in the internal dynamics of the category of sets; this is achieved on the basis of the forgetful functor, the mathematical expression of the mechanics of “forgetting.” In this way category theory, from its own unique perspective, provides a fascinating new insight into the relationship of the expressed level of mathematical reality described by abstract theories to the undifferentiated field of mathematical existence, the category of sets.
2.7 Category of All Categories

We have seen how category theory provides a universal theory of mathematical theories. With the continued development of category theory, there naturally arose a desire to provide a complete foundation for mathematics based solely on the abstract category-theoretic viewpoint. In this section we shall consider one such approach developed by William Lawvere in his theory of the category of all categories. The essence of this approach was to apply the category-theoretic viewpoint in a self-referral way to describe the structure of category theory itself.

We have seen how every abstract theory describes a category. Category theory is itself an abstract theory and therefore describes its own category of structures—the category of all categories. The category of all categories is a hypercategory, whose objects are all ordinary categories and whose arrows are all functors. The theory of the category of all categories studies this hypercategory using the language of category theory, that is, studying the way functors combine by the composition rule to yield new functors. The theory of the category of all categories thus presents the self-referral application of the viewpoint of category theory to describe the structure of category theory itself. The objective of this approach is to determine, by solely studying the relationships between functors, what the internal structure of each of the categories is: which is the category of sets, which is the category of groups, which is the category of topological spaces, and so on.

In Lawvere’s classic paper “The category of all categories as a foundation for mathematics” (1966), he showed how one could identify the category of sets within the category of all categories. Lawvere’s approach was to use functors to probe into the different categories to see what their internal structure was. To give a feeling for how this works, we shall survey the first few steps of Lawvere’s construction.

Let $U$ designate the category of all categories. Then $U$ has a terminal object, namely the category $1$, introduced in Section 2.1. For any category $K$ there is a unique functor $F$ from $K$ to $1$; this is the functor $F$ defined by $F(A) = M$ for every object $A$ in $K$, and $F(f) = 1_M$ for every arrow $f$ in $K$.

Now, if $K$ is any category, then for each object $A$ in $K$ there is a unique functor $F : 1 \to K$ that takes $M$ to $B$, namely the functor $F$
defined by $F(M) = A$ and $F(1) = 1_A$. Thus the different functors $F: 1 \to K$ can be used to represent each of the objects of the category $K$.

Next consider the category $2$ introduced in Section 2.1. It is possible to characterize this category $2$ in terms of the language of the category of all categories, using the category $1$ and special properties of functors from $1$ to $2$ and from $2$ to $2$.

Now if $K$ is any category, then for every arrow $f: A \to B$ in $K$ there is a unique functor $F$ from $2$ to $K$ that takes $n_{12}$ to $f$, namely the functor $F$ defined by $F(N_1) = A$, $F(N_2) = B$, and $F(n_{12}) = f$. Thus the different functors $F: 2 \to K$ can be used to represent each of the arrows of the category $K$.

Consider now the two functors $F_1$, $F_2$ from $1$ to $2$, where $F_1(M) = N_1$, and $F_2(M) = N_2$. These functors can be used to identify the source and target of each arrow in $K$. Thus, if $F: 2 \to K$ represents an arrow $f: A \to B$ of $K$, then $F \circ F_1: 1 \to K$ represents the source $A$ of $f$, and $F \circ F_2: 1 \to K$ represents the target $B$ of $f$.

At this point we have identified the objects of $K$, the arrows of $K$, and the source and target of each arrow. We next need to reconstruct the composition law for the arrows. For this we need the category $3$ introduced in Section 2.1.

First $3$ must itself be characterized in terms of properties of functors involving $1$, $2$ and $3$ itself. Next, if $f_1: A \to B$, $f_2: B \to C$, $f_3: A \to C$ are any three arrows of $K$ such that $f_3 = f_2 \circ f_1$, then there exists a unique functor $F$ from $3$ to $K$ such that $F(h_1) = g_1$, $F(h_2) = g_2$, $F(h_3) = g_3$, namely the functor $F$ defined by $F(L_1) = A$, $F(L_2) = B$, $F(L_3) = C$, $F(l_{12}) = f_1$, $F(l_{23}) = f_2$, and $F(l_{13}) = f_3$. Let $G_1$, $G_2$, $G_3$ be the functors from $2$ to $3$ that take $n_{12}$ to $l_{12}$, $l_{23}$, $l_{13}$ respectively. Then the functors $F \circ G_1$, $F \circ G_2$, $F \circ G_3$ represent $f_1$, $f_2$, $f_3$ respectively.

Thus all triples of functors from $2$ to $K$ of the form $(F \circ G_1, F \circ G_2, F \circ G_3)$, where $F$ ranges over all possible functors from $3$ to $K$, represent all possible triples $(f_1, f_2, f_3)$ of arrows of $K$ for which $f_3 = f_2 \circ f_1$. This approach therefore provides a complete description of the composition rule of the arrows in $K$.

In this way we see how the language of functors can be used to unfold the entire internal structure of a category $K$: its objects, its arrows, and the composition rule for the arrows. This knowledge of the internal structure of the categories can then be used to determine
which category is the category of sets, which is the category of groups, which is the category of topological spaces, and so on.

In this way the self-referral application of the category-theoretic viewpoint to describe the structure of category theory itself can be developed into a unified foundational theory of modern mathematics.

Chapter 3
Topos Theory

3.1 Definition of a Topos

We saw in the last chapter how category theory provides a “theory of mathematical theories” that has unified mathematical knowledge in a number of different ways. With the continued maturation of category theory, there naturally developed an interest in seeing to what extent category theory could provide a foundation for set-theory itself, the established foundational theory of modern mathematics.

Set theory describes the structure of the universe of sets. When the universe of sets is examined from the viewpoint of category theory, it is seen as the category of sets, as discussed in Chapter 2. The natural first step in providing a category-theoretic foundation for set theory is to formulate a theory of the category of sets, that is, to show how the structure of the category of sets can be axiomatized in the language of category theory. This was done in Lawvere’s (1964) classic paper “An elementary theory of the category of sets,” in which he presents eight axioms characterizing the structure of the category of sets.

The real breakthrough in the categorical approach to foundations, however, came in 1969 when Lawvere and Tierney introduced the concept of an elementary topos (Lawvere, 1971; Tierney, 1972). In the 1960’s there had been a number of developments in diverse areas of mathematics that converged upon this concept; these areas included algebraic geometry, sheaf theory, intuitionistic mathematics, and categorical logic.

A topos is a category that “resembles” the universe of sets. The defining properties of a topos do not uniquely characterize the category of sets, and toposes can differ from the category of sets in a number of fundamental ways. A topos nevertheless resembles the category of sets to the extent that set theory can be developed internally within any topos. The great power of topos theory has been in fact to provide mod-
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eels of a generalized type of set theory that have extraordinary properties not available in the familiar universe of sets. We shall begin our development of topos theory in this section by examining the definition of an elementary topos.

A topos \( T \) is defined to be a category satisfying certain axioms. The first three axioms simply tell us that \( T \) contains limits.

**Axiom 1.** \( T \) contains a terminal object.

**Axiom 2.** \( T \) contains all products of the form \( A \times B \).

**Axiom 3.** \( T \) contains all pullbacks.

From the above three axioms it follows that \( T \) contains all finite limits; that is, if \( D \) is a diagram in \( T \) containing a finite number of objects and arrows, then there exists a universal cone over \( D \).

The first three axioms are satisfied by almost all categories of mathematical interest. The next axiom is more distinctive.

**Axiom 4.** \( T \) is cartesian closed; that is, for every object \( A \), the direct product functor \(- \times A\) has a right adjoint \(-^A\).

As discussed in Chapter 2, this property means that the category \( T \) is complete in the sense that it contains objects \( B^A \) representing each of its hom-sets \( T(A, B) \). Of the examples considered in Chapter 2, only the category of sets \( \text{Set} \) and the category of abelian groups \( \text{Ab} \) are cartesian closed. (We should point out here that Axiom 4 can be formulated without explicit mention of functors. This fact is important because one of the objectives of the program of Lawvere and Tierney was to create a foundational theory using only first-order axioms, formulated in the language of category theory. Achieving this objective would avoid some of the undesirable features of second-order logic and would make it possible for the theory to be truly an alternative foundational theory for mathematics.)

The final axiom is the truly distinctive one and requires some motivation. This axiom asserts the existence of a special object in the topos called the **subobject classifier**. We shall now describe this concept.
We have already seen how the concept of a subobject can be described in the language of arrows; a subobject of $A$ is described by a monic arrow $B \to A$. This description is valid in any category.

In the category of sets, however, there is a second way to describe a subobject. Namely, if we start with a set $A$, then to form a subset $B$, we must make a choice, for each element of $A$, whether or not to include it in the subset $B$. Thus if we take a fixed set $\Omega$, containing two elements representing the truth values “true” and “false,” $\Omega = \{T, F\}$, then we can characterize a subset $B$ of $A$ by a function $f: A \to \Omega$. The function $f$ is simply defined by: $f(a) = T$ if $a$ is an element of $B$, and $f(a) = F$ if $a$ is not an element of $B$. In this way, each subset $B$ of $A$ corresponds to a unique function $f: A \to \Omega$.

We can describe the arrow $f$ in categorical terms in the following way. Let $1 = \{P\}$ designate any set containing exactly one element (that is, 1 is a terminal object in the category of sets). Let $t$ designate the “true” function $t: 1 \to \Omega$ that takes the single element of 1 to $T$; namely, $t(P) = T$. Let $g: B \to A$ be the monic arrow designating the subset $B$ of $A$. Putting these different arrows together, we obtain the commutative square (3.1).

\[
\begin{array}{ccc}
B & \longrightarrow & 1 \\
\downarrow g & & \downarrow t \\
A & \underset{f}{\longrightarrow} & \Omega
\end{array}
\] (3.1)

Furthermore, it is not too difficult to see that diagram (3.1) is actually a pullback diagram in the category of sets. This observation provides the motivation for the final axiom for a topos:

**Axiom 5.** There exists an object $\Omega$ (the subobject classifier) and an arrow $t: 1 \to \Omega$ (the true arrow) with the following property: For every object $A$ and every monic $g: B \to A$, there exists a unique arrow $f: A \to \Omega$ such that diagram (3.1) is a pullback.

This completes the definition of a topos. We shall examine several examples of toposes in the next section, and we shall then see how set theory can be developed internally within any topos.
3.2 Variable Sets

In the last section we introduced the concept of a topos. The defining properties of a topos were intended to capture several of the distinctive features of the category of sets. There are, however, many other examples of toposes as well. These examples “resemble” the category of sets to the extent that one can develop set theory internally within any topos. Nevertheless, they can differ from the category of sets in fundamental ways, giving rise to formulations of set theory strikingly different from the standard classical formulation. These toposes can be thought of as providing models of a generalized type of set theory in which sets have a variable structure. In this section we shall examine several examples of toposes that are based upon “variable” sets.

The familiar concept of a set is static in the sense that the elements of a set are fixed and nonchanging. To introduce variability into the concept of a set, we can think of a set as being evaluated at different stages, in which the elements can be different. The simplest case of a variable set would be to have two stages of evaluation, which we shall call $E_1$ and $E_2$. At each stage, we obtain an ordinary, static set. A variable set will then be represented by an ordered pair $\langle S_1, S_2 \rangle$ in which $S_1$ and $S_2$ are ordinary sets. The set $S_1$ will represent the set $S$ evaluated in stage $E_1$, and $S_2$ will represent the set $S$ evaluated at stage $E_2$. We can take all such ordered pairs $S = (S_1, S_2)$ of ordinary sets to be the elements of a category $K$.

The arrows of $K$ will then be the functions between these variable sets. The functions between variable sets will be variable functions. Suppose $S = (S_1, S_2)$ and $T = (T_1, T_2)$ are each variable sets. To say that $f$ is a function from $S$ to $T$ will mean that, in stage $E_1$, $f$ will be an ordinary function from $S_1$ to $T_1$, and that, in stage $E_2$, $f$ will be an ordinary function from $S_2$ to $T_2$. This means that a variable function $f: S \to T$ will be represented by an ordered pair of ordinary functions $(f_1, f_2)$, where $f_1 : S_1 \to T_1$ and $f_2 : S_2 \to T_2$.

In this way we obtain a category called $\text{Set}^2 = \text{Set} \times \text{Set}$, whose objects are all ordered pairs of sets and whose arrows are all ordered pairs of functions. It is not difficult to verify that this category is a topos. The subobject classifier is $\Omega = \Omega_{\text{Set}} \times \Omega_{\text{Set}}$, where $\Omega_{\text{Set}} = \{T, F\}$ is the subobject classifier in $\text{Set}$. This means that $\Omega = \{(T, T), (T, F), (F, T), (F, F)\}$. If $S = (S_1, S_2)$ and $T = (T_1, T_2)$ then the exponential $S^T = (S_1^{T_1}, S_2^{T_2})$. 

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We can generalize the above construction to three stages of evaluation, four stages of evaluation, and so on. In the case of $n$ stages of evaluation, we obtain a topos whose objects are all possible $n$-tuples of sets $(S_1, S_2, \ldots, S_n)$, and whose arrows $f : S \to T$ are $n$-tuples of functions $(f_1, \ldots, f_n)$ with $f_1 : S_1 \to T_1, \ldots, f_n : S_n \to T_n$. This category is designated $\text{Set}^n = \text{Set} \times \cdots \times \text{Set}$.

We shall now consider the main example of this section, in which the notion of variability is extended to include a continuum of stages of evaluation. We shall take as our stages of evaluation all possible points on a continuous segment of the number line; these points will correspond to all real numbers between 0 and 1. Thus, for each real number $r$ between 0 and 1, we shall have a stage of evaluation $E_r$.

Suppose now $S$ is a variable set. This means that at every stage $E_r$ we will have an ordinary set $S_r$. In this case, however, we shall add a new requirement to respect the continuous nature of the stages of evaluation: we shall require that the sets $S_r$ vary in a continuous way. The precise requirement is that sets $S_r$ are connected together in a way that forms a type of topological structure called a sheaf. We shall not give a formal topological definition of a sheaf but will be content to give an intuitive sense of the meaning of this concept.

Suppose $a \in S_r$, that is, $a$ is an element of $S$ in stage $r$. Then it is required that, whenever the number $t$ is sufficiently close to $r$, there should be a well-defined element $a_t \in S_t$ corresponding to the element $a \in S_r$. We think of $a_t$ as representing the “same” element as $a$. This means that the elements vary continuously; if we have an element $a$ in a given stage of evaluation, the “same” element will always be found at sufficiently close stages of evaluation.

We can draw diagrams to represent sheaves. We first draw a line representing the set of real numbers $X$ lying between 0 and 1. Above each number $r$ we have discrete points representing the different elements of $S_r$, called the stalk at $r$. These points are connected in horizontal layers, reflecting the way each point has nearby points on either side corresponding to the same element of $S$. We have given eight examples of such diagrams in Figure 3.1.

To describe the structure of these sheaves in somewhat more detail, it will be helpful to use mathematical notation for intervals of real numbers.
We recall that the real numbers structure a continuous number line. Suppose $a$ and $b$ are two real numbers and $a < b$. The open interval $(a, b)$ is defined to be the set of all real numbers lying between $a$ and $b$, but not including $a$ or $b$, that is, $(a, b) = \{x : a < x < b\}$.

A set of real numbers is said to be open if it can be expressed as a union of open intervals. The empty set $\emptyset$ is also considered open.

The closed interval $[a, b]$ is defined to be the set of all real numbers lying between $a$ and $b$, including both $a$ and $b$, that is, $[a, b] = \{x : a \leq x \leq b\}$. Thus $2 \in [1, 2]$, but $2 \notin (1, 2)$.

One can also consider intervals that are half-open and half-closed. We define the interval $[a, b)$ to be the set of all real numbers lying between $a$ and $b$, including $a$ but excluding $b$, that is, $[a, b) = \{x : a \leq x < b\}$. We similarly define the interval $(a, b]$ to be the set of all real numbers lying between $a$ and $b$, including $b$ but excluding $a$, that is, $(a, b] = \{x : a < x \leq b\}$.

We can now describe the sheaves in Figure 3.1. In all cases, the base space $X$ is taken to be the open interval $(0, 1)$.

The sheaf $S_0$ is the simplest: the empty sheaf. Here the stalk $S_r$ is the empty set for every point $r \in (0, 1)$: $S_r = \emptyset$. Consequently there is nothing to draw!

In the sheaf $S_1$, for every point $r \in (0, 1)$ the stalk $S_r$ contains a single point $a$, that is, $S_r = \{a\}$. These points are connected together in a continuous line. We therefore represent this sheaf by drawing a single horizontal line covering the entire interval $(0, 1)$. This sheaf represents a set containing a single, fixed element in all stages of evaluation.

In the sheaf $S_2$, the stalk $S_r$ contains two points for every $r \in (0, 1)$. These points are connected together in two continuous lines, labeled $A$ and $B$, each covering the entire interval $(0, 1)$. This sheaf represents a set containing two fixed elements in all stages of evaluation.

The sheaf $S_3$ is the first example where the elements are really different in different stages of evaluation. Here the stalk $S_r$ contains a single point $a$ for $r \in (0, 0.4)$, and the stalk $S_r$ is empty for $r \in [0.4, 1)$. We represent this sheaf graphically by drawing a horizontal line segment covering precisely the open interval $(0, 0.4)$. 

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Figure 3.1 Examples of sheaves.
The reader might perhaps feel that this example violates the intuitive concept of a “continuously” variable set, as an element seems to suddenly pop into existence at the point $r = 0.4$. But let us examine more closely what is happening near the point $r = 0.4$. At the point $r = 0.4$, the stalk is empty, that is, $S_{0.4} = \emptyset$. The stalk is likewise empty for all $r$ to the right of 0.4. For all values of $r < 0.4$, the stalk contains a single point.

Now the criterion of continuous variability is the following: If $a \in S_r$, then there must exist corresponding elements of $S_t$ for all $t$ sufficiently close to $r$. But if $a \in S_r$, then $r$ must be less than 0.4 (because $S_r$ contains no elements for $r \geq 0.4$). If we require $t$ to be close enough to $r$, then $t$ also will be less than 0.4, and therefore $S_t$ will contain a point $a_t$ corresponding to $a$. For example, if $r = 0.399$, we can require $t$ to be within distance 0.000001 from $r$; for all such values of $t$, we will then have $t < 0.4$. Our requirement of continuous variability is therefore satisfied. This sheaf represents a set that is empty in stages of evaluation $r \in [0.4, 1)$, and that contains a single element in stages of evaluation $r \in (0, 0.4)$.

The sheaf $S_4$ is structured in two unconnected pieces: one horizontal piece $A$ lying above the interval $(0.1, 0.4)$ and one horizontal piece $B$ lying above the interval $(0.6, 0.8)$.

The sheaf $S_5$ is structured in three pieces: a piece $A$ covering the interval $(0, 0.3)$, a piece $B$ covering the interval $(0.1, 0.5)$, and a piece $C$ covering the interval $(0.8, 1)$. In this example, the stalk $S_r$ contains a single point for $r \in (0, 0.1]$, two points for $r \in (0.1, 0.3)$, one point for $r \in [0.3, 0.5)$, no points for $r \in [0.5, 0.8]$, and one point for $r \in (0.8, 1)$.

The sheaf $S_6$ is structured in two unconnected pieces: a piece $A$ covering the interval $(0, 0.5)$, and a piece $B$ covering the interval $(0.5, 1)$. Here the $S_r$ stalk is empty at the value $r = 0.5$, $S_{0.5} = \emptyset$, and the stalk contains a single point for every other value of $r$.

The final example illustrates a phenomenon whereby two distinct elements can merge into a common element. Here the stalk $S_r$ consists of two elements for $r \in (0, 0.5]$, and consists of a single element for $r \in (0.5, 1)$. We note that there are two distinct horizontal sections, $A$ and $B$, over the open interval $(0, 0.5)$, and a single horizontal section $C$ over the open interval $(0.5, 1)$. There are two distinct points in the sheaf lying above the point 0.5: $S_{0.5} = \{P_1, P_2\}$. One of these points, $P_1$, con-
nects the interval $A$ to the interval $C$, and the other point, $P_1$, connects the interval $B$ to the interval $C$. Here the two distinct points to the left of 0.5 become the same point to the right of 0.5. At $r = 0.5$, the two points are still distinct. If this example seems to have an unintuitive geometrical flavor, it is because the topology of the sheaf is not “Hausdorff.” (Familiar spaces, like a line, a plane, 3-space, and most spaces used in applications, have the Hausdorff property and therefore do not exhibit such pathologies.)

These examples should give the reader some sense for the meaning of sheaves as continuously variable sets. We next describe the “functions” from one sheaf to another.

In our discussion of products $\text{Set}^n$, the functions were just $n$-tuples of ordinary functions. The generalization to sheaves is to define a function from a sheaf $S$ to a sheaf $T$ to consist of a continuum of functions $f_r, r \in (0, 1)$, where $f_r : S_r \rightarrow T_r$ is a function from the stalk $S_r$ of $S$ at $r$ to the stalk $T_r$ of $T$ at $r$. We require further that these functions respect the continuous structure of the sheaves. This means the following.

Suppose $f_r(a) = b$, where $a \in S_r$ and $b \in T_r$. Then for all $t$ sufficiently close to $r$ we require that $f_t(a_t) = b_t$, where $a_t \in S_t$ is the element of $S_t$ corresponding to $a$, and $b_t \in T_t$ is the element of $T_t$ corresponding to $b$. This means that if $f$ takes $a$ to $b$, it must take nearby elements corresponding to $a$ to nearby elements corresponding to $b$. Such a function $f$ is called a morphism of sheaves.

If we think of sheaves in terms of our diagrams, then a sheaf morphism is just a vertical projection of one sheaf onto another. The continuity requirement is that no continuous horizontal section gets “torn” into two or more pieces.

For example, there is no morphism from the sheaf $S_3$ to the sheaf $S_5$. We can see this as follows. A morphism would have to take the continuous section $A$ of $S_3$ over $(0, 0.4)$ to a continuous piece of $S_5$ lying above $(0, 0.4)$. But $S_5$ has no continuous section covering all of $(0, 0.4)$.

We can analyze this situation in somewhat more detail as follows. Suppose $f$ is any “vertical” function from $S_3$ to $S_5$. Let $a$ be the point of $S_3$ lying above $r = 0.05$. Then $f(a)$ must be the point $c$ of $S_5$ lying above $r = 0.05$, where $c$ is an element of the piece $A$ of $S_5$. Similarly, let $b$ be the point of $S$ lying above $r = 0.35$; then $f(b)$ must be the point $d$ of $S_5$ lying
above \( r = 0.35 \), where \( d \) is an element of the piece \( B \) of \( S5 \). This means that the function \( f \) must “jump” from the \( A \) piece to the \( B \) piece for some value of \( r \) lying between 0.05 and 0.35. But this is incompatible with the continuity requirement: If \( e \) is the point of \( S3 \) lying above \( r \), and \( f(e) \) is an element of \( A \), then all nearby points of \( S3 \) must be mapped into \( A \). Similarly, if \( f(r) \) is an element of \( B \), then all nearby points of \( S3 \) must be mapped into \( B \). This disallows the possibility of jumping from \( A \) to \( B \) at any intermediate value.

Having introduced the concept of a function between variable sets, we can consider now a category whose objects are all such sets and whose arrows are all such functions. In the case we have been considering, we obtain the category \( \text{Shv}(X) \), whose objects are all sheaves over \( X = (0, 1) \), and whose arrows are all sheaf morphisms. This category is a topos.

The initial object \( 0 \) of this topos is the empty sheaf, \( S0 \). The unique arrow from \( S0 \) to any sheaf \( A \) is the function defined by \( f_r = \) the empty function, for every \( r \in (0, 1) \).

The terminal object \( 1 \) of \( \text{Shv}(X) \) is the sheaf \( S1 \), consisting of a single section covering the entire interval \((0,1)\). For any sheaf \( A \), the unique arrow from \( A \) to \( S1 \) is the function defined by \( f_r = \) the unique point of \( S1 \) lying above \( r \).

In the category of sets, a set consisting of a single point, \( \{P\} \), is a terminal object. In the topos \( \text{Shv}(X) \), the sheaf 1 plays the role of a set consisting of a single point.

We now inquire: What are the different subobjects of 1 in \( \text{Shv}(X) \)? In the category of sets, the terminal object \( 1 = \{P\} \) has only two subobjects: the empty set \( \varnothing \) and 1 itself. In \( \text{Shv}(X) \), however, the terminal object 1 has many different subobjects.

We recall that subobjects are described in general by monic arrows. A subobject of 1 is thus described by a monic arrow \( f: A \to 1 \). It can be shown that the subobjects of 1 correspond precisely to the open subsets of \((0, 1)\).

An open subset of \((0, 1)\) is by definition a subset of \((0, 1)\) that can be expressed as a (possibly infinite) union of open intervals. The empty set is also considered open. Suppose now \( U \) is an open subset of \((0, 1)\). We can consider the sheaf \( S(U) \) whose stalk \( S_r(U) \) consists of a single point for
every $r \in U$ and is empty for every $r \not\in U$. For example, if $U = (0.1, 0.4) \cup (0.6, 0.8)$, then $S^U$ is the sheaf $\mathcal{S}_4$.

If now $f$ is the unique arrow $S^U \to 1$, then $f$ is clearly one-to-one as a function (since each stalk contains at most one element), and $f$ is therefore easily seen to be monic. The arrow $f$ therefore describes a subobject of $1$. It can further be shown that every monic arrow arises in this way from some open subset of $(0, 1)$.

Thus, every open subset of $(0, 1)$ determines a distinct subobject of $1$. There are therefore infinitely many subobjects of $1$ in $\text{Shv}(X)$, infinitely many “sets” lying in the gap between the “empty set,” $0$, and the “set” containing a single point, $1$. These intermediate sets can be thought of as sets containing a “partially existing” point, sets that are empty in certain stages of evaluation but that contain a single point in other stages of evaluation.

We inquire finally: What is the subobject classifier, $\Omega$? We know from the defining property of $\Omega$ that there is a unique arrow $1 \to \Omega$ corresponding to each subobject $A$ of $1$. Now for any sheaf $S$, an arrow $1 \to S$ must take the single, connected piece of $1$ to a continuous horizontal “slice” of $S$ covering the entire interval $(0, 1)$. Such a slice is called a global section of $S$. (More generally, if $U$ is any open subset of $X$, then an arrow $S^U \to S$ is called a section of $S$ over $U$ and represents a horizontal slice of $S$ over the subset $U$ of $X$.)

The arrows $1 \to S$ thus correspond to all the global sections of $S$. This means in particular that the arrows from $1$ to $\Omega$ correspond to all the different possible global sections of $\Omega$. Thus, for every subobject of $1$ there corresponds a unique global section of $\Omega$.

In particular, to the empty subobject $0 \to 1$, there corresponds a global section $f_0 : 1 \to \Omega$, and to the identity subobject $1 \to 1$, there corresponds a global section $f_1 : 1 \to \Omega$. But consider now the subobject $A \to 1$ corresponding to the open subset $U = (0, 0.4)$, that is, the sheaf $\mathcal{S}_3$. This corresponds to a global section $f_U : 1 \to \Omega$. The “subset” $A$ is identical to the subset $1$ (containing a single point) in all stages of evaluation $r \in (0, 0.4)$, and this has as a consequence that the section $f_U$ is identical to the section $f_1$ above the subinterval $(0, 0.4)$. Likewise, the subset $A$ is identical to the subset $0$ (the empty set) in all stages of evaluation $r \in (0.4, 1)$, and this has as a consequence that the section $f_U$ is identical to the section $f_0$ above the subinterval $(0.4, 1)$. The point of
the section \( f_U \) lying above \( r = 0.4 \) then “glues” together these two pieces to make one continuous section.

From this we see that in the structure of \( \Omega \), the 0 and 1 sections are not disconnected, but are glued together. So far we have considered only three of the uncountable infinity of subobjects of 1. When we consider that each has its own global section, we begin to appreciate the complexity of the topological structure of \( \Omega \), the “set” in \( \text{Shv}(X) \), corresponding to the simple two-element set \([T, F]\) in the category of sets.

There is in fact a simple way of describing the sheaf \( \Omega \), but this involves a categorical characterization of sheaves as functors. This we shall now consider.

Suppose we are given a sheaf \( S \) over the base space \( X = (0, 1) \). Let \( T \) be the set of all open subsets of \( X \). Then there is a natural partial ordering on \( T \) corresponding to the inclusion relation \( \supseteq \), namely, \( U \subseteq V \) if \( U \supseteq V \). Let \( K \) be the category describing this partial ordering; that is, the objects of \( K \) are all possible open subsets of \( X \), and there is a unique arrow \( f: U \rightarrow V \) if \( U \supseteq V \) and there is no arrow from \( U \) to \( V \) otherwise. For any sheaf \( S \) over \( X \), one can define a functor \( F_S \) from \( K \) to \( \text{Set} \), as follows:

1. For any object \( U \) in \( K \), \( F(U) \) is defined to be the set of all sections of \( S \) over \( U \).
2. Suppose \( f: U \rightarrow V \) is an arrow of \( K \). Then \( F(f) \) is the function from \( F(U) \) to \( F(V) \) defined as follows. Since \( f: U \rightarrow V \), we must have \( U \supseteq V \), and therefore there will exist a unique morphism \( g: S^U \rightarrow S^V \); \( g \) simply maps each point of \( S^V \) to the point of \( S^U \) lying above the same point of \( V \). Suppose now \( b \) is an element of \( F(U) \); then \( b \) is an arrow \( b: S^U \rightarrow S \). Let \( k = b \circ g \). Then \( k \) is an arrow from \( S^V \) to \( S \), and therefore \( k \) is an element of \( F(V) \). We define \( F(f)(b) = k \).

In this way, every sheaf \( S \) gives rise to a functor \( F: K \rightarrow \text{Set} \). Furthermore, morphisms of sheaves can be shown to correspond to natural transformations between the corresponding functors. The category \( \text{Shv}(X) \) thereby can be identified with a subcategory of the functor category \( \text{Set}^K \).
The subobject classifier $\Omega$ can be shown to correspond to the following functor $F_\Omega : K \to \text{Set}$.

1. If $U$ is an object of $K$, then $F_\Omega(U)$ is the set of all open subsets of $U$.
2. If $f : U \to V$ is an arrow of $K$, so that $U \supseteq V$, then $F_\Omega(f)$ is the function from $F(U)$ to $F(V)$ that takes each subset $W$ of $U$ to $W \cap V$.

For any topological space $X$, the category $\text{Shv}(X)$ of sheaves over $X$ is always a topos; a topos of this form can always be identified with a suitable subcategory of the functor category $\text{Set}^K$ where $K$ is the category of open subsets of $X$. It is furthermore true that for any category $K$, the functor category $\text{Set}^K$ is always a topos. If one puts a Grothendieck topology on a category $K$, and considers the subcategory of $\text{Set}^K$ consisting of continuous functors, then one obtains a special type of topos called a Grothendieck topos. Sheaves are examples of Grothendieck toposes, but there are many other important examples as well. In fact most of the examples of toposes that have been utilized in applications of topos theory are Grothendieck toposes. We shall discuss this subject further in Section 3.9, where we shall consider the mechanics of creating toposes displaying desired values of organizing power.

### 3.3 Internal Logic

As we continue our development of topos theory, we shall see how any topos can be understood as a generalized universe of sets. This understanding requires, however, that the topos be approached on its own level, in terms of its own internal logic, rather than the familiar two-valued “true-false” logic of classical mathematics. In this section we shall describe the internal logic of a topos lying at the basis of this development.

We saw in Section 3.1 that the subobject classifier $\Omega$ in the category of sets can be identified with the two-element set $\{T, F\}$, representing the two truth values “true” and “false” of classical logic. Suppose $S$ is any set, and we wish to form a subset $B$ of $S$. Then for every element $a \in S$, we must make a choice whether or not to include $a$ in the subset; that is, we must make a choice whether to make the statement
a \in B$ true or false. The fact that there are precisely two elements in the subobject classifier $\Omega$ is intimately linked to the two-valued nature of classical logic, which lies at the basis of the development of ordinary set theory.

The subobject classifier in the category of sets thus naturally represents the “set of truth values” for classical logic, on which basis set theory is developed. In general, for any topos, the subobject classifier $\Omega$ can be understood as representing a “set of truth values” for a structure of logic appropriate to the topos, called the topos’s *internal logic*. This internal logic has an algebraic structure analogous to the boolean algebra structure of the propositional calculus of classical logic. We shall begin by reviewing the structure of the propositional calculus, and we shall then see how this structure can be generalized to an arbitrary topos.

The *propositional calculus* describes the *logical connectives*: $\land$ (and), $\lor$ (or), $\neg$ (not), $\rightarrow$ (implies). The role of the logical connectives in mathematical logic is analogous to the role of the operations of addition and multiplication in ordinary arithmetic; just as these arithmetic operations are used to combine numbers to yield new numbers, the logical connectives are used to combine propositions to yield new propositions. The “values” of propositions, however, are not numerical, but are logical. In classical logic, propositions can have two possible values: the truth values *true*, $T$, and *false*, $F$.

For example, consider the two propositions “$1 + 1 = 2$” and “$3$ is an even number.” The first is true and the second is false. Call the first proposition $A$ and the second $B$. Then $A$ has truth value $T$ and $B$ has truth value $F$.

Using the logical connectives, we can build new propositions from $A$ and $B$, namely $A \land B, A \lor B, A \rightarrow B, B \rightarrow A, \neg A, \neg B$. The proposition $A \land B$ asserts, “$1 + 1 = 2$ and $3$ is an even number”; the proposition $A \rightarrow B$ asserts, “$1 + 1 = 2$ implies $3$ is an even number”; the proposition $\neg A$ asserts, “it is not the case that $1 + 1 = 2$,” that is, $1 + 1 \neq 2$; and so on.

If we combine propositions using the logical connectives, the truth value of the resultant proposition is determined by the truth values of its constituent propositions according to the *truth tables* of Figure 3.2.
In the example we have been considering,

\[ A \land B = T \land F = F \]
\[ A \lor B = T \lor F = T, \]

and so on. This use of the truth tables is just like using addition and multiplication tables for a number system such as \( \mathbb{Z}_5 \). We can think of the truth tables as describing a mathematical structure having two elements, \( T \) and \( F \), and four operations, \( \land, \lor, \neg, \Rightarrow \). This particular structure describes the algebraic structure of classical logic and therefore has special significance in the study of the foundations of mathematics.

We wish to see now how the algebraic structure of the propositional calculus can be developed internally within any topos. This will involve looking at the familiar propositional calculus of classical logic in the context of the category of sets, and seeing how the logical connectives \( \land, \lor, \neg, \Rightarrow \) can be described by suitable arrows in the category of sets. We shall then find an appropriate categorical description of these arrows that can be applied to any topos.

We have seen that the subobject classifier \( \Omega \) in the category of sets is a two-element set, whose elements can be equated with the two truth values of classical logic: \( \Omega = \{ T, F \} \). The truth tables of figure 3.2 then simply describe the propositional connectives \( \land, \lor, \neg, \Rightarrow \) by functions: \( \land : \Omega \times \Omega \to \Omega, \lor : \Omega \times \Omega \to \Omega, \neg : \Omega \to \Omega, \Rightarrow : \Omega \times \Omega \to \Omega \). For example, the connective \( \land \) is described by the function:

\[ \land((T, T)) = T, \]
\[ \land((T, F)) = F, \]
\[ \land((F, T)) = F, \]
\[ \land((F, F)) = F. \]
Now in any topos, one can construct suitable arrows to represent the propositional connectives $\land, \lor, \neg, \Rightarrow$. If $\Omega$ is the subobject classifier, these arrows will have the following source and target:

$$
\land : \Omega \times \Omega \to \Omega,
\lor : \Omega \times \Omega \to \Omega,
\neg : \Omega \to \Omega,
\Rightarrow : \Omega \times \Omega \to \Omega
$$

The “and” arrow. We construct the “and” arrow $\land : \Omega \times \Omega \to \Omega$ as follows: In the category of sets, we have $\land((T, T)) = T$, $\land((T, F)) = \land((F, T)) = \land((F, F)) = F$. This means that the “and” arrow $\land : \Omega \times \Omega \to \Omega$, in the category of sets, is the classifying map for the subset $\{(T, T), (T, F), (F, T), (F, F)\}$ of $\Omega \times \Omega$. Now the subset $\{(T, T)\}$ of $\Omega \times \Omega$ is described by the monic arrow $(t, t) : 1 \to \Omega \times \Omega$, since the arrow $(t, t)$ is defined by $(t, t)(P) = (t(P), t(P)) = (T, T)$. We thus make the following definition, for an arbitrary topos.

**Definition.** The “and” arrow $\land : \Omega \times \Omega \to \Omega$ is the classifying arrow of $(t, t) : 1 \to \Omega \times \Omega$.

The “or” arrow. We construct the “or” arrow $\lor : \Omega \times \Omega \to \Omega$ as follows: In the category of sets, we have $\lor((T, T)) = \lor((T, F)) = \lor((F, T)) = T$, $\lor((F, F)) = F$. This means that the “or” arrow $\lor : \Omega \times \Omega \to \Omega$, in the category of sets, is the classifying map for the subset $\{(T, T), (T, F), (F, T), (F, F)\}$ of $\Omega \times \Omega$. We can express $A$ as the union of the two sets $A_1 = \{(T, T), (T, F)\}$ and $A_2 = \{(T, T), (F, T)\}$; that is, $A = A_1 \cup A_2$.

Consider the function $g : \Omega \to \Omega$ defined by $g(T) = g(F) = T$. The function $g$ is simply the composite $t \circ h : \Omega \to 1 \to \Omega$, where $t$ is the true arrow $t : 1 \to \Omega$ and $h$ is the (unique) arrow $\Omega \to 1$.

Let $1_\Omega$ be the identity function $1_\Omega : \Omega \to \Omega$, so $1_\Omega(T) = T$, $1_\Omega(F) = F$. The function $(1_\Omega, g) : \Omega \to \Omega \times \Omega$ acts as follows: $(1_\Omega, g)(T) = (1_\Omega(T), g(T)) = (T, T)$, $(1_\Omega, g)(F) = (1_\Omega(F), g(F)) = (F, T)$. Thus $(1_\Omega, g) : \Omega \to \Omega \times \Omega$ is a monic arrow describing the subset $A_2 = \{(T, T), (F, T)\}$ of $\Omega \times \Omega$.

Likewise, the arrow $(g, 1_\Omega) : \Omega \to \Omega \times \Omega$ is a monic arrow describing the subset $A_1 = \{(T, T), (T, F)\}$ of $\Omega \times \Omega$. The union $A_1 \cup A_2$ is thus the
The image of the arrow $k : \Omega \amalg \Omega \to \Omega \times \Omega$ determined by the coproduct diagram (3.2).

The image of the arrow $k$ can be described categorically using the $\exists_k$ functor as discussed in Section 2.5. We therefore make the following definition, for an arbitrary topos.

**Definition.** The “or” arrow $\vee : \Omega \times \Omega \to \Omega$ is the classifying arrow of the image of the arrow $k : \Omega \amalg \Omega \to \Omega \times \Omega$ determined by the coproduct diagram (3.2), where $g : \Omega \to \Omega$ is the composite $t \circ h : \Omega \to 1 \to \Omega$.

**The “false” arrow.** Before proceeding to “not,” we must characterize the “false” arrow $f : 1 \to \Omega$. In the category of sets, the false arrow is the function $f : 1 \to \Omega$ defined by $f(P) = F$. The false arrow can be seen to be the classifying arrow for the empty subset of 1, described by the monic arrow $0 \to 1$, where $0$ is the empty set ($0 \to 1$ is the empty function). The empty set 0, we recall, is the initial object in the category of sets. We thus make the following definition, for an arbitrary topos.

**Definition.** The “false” arrow $f : 1 \to \Omega$ is the classifying arrow of the subobject $0 \to 1$, where $0$ is the initial object of the topos.

**The “not” arrow.** We construct the “not” arrow $\neg : \Omega \to \Omega$ as follows: In the category of sets, we have $\neg(T) = F$, $\neg(F) = T$. This means that the “not” arrow $\neg : \Omega \to \Omega$, in the category of sets, is just the classifying arrow of the subset $\{F\}$ of $\Omega$. This subset is described by the false arrow $f : 1 \to \Omega$, defined by $f(P) = F$. We thus make the following definition, for any arbitrary topos.

**Definition.** The “not” arrow $\neg : \Omega \to \Omega$ is the classifying arrow of the false arrow $f : 1 \to \Omega$. 

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The “implies” arrow. We construct the “implies” arrow $\Rightarrow$ as follows. In the category of sets, we have $\Rightarrow(((T, T)) = \Rightarrow((F, T)) = \Rightarrow((F, F)) = T, \Rightarrow((T, F)) = F$. (The idea is that an implication $P \Rightarrow Q$ is true unless the premise $P$ is true and the conclusion $Q$ is false.) We see that $P \Rightarrow Q$ is true precisely when $P \land Q = P$. This means that $\Rightarrow : \Omega \times \Omega \to \Omega$ is the classifying arrow of the subset $A$ of $\Omega \times \Omega$ consisting of those pairs $(P, Q)$ such that $P \land Q = P$, that is, $\land((P, Q)) = \pi_1((P, Q))$. The subset $A$ is thus the equalizer of the two functions $\land$ and $\pi_1$, from $\Omega \times \Omega$ to $\Omega$. We thus make the following definition for an arbitrary topos.

**Definition.** The “implies” arrow $\Rightarrow : \Omega \times \Omega \to \Omega$ is the equalizer of $\land, \pi_1 : \Omega \times \Omega \to \Omega$.

Once we have defined the internal operations $\land, \lor, \neg, \Rightarrow$ on $\Omega$, we can explore the algebraic properties of these operations. In classical logic, the propositional connectives satisfy the *boolean-algebra* axioms:

1. **Axiom 1** (commutative law for $\land$). $P \land Q = Q \land P$.
2. **Axiom 2** (commutative law for $\lor$). $P \lor Q = Q \lor P$.
3. **Axiom 3** (associative law for $\land$). $P \land (Q \land R) = (P \land Q) \land R$.
4. **Axiom 4** (associative law for $\lor$). $P \lor (Q \lor R) = (P \lor Q) \lor R$.
5. **Axiom 5** (distributive law). $P \land (Q \lor R) = (P \land Q) \lor (P \land R)$.
6. **Axiom 6** (absorption law). $P \lor (Q \land P) = (P \lor Q) \land P = P$.
7. **Axiom 7**. $P \land T = P \lor F = P$.
8. **Axiom 8**. $P \lor \neg P = T$.
9. **Axiom 9**. $P \land \neg P = F$.
10. **Axiom 10**. $P \Rightarrow Q = \neg P \lor Q$. 

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We notice some similarity to the commutative ring axioms. If we replace $\wedge$ by $\cdot$ and $\vee$ by $+$, then Axioms 1–5 correspond to the commutative, associative and distributive laws for a commutative ring. If we replace $T$ by 1 and $F$ by 0, then Axiom 7 corresponds to the identity laws: $a \cdot 1 = a + 0 = a$. The absorption law (6), however, corresponds to the relation $a + (b \cdot a) = (a + b) \cdot a = a$, which is not valid in a commutative ring.

Axiom 10 shows how implication $\Rightarrow$ can be defined in terms of $\neg$ and $\lor$, and hence need not be taken as a fundamental operation in the theory of boolean algebras.

Just as there are many different examples of commutative rings, there are many different examples of boolean algebras in mathematics. In particular, the algebra of subsets of a given set is always a boolean algebra. This boolean algebra is defined as follows. Let $\mathcal{S}$ be any set, and let $P(\mathcal{S})$ designate the power set of $\mathcal{S}$; that is, $P(\mathcal{S})$ is the set consisting of all possible subsets of $\mathcal{S}$. The power set $P(\mathcal{S})$ can be made into a boolean algebra through the following interpretations of $\wedge$, $\lor$, $\neg$, $\Rightarrow$, $T$, $F$:

1. $A \land B$ is interpreted as the intersection $A \cap B$, the set of all points both in $A$ and in $B$.
2. $A \lor B$ is interpreted as the union $A \cup B$, the set of all points either in $A$ or in $B$.
3. $\neg A$ is interpreted as the complement $C(A)$, the set of all points of $\mathcal{S}$ not in $A$.
4. $A \Rightarrow B$ is interpreted as $\neg A \cup B$, that is as $C(A) \cup B$.
5. $T$ is interpreted as $\mathcal{S}$.
6. $F$ is interpreted as the empty set, $\emptyset$.

In particular, if we take $\mathcal{S}$ to be a set consisting of a single point, then the power set $P(\mathcal{S})$ contains exactly two elements, and the boolean algebra one obtains turns out to be isomorphic to the $\{T, F\}$ boolean algebra of classical logic.

Now it can be shown that in any topos the internal logical operations $\land$, $\lor$, $\neg$, $\Rightarrow$ satisfy the following axioms:
Axioms (1)–(5) as before.

**Axiom 6′.** \( P \Rightarrow P = T \).

**Axiom 7′.** \( P \land (P \Rightarrow Q) = P \land Q \).

**Axiom 8′.** \( Q \land (P \Rightarrow Q) = Q \).

**Axiom 9′.** \( P \Rightarrow (Q \land Z) = (P \Rightarrow Q) \land (P \Rightarrow Z) \).

**Axiom 10′.** \( \neg P = P \Rightarrow F \).

Because the operations \( \land, \lor, \neg, \Rightarrow \) are internal, these relationships are to be understood as statements that certain diagrams commute, just as the axioms for an internal monoid or internal ring are interpreted as assertions that certain diagrams commute.

The axioms in this second list describe a type of algebraic structure called a **Heyting algebra**. Every boolean algebra is a Heyting algebra, but the converse is not true; a Heyting algebra need not be a boolean algebra.

The Heyting algebra axioms describe the propositional calculus of **intuitionistic logic**, discussed in the next section. In intuitionistic logic, the **law of the excluded middle**, \( P \lor \neg P = T \), is not valid.

It is not difficult to show that if a Heyting algebra satisfies the additional axiom \( P \lor \neg P = T \), then it is a boolean algebra. An equivalent condition is the **double negation law**: \( \neg\neg P = P \). Any Heyting algebra that satisfies the double negation law likewise must be a boolean algebra.

A topos whose internal logic satisfies either (and hence both) of these equivalent conditions (excluded middle, double negation) is called a **boolean topos**. Some toposes, such as the category of sets, are boolean, but others, such as the topos \( \text{Shv}(X) \) considered in Section 3.2, are not.

We have said that the subobject classifier \( \Omega \) represents the “set of truth values” for a topos’s internal logic. \( \Omega \) represents this set of truth values internally, as an object of the topos itself; \( \Omega \) is not an ordinary “classical” set. It is possible, however, to find an ordinary set \( S \) that also in a sense represents the set of truth values of the topos. To motivate this construction, we shall consider the situation in the category of sets.
Let \( A = \{ P \} \) be a set containing a single point \( P \). Suppose we wish to form a subset \( B \) of \( A \). We can think of the process of forming the subset \( B \) as the process of assigning a truth value to the formula \( P \in B \). If we assign this formula the truth value “true,” then we obtain the subset \( \{ P \} = A \), whereas if we assign this formula the truth value “false,” we obtain the empty subset \( \varnothing \). The two possible subsets correspond to the two truth values of classical logic.

Suppose, however, there were additional intermediate truth values lying between true and false. Then, by assigning these truth values to the formula \( P \in B \), we would obtain additional intermediate subsets of \( A \) lying between \( A \) itself and \( \varnothing \).

Now suppose \( T \) is any topos. The terminal object \( 1 \) of \( T \) then plays the role of the one-element set \( A \) in the category of sets. The above observation then suggests that the subobjects of \( 1 \) can be taken to represent the different possible truth values of the topos’s internal logic. The set \( S \) of subobjects of \( 1 \) then represents the set of truth values for the topos’s logic.

The set \( S \) is an ordinary, classical set. The internal operations \( \land, \lor, \neg, \Rightarrow \) give rise to ordinary operations on the set \( S \) in the following way.

We recall that the subobjects of \( 1 \) correspond to all the different possible arrows from \( 1 \) to \( \Omega \). If \( A \) is a subobject of \( 1 \), then we shall use the notation \( f_A : 1 \to \Omega \) to designate the corresponding arrow. The operations \( \land, \lor, \neg, \Rightarrow \) on the set \( S \) are defined as follows.

Suppose \( A \) and \( B \) are each elements of \( S \). Let \( (f_A, f_B) : 1 \to 1 \times 1 \) be the direct product of the arrows \( f_A \) and \( f_B \) as described in Section 2.3. Then

1. \( A \land B \) is the subobject of \( 1 \) corresponding to the arrow \( \land \circ (f_A, f_B) : 1 \to \Omega \).
2. \( A \lor B \) is the subobject of \( 1 \) corresponding to the arrow \( \lor \circ (f_A, f_B) : 1 \to \Omega \).
3. \( \neg A \) is the subobject of \( 1 \) corresponding to the arrow \( \neg \circ f_A : 1 \to \Omega \).
4. \( A \Rightarrow B \) is the subobject of \( 1 \) corresponding to the arrow \( \Rightarrow \circ (f_A, f_B) : 1 \to \Omega \).
Because the internal operations $\wedge, \lor, \neg, \Rightarrow$ satisfy the Heyting algebra axioms, it follows that the operations defined above on $S$ likewise will satisfy the Heyting algebra axioms. The subobject 1 represents the truth value $T$, and the subobject 0 represents the truth value $F$. In this way we obtain a “real” Heyting algebra representing the internal logic of the topos.

Let us see what this algebra is in the examples considered in the preceding section. Consider first the topos $T = \text{Set}^2 = \text{Set} \times \text{Set}$. Let $B = \{P\}$ be a set containing a single point; then $1 = (B, B)$ is a terminal object in $T$. The terminal object contains four subobjects: $(B, B), (\emptyset, \emptyset), (B, \emptyset), \text{ and } (\emptyset, B)$. There are thus four truth values. The logical operations in this example satisfy the boolean algebra axioms; the boolean algebra one obtains is in fact isomorphic to the direct product $\{T, F\} \times \{T, F\}$, the direct product of the classical two-valued boolean algebra with itself. The topos $T$ in this example is governed by classical logic, even though there are more than two truth values.

In the case of $n$ stages of evaluation, where $T = \text{Set}^2$, one similarly obtains a boolean algebra containing $2^n$ elements.

In the case of $\text{Shv}(X)$, the category of sheaves over the interval $X = (0, 1)$, the situation is strikingly different. We have seen in this example that the subobjects of 1 correspond to all the different possible open subsets of $(0,1)$. The set $S$ of truth values can thus be taken to be the set of all open subsets of $(0,1)$. The logical operations $\wedge, \lor, \neg, \Rightarrow$ then act as follows:

(1) $A \wedge B = A \cap B =$ the intersection of the open sets $A$ and $B$ (all points that are both in $A$ and in $B$).

(2) $A \lor B = A \cup B =$ the union of the open sets $A$ and $B$ (all points either in $A$ or in $B$).

(3) $\neg A = \text{int}(C(A)) =$ the interior of the complement of $A =$ the largest open set contained in the complement $X \setminus A$.

(4) $A \Rightarrow B =$ the largest open set $C \subseteq X$ such that $C \cap A \subseteq B$.

Let us consider what (3) means. The complement $C(A) = X \setminus A$ is simply the set of all points in $X$ that are not in $A$. Suppose $A$ is the open interval $(\frac{1}{2}, 1)$. Then $C(A) = (0, \frac{1}{2})$, that is, $C(A)$ contains the point $\frac{1}{2}$ because $A$ does not contain $\frac{1}{2}$. But the set $(0, \frac{1}{2})$ is not an open set, and
truth values are represented only by open sets. It is therefore necessary
to take the largest open subset of \((0, \frac{1}{2}]\) in defining the truth value of
\(\neg A\); this largest open subset is simply \((0, \frac{1}{2})\). Thus, if \(A = (\frac{1}{2}, 1)\), \(\neg A = (0, \frac{1}{2})\).

Since both \(X\) and \(\emptyset\) are open, and \(C(X) = \emptyset\) and \(C(\emptyset) = X\), it follows
that \(\neg T = F\) and \(\neg F = T\). We see, however, that in this example the
law of the excluded middle, \(P \lor \neg P = T\), does not hold! For example,
take \(A = (\frac{1}{2}, 1)\). Then we saw above that \(\neg A = (0, 1/2)\), and therefore
\(A \lor \neg A = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)\). Thus \(A \lor \neg A\) excludes the point \(\frac{1}{2}\), and hence
\(A \lor \neg A \neq X\). This means \(A \lor \neg A \neq T\) (because \(X = T\)).

What is \(\neg(A \lor \neg A)\)? The complement \(C(A \lor \neg A)\) consists of the single
point \(\frac{1}{2}\), so the largest open subset of \(C(A \lor \neg A)\) is the empty set,
\(\emptyset\). Hence \(\neg(A \lor \neg A) = \emptyset = F\). This in fact turns out to be true for every
open set \(B\), that is, \(\neg(B \lor \neg B) = F\). Negating once more, we obtain \(\neg\neg(A
\lor \neg A) = \neg F = T\).

Thus the operation of double negation in some sense restores the
validity of the law of the excluded middle, that is, the validity of clas-
sical logic. This turns out to be a very important feature of the internal
logic of toposes. The operation of double negation, \(\neg\neg : \Omega \to \Omega\), can be
used to transform any topos into a boolean topos, for which the internal
logic is governed by the classical laws of boolean propositional calculus.

### 3.4 Intuitionism

In the last section we saw that every topos has its own internal logic,
but that this internal logic is governed in general by the laws of intu-
itionistic logic; only in special cases is classical boolean logic valid. In
this section we shall briefly review the development of intuitionistic
mathematics. We shall later see how topos theory has fully integrated
intuitionistic mathematics into the framework of classical mathematics.

The intuitionistic school of mathematics was founded by the Dutch
mathematician L. E. J. Brouwer (1908) during the early decades of this
century. The roots go back, however, to Kronecker and the great con-
troversy he had with Georg Cantor, the founder of set theory.

The basic tenet of intuitionism is that nonconstructive methods of
reasoning about infinite totalities are not ultimately meaningful. A
mathematical construction, to be meaningful, must in some sense be actualizable on the level of the human intellect. The intuitionists
rejected, for example, mathematical proofs based upon the axiom of choice. The axiom of choice allows one to “create” a set on the basis of an infinite number of simultaneous choices. This allows one to infer the existence of sets having quite unusual properties—sets which cannot be “constructed” or presented concretely.

One famous example is the sets that arise from the Banach-Tarski paradox; using the axiom of choice, it can be shown that a sphere can be decomposed into four pieces that can then be reassembled, using only rigid motions (no distortion or stretching) to form two spheres the same size as the original. These pieces cannot be “visualized” or in any way concretely exhibited, but they can be proved to exist using the axiom of choice as an abstract existential principle.

This reasoning is precisely the type of argument that was rejected by the intuitionists. For the intuitionists, one is justified in asserting an object exists only if one can show how it can be concretely constructed; in this example, this requirement would mean providing a criterion that could be effectively applied to determine whether any given point on the sphere is in a particular subset.

The constructive approach of the intuitionists not only involved rejecting non-constructive principles of set theory, such as the axiom of choice, but further involved a modification of the very structure of logic. Consider the following example of a classical mathematical proof.

Theorem. There exist irrational numbers $r, s$ such that $rs$ is rational.

Proof: It is easily shown that $\sqrt{2}$ is irrational. Now let $t = \sqrt{2}^{\sqrt{2}}$. If $t$ is rational, then take $r = s = \sqrt{2}$. If $t$ is irrational, then take $r = t, s = \sqrt{2}$.

The intuitionists would reject the above proof, because it does not provide specific irrational numbers $r$ and $s$ with the required property. To complete the proof, one would have to determine whether or not $t$ was rational, a challenging problem in number theory!

If we analyze the logical structure of the above classical proof, we see it has the following form:
$P \lor \neg P$ (either $t$ is rational or $t$ is irrational)

$P \Rightarrow Q$

$\neg P \Rightarrow Q$

Therefore, $Q$.

The step rejected by the intuitionists is $P \lor \neg P$, the law of the excluded middle. Intuitionistic logic thus has a different logical structure than classical logic. In intuitionistic logic, one can infer a disjunction, $A \lor B$, only by either giving a proof of $A$, or giving a proof of $B$.

Brouwer’s development of intuitionistic mathematics gave rise to a formulation of mathematics in some ways drastically different from classical mathematics. For example, in intuitionistic analysis, one can prove that every function from the reals to the reals is continuous! On one level, classical and intuitionistic mathematics presented two competing formulations of mathematics, each claiming to be the “true” mathematics. On another level, it was apparent that the intuitionists were using mathematical language in a somewhat different way than the classical mathematicians, and there naturally developed an interest in “understanding” intuitionistic mathematics from a classical perspective. This involved constructing classical models for intuitionistic logic.

A first step in this direction was the formalization of intuitionistic logic by the Dutch mathematician Arend Heyting in the 1930s. Heyting (1930) showed in particular how the intuitionistic propositional calculus could be axiomatized as a Heyting algebra, and it was soon realized that there were natural topological models for this type of algebraic structure, such as the open subsets of $(0, 1)$ as we described in Section 3.3.

The crucial element in creating models for intuitionistic mathematics came from formalizing the concept of the “creative subject,” introduced by Brouwer. Brouwer’s formulation of intuitionistic mathematics reflected a change of viewpoint that shifted emphasis from the known to the knower, the subject. The concept of “creative subject” is that of an idealized mathematician, who extends both his knowledge and his universe of discourse in the course of time. For a particular mathematical principle that becomes established, there is a specific time at which it first becomes established, but it remains valid then for all future times.
In 1965 Saul Kripke constructed a semantics for intuitionistic logic based upon a formalization of the concept of the creative subject, called "stages of knowing" (Kripke, 1965). We shall describe the Kripke semantics in some detail because it provides the basis for the development of the sheaf semantics of a topos.

In the Kripke semantics, there are specified different stages of knowing, representing different stages of the knower or "creative subject." For each stage of knowing, there is a specific universe of discourse, and certain specified relationships within that universe. The stages of knowing are partially ordered, corresponding to the way knowledge can evolve in different directions. The universe of discourse at a given stage is a subset of the universe at all later stages, and the specified relationships at a later stage extend those at a given stage in a consistent way. This type of structure is called a Kripke model.

We shall give a simple example of a Kripke model for a commutative ring structure. We shall take three stages of knowing: $K_1$, $K_2$, and $K_3$, with the partial ordering defined by $K_1 < K_2$ and $K_1 < K_3$. At each of these three stages of knowing, the universe of discourse will be a classical ring structure: at $K_1$ the ring $R_1 = \mathbb{Z}$, at $K_2$ the ring $R_2 = \mathbb{Z}(i)$, and at $K_3$ the ring $R_3 = \mathbb{Z} \times \mathbb{Z}$. Because $K_2$ is a later stage than $K_1$ we must specify how the ring $R_1$ is to be included as a subring of $R_2$; this will be the standard identification of the ordinary integer $n$ with the Gaussian integer $n + 0 \cdot i$. Likewise, because $K_3$ is a later stage than $K_1$ we must specify how the ring $R_1$ is to be included as a subring of $R_3$. For this, we shall identify the integer $n$ with the element $(n, 0)$ in $R_3$. This will preserve the basic arithmetic relationships between the elements of $R_1$ in both $R_2$ and $R_3$.

Now the Kripke semantics gives a precise set of rules for sequentially determining which formulas hold or are true at each stage of knowing of a Kripke model. The motivation for these rules comes from Brouwer's concept of the creative subject. One fundamental idea is that a formula, once validated at a particular stage, must remain valid at all later stages. This expresses the theme of "infallibility" of mathematical proof: once a mathematical proposition is "proved," then it is established for all time. At the same time, the semantics reflects the constructive nature of intuitionistic mathematics: at a given stage, one is only justified in asserting that an object exists having a given property.
if there is available an object already constructed at that stage having
the given property.

The rules are themselves formulated inductively. The idea is that all
formulas are built up in a sequence of steps starting from atomic for-
mulas. In the example we are considering, the relevant language is the
language of rings, and the atomic formulas in this case are simply equa-
tions involving variables, constant symbols representing specific elements
of the ring at the given stage, and the operation symbols, +, ·, and –. An
example of an atomic formula is the formula $x + y = x \cdot (z - w \cdot w)$.

An example of an atomic formula available at stage $K_2$ is $x + i = 3 \cdot z$.
This formula is not available at stages $K_1$ or $K_3$ because the element $i$
does not belong to the rings $R_1$ or $R_3$.

From these atomic formulas one builds up more complex formulas
using the propositional connectives $\land$, $\lor$, $\neg$, $\Rightarrow$, as well as the logical
quantifiers $\forall$ (for all) and $\exists$ (there exists). The following are examples
of such formulas:

$$\forall x \exists y (y + x = 0).$$
*For all $x$ there exists a $y$ such that $y + x = 0$.*

$$\exists x (x + x = 0 \land x \cdot x = 1)$$
*There exists an $x$ such that $x + x = 0$ and $x \cdot x = 1$.*

The rules for the Kripke semantics tell us when a complex formula
holds at a given stage, based upon knowledge of whether its component
formulas hold at given stages. By applying these rules over and over
again, everything is ultimately determined by the arithmetic properties
of the mathematical structures at the different stages of knowing (in
our example, the three rings $R_1$, $R_2$, and $R_3$).

The precise formulation of the rules involves the distinction between
free and bound occurrences of variables in formulas; this we shall now
declare.

Every occurrence of a variable in a formula is either free or bound. A
*free* occurrence is one for which we are free to substitute any value. A
*bound* occurrence is one that is “trapped” by a quantifier. In any for-
mula without quantifiers, such as $x = y$, $x \in y$, $x \cdot y = 0$, $x + y = z$, all the
variables are free. In the formula $\forall x (x = y)$, $y$ is free and $x$ is bound. In the formula $\forall x \exists y (y + x = 0)$, both $x$ and $y$ are bound.

Now suppose $x$, $y$, $z$ are variables, and consider the following formulas:

\[
\begin{align*}
(i) & \quad x \cdot y = x + y \\
(ii) & \quad \exists y (x \cdot y = x + y) \\
(iii) & \quad \exists y \forall x (x \cdot y = x + y).
\end{align*}
\]

In formula (i), both $x$ and $y$ are free. The formula does not receive a truth value until we substitute values for both $x$ and $y$. Suppose, for example, that we are considering this formula in the context of the ring of integers, $\mathbb{Z}$. Then if we substitute 2 for $x$ and 2 for $y$, we obtain the formula $2 \cdot 2 = 2 + 2$, which is true; if we substitute, however, 1 for $x$ and 3 for $y$ we obtain the formula $1 \cdot 3 = 1 + 3$, which is false.

In formula (ii), $x$ is free and $y$ is bound. As soon as we substitute a value for $x$, the formula becomes either true or false. For example, if we substitute 0 for $x$, we obtain the formula $\exists y (0 \cdot y = 0 + y)$, which is true ($y = 0$ works), but if we substitute 3 for $x$, we obtain the formula $\exists y (3 \cdot y = 3 + y)$, which is false (no integer $y$ has the required property).

In formula (iii) both $x$ and $y$ are bound. The formula has no free variables and has truth value “false.”

The general situation is the following. Suppose the formula $\phi(x_1, \ldots, x_n)$ has free variables $x_1, \ldots, x_n$, and suppose $a_1, \ldots, a_n$ are names of specific elements of a ring $R$. Let $\phi(a_1, \ldots, a_n)$ designate the formula obtained by substituting $a_i$ for all free occurrences of $x_i$ in $\phi$, $a_2$ for all free occurrences of $x_2$ in $\phi$, and so forth. Then the formula $\phi(a_1, \ldots, a_n)$ has a well-defined truth value (either $T$ or $F$). The original formula $\phi(x_1, \ldots, x_n)$ containing free variables $x_1, \ldots, x_n$ is said to be valid in $R$ if the resulting formula $\phi(a_1, \ldots, a_n)$ is true for all possible substitutions of names for specific elements $a_1, \ldots, a_n$ of $R$. For example, the formula $x + y = y + x$ is valid in any commutative ring; the formula $x + x = 0$ is valid in the ring $\mathbb{Z}_2$, but not in the ring $\mathbb{Z}$.

The rules for the Kripke semantics provide criteria for determining whether a formula of the form $\phi(a_1, \ldots, a_n)$ holds (that is, is “true”) at a given stage of knowing $K$, where $a_1, \ldots, a_n$ are names of specific elements at stage $K$. The rules are formulated in a way that tells us when
a complex formula holds at a given stage, based upon knowledge of whether its component formulas hold at given stages. By applying these rules over and over again, everything is ultimately determined by the arithmetic properties of the mathematical structures at the different stages of knowing (in our example, the three rings $R_1, R_2,$ and $R_3$).

We shall state now the rules and then we shall see how they apply to our example.

1. **∧ rule.** \( \phi(a_1, \ldots, a_n) \land \psi(b_1, \ldots, b_m) \) holds at stage \( \alpha \) if and only if \( \phi(a_1, \ldots, a_n) \) holds at stage \( \alpha \) and \( \psi(b_1, \ldots, b_m) \) holds at stage \( \alpha \).

2. **∨ rule.** \( \phi(a_1, \ldots, a_n) \lor \psi(b_1, \ldots, b_m) \) holds at stage \( \alpha \) if and only if \( \phi(a_1, \ldots, a_n) \) holds at stage \( \alpha \) or \( \psi(b_1, \ldots, b_m) \) holds at stage \( \alpha \).

3. **⇒ rule.** \( \phi(a_1, \ldots, a_n) \Rightarrow \psi(b_1, \ldots, b_m) \) holds at stage \( \alpha \) if and only if for every stage \( \beta \geq \alpha \), if \( \phi(a_1, \ldots, a_n) \) holds at stage \( \beta \), then \( \psi(b_1, \ldots, b_m) \) holds at stage \( \beta \).

4. **¬ rule.** \( \neg \phi(a_1, \ldots, a_n) \) holds at stage \( \alpha \) if and only if for no \( \beta \geq \alpha \) does \( \phi(a_1, \ldots, a_n) \) hold at stage \( \beta \).

5. **∀ rule.** \( \forall x \phi(x, a_1, \ldots, a_n) \) holds at stage \( \alpha \) if and only if for every stage \( \beta \geq \alpha \) and every element \( b \) in the domain at stage \( \beta \), \( \phi(b, a_1, \ldots, a_n) \) holds at stage \( \beta \).

6. **∃ rule.** \( \exists x \phi(x, a_1, \ldots, a_n) \) holds at stage \( \alpha \) if and only if there is some element \( b \) in the domain at stage \( \alpha \) such that \( \phi(b, a_1, \ldots, a_n) \) holds at stage \( \alpha \).

It should be noticed that in rules (3), (4), and (5), to determine whether a given formula holds at stage \( \alpha \), it is necessary to know whether its component formulas hold in stages \( \beta \geq \alpha \), that is, whether its component formulas hold in all later stages. The rationale for this is the following.

Consider rule (5). As mentioned above, if a formula is declared true at a given stage, it must remain true at all later stages. Now consider the formula \( \forall x \phi(x) \). Even if we know that the formula \( \phi(x) \) is true for all \( x \) at stage \( \alpha \), this does not mean that the formula will remain true for all elements at later stages, since the universe of discourse is constantly expanding. If there is some element \( b \) at some later stage for which \( \phi(b) \)
is false, then the formula $\forall x \phi(x)$ will be invalidated at that later stage. Thus, to assert the truth of $\forall x \phi(x)$ at a given stage one must know that it will never be invalidated at a later stage. This is the rationale for rule (5).

The reasoning is similar for rules (3) and (4). In the case of rule (4), it is not enough to know that $\phi$ does not hold at stage $\alpha$ to assert that $\neg \phi$ holds at stage $\alpha$. We must know that $\phi$ will not become true at any later stage; if it did, it would invalidate $\neg \phi$, but $\neg \phi$, once true, must be true for all time.

We repeat that the rules (1)–(6) above are to be applied to formulas for which names of elements are substituted for all free variables. On this basis, we define the concept of validity for formulas containing free variables as follows.

*Definition.* The formula $\phi(x_1, \ldots, x_n)$, containing the free variables $x_1, \ldots, x_n$, is said to be *valid* in a Kripke model $M$ if, for every stage of knowing $K$ and every $n$-tuple $a_1, \ldots, a_n$ of elements at stage $K$, the formula $\phi(a_1, \ldots, a_n)$ holds at stage $K$.

This definition corresponds to the ordinary notion of validity of a formula containing free variables, except that one has to consider all the different stages of knowing. In the case when the formula $\phi$ contains no free variables, then $\phi$ is valid precisely if it holds in all stages of knowing.

With the above definition of validity, it can be shown that Kripke models are governed by intuitionistic logic in the following sense. If the formulas $\phi_1, \ldots, \phi_n$ are all valid in a Kripke model $M$, and if the formula $\psi$ can be derived from these formulas using the rules of inference of intuitionistic logic, then $\psi$ also will be valid in $M$.

Let us see how the rules for the Kripke semantics can be applied to the example introduced above. Consider first the formula $\phi_1: \exists x(x^2 + 1 = 0)$. From rule (5) it follows that this formula holds at stage $K_i$ if and only if there is an element $a \in R_i$ such that $a^2 + 1 = 0$. Such an element $a$ exists in $R_i$, namely $i$, but no such element exists in $R_1$ or $R_3$. The formula $\phi_1$ therefore holds in stage $K_2$ but does not hold in stages $K_1$ or $K_3$.

Consider next the formula $\neg \phi_1: \neg \exists x(x^2 + 1 = 0)$. From rule (4) it follows that the formula $\neg \phi_1$ holds in stage $K_1$ if and only if $\phi_1$ does not
hold in each stage ≥ \( K_1 \), namely, in the stages \( K_1, K_2, K_3 \). Since \( \phi_1 \) holds in stage \( K_2 \), this condition is not satisfied, and therefore the formula \( \neg \phi_1 \) does not hold in stage \( K_1 \).

Applying the rule (4) in stages \( K_2 \) and \( K_3 \) we see that \( \neg \phi_1 \) does not hold in stage \( K_2 \) but that it does hold in stage \( K_3 \).

Consider next the formula: \( \phi_1 \lor \neg \phi_1 \). Using rule (2), we see that this formula does not hold in stage \( K_1 \), but that it does hold in stages \( K_2 \) and \( K_3 \). This shows that the law of the excluded middle is not valid in this Kripke model! But this should not be too surprising, since the whole idea of the Kripke semantics was to provide models for intuitionistic logic rather than classical logic.

Consider now the formula: \( \neg(\phi_1 \lor \neg \phi_1) \). Applying rule (4) once more, we see that this formula is not valid in any of the stages \( K_1, K_2, \) or \( K_3 \). Finally, consider the formula: \( \neg \neg(\phi_1 \lor \neg \phi_1) \). A final application of rule (4) shows that this formula is valid in all three stages of knowing \( K_1, K_2, K_3 \). This is an expression of the phenomenon alluded to in Section 3.3, whereby the double negation operator “\( \neg \neg \)” restores the validity of classical logic; if a formula \( \phi \) is a tautology of classical logic, then \( \neg \neg \phi \) will be a tautology of intuitionistic logic.

The Kripke semantics provided a significant advance in integrating intuitionism into the framework of classical mathematics by showing how one can in principle construct meaningful models for intuitionistic theories. Kripke’s original formulation of the semantics was however not yet adequate to provide models for the intuitionistic theories of real mathematical interest. For this it was necessary to have models for higher-order intuitionistic logic. That is, it is not enough to have variables that can range over the elements of the domain under consideration. Rather, it is necessary to have variables that can range over arbitrary subsets of the domain. In our example, this would be the distinction between a variable \( x \) referring to elements of a ring, and a variable, say \( U \), referring to arbitrary sets of elements of a ring. For the complete development of higher-order logic, one must be able to talk about sets of elements, sets of sets of elements, and so on. The complete development of higher-order logic thus amounts essentially to the development of set theory. What was required to truly integrate intuitionistic mathematics into the framework of classical mathematics was to provide models for intuitionistic set theory. Topos theory pro-
vides precisely the required models. In the following sections we shall see how intuitionistic set theory can be formulated internally within a topos, and then how the Kripke semantics can be adapted to the topos to provide a meaningful model for the intuitionistic theory.

3.5 The Mitchell-Bénabou Language
In this and the following sections, we shall show how set theory can be developed internally in any topos. This set theory will be formulated in a special language called the Mitchell-Bénabou language of the topos. The Mitchell-Bénabou language differs somewhat from the familiar language of axiomatic set theory; it is an example of a typed language. We shall begin by describing this concept.

In the familiar formulation of set theory, one uses variables $x$, $y$, $z$, . . . , that can range in principle over all possible sets. This is an example of an untyped language: there is a single type of variable that in principle can talk about anything.

In a typed language, there are different types of variables, with each type restricted to a specific domain of discourse. The set-theoretic language most appropriate to an arbitrary topos turns out to be a typed language. There is a distinct type corresponding to each object of the topos. Thus, for each object $A$ of the topos, there is a distinct collection of variables of type $A$. The idea is that the variables of type $A$ should refer to “elements” of the “set” $A$.

Of course, at this point we do not know what it means to talk about elements of an object $A$. All we know is that $A$ is an object in a category; it is not presented to us as a set. Nevertheless, we can still formally develop a language treating the object $A$ as if it were a set, and then later see whether there is some meaningful way to interpret this language. This will be our approach. The meaning of the language will ultimately be unfolded from the Kripke semantics, which will reveal how each object $A$ of the topos can in fact be understood to be a variable set having real, but variable, elements.

The construction of the Mitchell-Bénabou language thus begins by assigning, to each object $A$ of the topos, a sequence of variables $x$, $y$, $z$, . . . of type $A$, with the intuitive understanding that these variables will refer to elements of $A$. These variables are special examples of terms of type $A$, symbolic expressions designating elements of $A$. 
We next introduce, for each arrow of the topos, a special function symbol in the language. This function symbol is used to describe the arrow using functional notation, just as if it were an ordinary function. We shall use the same letter to designate the arrow as the corresponding function symbol. Thus, if \( f : A \rightarrow B \) is any arrow, and if \( s \) is a term of type \( A \), then \( f(s) \) is a term of type \( B \). We think of \( f(s) \) as designating the element of \( B \) obtained by applying the “function” \( f \) to the element \( s \) of \( A \). For example, if \( f : A \rightarrow B \), \( g : B \rightarrow C \), and \( x \) is a variable of type \( A \), then \( f(x) \) is a term of type \( B \), and \( g(f(x)) \) is a term of type \( C \).

We further introduce a notation for ordered pairs. If \( r \) is a term of type \( A \) and \( s \) is a term of type \( B \), then \((r, s)\) is a term of type \( A \times B \). The idea is that if \( r \) designates an “element” of \( A \), and \( s \) designates an “element” of \( B \), then the ordered pair \((r, s)\) is an “element” of the cartesian product \( A \times B \).

All terms are built up from variables, functional notation, and ordered pair notation by repeated application of the above two rules. For example, if \( x \) is a variable of type \( A \), \( y \) a variable of type \( B \), \( f : A \rightarrow C \), \( g : C \times B \rightarrow D \), then \((x, g((f(x), y)))\) is a term of type \( A \times D \).

From terms, one constructs atomic formulas. There are two kinds of atomic formulas in this language: equations and membership relations.

The rule for equations is simple. If \( r \) and \( s \) are terms of the same type, then \( r = s \) is an atomic formula. For example, if \( x \) is a variable of type \( A \), \( y \) is a variable of type \( B \), and \( z \) is a variable of type \( A \times B \), then \((x, y) = z\) is an atomic formula.

The rule for membership relations is based upon the power-set operation for a topos. This we shall now describe.

Suppose \( A \) is any object of the topos \( T \). Then the different subobjects of \( A \) correspond to the different possible arrows from \( A \) to \( \Omega \), that is, the different possible elements of the hom-set \( T(A, \Omega) \). Thus the hom-set \( T(A, \Omega) \) naturally represents the “set of all subobjects of \( A \).” Because the category \( T \) is cartesian closed, there is an object—denoted \( \Omega^4 \)—in \( T \) that internally represents the hom-set \( T(A, \Omega) \). The object thus internally represents the “set of all subsets of \( A \),” that is, it represents the power set of \( A \). We shall use the notation \( P(A) \) to designate the object \( \Omega^4 \) and shall call it the power object of \( A \).

Suppose now \( r \) is a term of type \( A \), and \( s \) is a term of type \( P(A) \). Then the “values” of \( s \) are “elements” of \( P(A) \), that is, they are “subsets” of
$A$; the “values” of $r$ are “elements” of $A$. It is therefore appropriate to ask whether the element $r$ is a member of the subset $s$, that is, whether $r \in s$. We therefore take as atomic formulas any expression of the form $r \in s$, where $r$ is a term of type $A$, and $s$ is a term of type $P(A)$. This ability to internally express the membership relation, $\in$, lies at the basis of the development of set theory within a topos.

From the two types of atomic formulas described above, all formulas of the Mitchell-Bénabou language are sequentially constructed using the propositional connectives $\land$, $\lor$, $\neg$, $\Rightarrow$, and the quantifiers $\forall$, $\exists$. Following are some examples of such formulas.

Suppose $x$, $z$ are variables of type $A$, and $y$ is a variable of type $P(A)$. Then the following is a formula of the Mitchell-Bénabou language:

$$\forall x \exists y (x \in y \land \forall z (z \in y \Rightarrow z = x))$$

For every $x$ there exists a $y$ such that:

$x \in y$ and for every $z$, if $z \in y$ then $z = x$.

This formula expresses the familiar principle of set theory that for every element $x$ of $A$ one can form the set $y = \{x\}$ whose only element is $x$.

Of course, the rules for generating formulas are not restricted to “reasonable” formulas. The following is an example of a formula that will be false for any reasonable interpretation of the Mitchell-Bénabou language:

$$\forall x \forall z (x = z \Rightarrow \neg x = z)$$

For every $x$ and $z$, if $x = z$ then $x \neq z$.

We have seen in this section how one can associate with any topos a set-theoretic language, its Mitchell-Bénabou language. The idea of this language is to treat the objects of the topos as if they were sets and the arrows of the topos as if they were functions. We see that the language is tailor-made to the topos; each object of the topos has its own variables, and each arrow of the topos has its own function symbol in the language. However, for the language to be meaningful, it must be shown that the language can be interpreted in a consistent way as statements about the structure of the topos.
A topos is presented as a collection of objects and arrows; the objects of a topos are not presented as collections of elements, and for this reason the Mitchell-Bénabou language, which talks about the elements of the objects, has no obvious interpretation. The meaning of the Mitchell-Bénabou language is deeply hidden in the internal structure of the topos; two successive stages of interpretation are required to make this meaning explicit. The first stage is provided by the internal interpretation of the language, and the second is provided by the sheaf semantics of the topos, an adaptation of the Kripke semantics for intuitionistic logic. These we shall consider in the following sections.

3.6 Internal Interpretation
In this section we shall present the first stage of unfolding the meaning of the Mitchell-Bénabou language. This will provide a formal, internal interpretation of the language. To motivate this development, we shall begin by examining the process of interpretation of an ordinary algebraic formula in the category of sets. As an example, we shall consider the formula \( y = x + 1 \), where the variables \( x \), \( y \) are understood to range over the integers, \( \mathbb{Z} \). That is, \( x \) and \( y \) are taken to be variables of type \( \mathbb{Z} \).

The formula \( y = x + 1 \), in itself, is neither true nor false. If we substitute numerical values for the variables \( x \) and \( y \), then the formula becomes either true or false. For example, if we substitute 2 for \( x \) and 3 for \( y \), then we obtain the formula \( 3 = 2 + 1 \), which is true. If we substitute, however, 3 for \( x \) and 8 for \( y \), then we obtain the formula \( 8 = 3 + 1 \), which is false.

The logical interpretation of the formula \( y = x + 1 \) can thus be described by a function \( g \) from the cartesian product \( \mathbb{Z} \times \mathbb{Z} \) to the set of truth values \( \Omega = \{T, F\} \). The function \( g : \mathbb{Z} \times \mathbb{Z} \rightarrow \Omega \) is defined by:

\[
g((a, b)) = T \text{ if } b = a + 1 \\
g((a, b)) = F \text{ if } b \neq a + 1.
\]

Thus \( g((2, 3)) = T \), \( g((3, 8)) = F \), and so on.

The function \( g \) is clearly the classifying function of the subset \( S \) of \( \mathbb{Z} \times \mathbb{Z} \) consisting of all pairs \( (a, b) \) such that \( b = a + 1 \). Suppose
$f : \mathbb{Z} \to \mathbb{Z}$ is the function defined by $f(a) = a + 1$. Then $S$ is just the equalizer of the two functions, $\pi_2 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ and $f \circ \pi_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, where $\pi_2((a, b)) = b$, and $f \circ \pi_1((a, b)) = f(a) = a + 1$.

This observation allows us to define the internal interpretation of equational atomic formulas of the Mitchell–Bénabou language within a topos. As an example, consider the formula $y = f(x)$, where $x$ is a variable of type $A$, $y$ is a variable of type $B$, and $f : A \to B$ is an arrow from $A$ to $B$, that is, $f : A \to B$. Since $x$ is understood to range over the elements of $A$, and $y$ is understood to range over the elements of $B$, and the object $\Omega$ represents the “set” of truth values, the internal interpretation of the formula $y = f(x)$ will be described by an arrow $g : A \times B \to \Omega$.

By analogy with the situation in the category of sets, we define the internal interpretation of the formula $y = f(x)$ to be the arrow $g$ constructed as follows. The arrow $g : A \times B \to \Omega$ is the classifying arrow of the equalizer of $\pi_2 : A \times B \to B$ and $f \circ \pi_1 : A \times B \to B$; that is, if $h : C \to A \times B$ is the equalizer of $\pi_2$ and $f \circ \pi_1$, then $g : A \times B \to \Omega$ is the classifying arrow of $h$:

$$G \xrightarrow{h} A \times B \xrightarrow{\pi_2} \Omega \xleftarrow{f \circ \pi_1}$$

Thus the internal interpretation of a formula is always defined to be a certain arrow of the topos, with target $\Omega$.

As a second example, consider the formula $z = (x, y)$, where $x$ is a variable of type $A$, $y$ a variable of type $B$, and $z$ a variable of type $A \times B$. Then it is not difficult to verify that the proper definition of the internal interpretation of the formula $z = (x, y)$ is the classifying arrow $g$ of the equalizer of $\pi_3 : A \times B \times (A \times B) \to A \times B$ and $(\pi_1, \pi_2) : A \times B \times (A \times B) \to A \times B$. That is, if $h : C \to A \times B \times (A \times B)$ is the equalizer of $\pi_3$ and $(\pi_1, \pi_2)$, then $g : A \times B \times (A \times B) \to \Omega$ is the classifying arrow of $h$.

As a third example, consider the formula $f((x, y)) = h(z)$, where $x$ is a variable of type $A$, $y$ is a variable of type $B$, $z$ is a variable of type $C$, $f : A \times B \to D$, $h : C \to D$. Then the internal interpretation of the formula is the classifying arrow $g : A \to \Omega$ of the equalizer of $f : A \to A$ and the identity arrow $\text{id}_A : A \to A$.

As a final example, consider the formula $f((x, y)) = h(z)$, where $x$ is a variable of type $A$, $y$ is a variable of type $B$, $z$ is a variable of type $C$, $f : A \times B \to D$, $h : C \to D$. Then the internal interpretation of the formula is the classifying arrow $g : A \times B \times C \to \Omega$ of the equalizer of $f \circ (\pi_1, \pi_2) : A \times B \times C \to D$ and $h \circ \pi_3 : A \times B \times C \to D$. 

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We next consider interpretation of atomic formulas involving the
membership relation, formulas of the form \( r \in s \), where \( r \) is a term of
type \( A \) and \( s \) is a term of type \( P(A) \). We recall that the object \( P(A) = \Omega^A \) where the exponential functor \(-^A\) is the right adjoint of the product
functor \(- \times A\). The situation here is somewhat complex, so we shall first
examine it in detail in the category of sets.

In the category of sets, the object \( \Omega^A \) is the hom-set consisting of all
possible functions \( f \) from \( A \) to \( \{T, F\} \). As we discussed in Section 2.6,
the counit transformation \( \epsilon : A \times \Omega^A \to \Omega \) is the “evaluation arrow”
\( \text{ev}((a, f)) = f(a) \).

Suppose now \( x \) is a variable of type \( A \) and \( y \) is a variable of type
\( P(A) = \Omega^A \). What is the interpretation map of \( x \in y \) in the category of
sets? The values of \( x \) are elements of the set \( A \), and the values of \( y \) are
elements of the set \( \Omega^A \), that is, they are functions from \( A \) to \( \Omega \). If the
value of \( y \) is the function \( f \), then the subset designated by \( y \) is the subset
of \( A \) having \( f \) as its classifying map. To distinguish the subset from the
function we shall write \( Bf \) for the subset designated by the function \( f \).
The formula \( x \in y \) is a formula of the Mitchell-Bénabou language. Its
intended meaning, in the category of sets, is \( x \in B_y \), where the “\( \in \)” here
designates the ordinary membership relation of set theory. Now by the
definition of the classifying function of a subset, \( a \in B \) if and only if
\( f(a) = T \), that is, if and only if \( \text{ev}((f, a)) = T \). This means that the subset
of \( A \times \Omega^A \) consisting of all ordered pairs \((a, f)\) such that \( a \in B \), is the
subset classified by the evaluation function \( \text{ev} : A \times \Omega^A \to \Omega \). Thus the
intended interpretation of the formula \( x \in y \) of the Mitchell-Bénabou
language, in the category of sets, is described precisely by the evaluation
function \( \text{ev} : A \times \Omega^A \to \Omega \). We therefore make this the definition
of the internal interpretation of the membership relation in an arbitrary
topos. If \( x \) is a variable of type \( A \), and \( y \) is a variable of type \( P(A) \), then
the internal interpretation of the formula \( x \in y \) is defined to be the
evaluation arrow \( \text{ev} : A \times \Omega^A \to \Omega \).

We next consider the interpretation of formulas built up from atomic
formulas by means of the logical connectives \( \land, \lor, \neg, \Rightarrow \). We shall use
the internal propositional calculus to interpret these formulas.

As an example, consider the formula \( z = f(x) \land b(y) = y \) where \( x \) is
of type \( A \), \( z \) of type \( B \), and \( y \) of type \( C \). The interpretation will be an
arrow \( g : A \times B \times C \to \Omega \). We build up \( g \) from the interpretations of
the component atomic formulas \( z = f(x) \) and \( b(y) = y \), together with the “and” arrow \( \land : \Omega \times \Omega \rightarrow \Omega \).

We shall first see what the interpretation of this formula is in the category of sets. The interpretation of the formula \( z = f(x) \land b(y) = y \) is the function \( g : A \times B \times C \rightarrow \Omega \) defined by \( g(a, b, c) = T \) if \( b = f(a) \) and \( h(c) \) is defined to be the function \( \Omega \times \Omega \rightarrow \Omega \). The interpretation \( k : A \times B \rightarrow \Omega \) of \( z = f(x) \) is defined by: \( k(a, b) = T \) if \( b = f(a) \) and \( k(a, b) = F \) otherwise. The interpretation \( j : C \rightarrow \Omega \) of \( b(y) = y \) is defined by \( j(c) = T \) if \( b(c) = c \) and \( j(c) = F \) otherwise. Comparing these functions, we see that \( g(a, b, c) = T \) if both \( k(a, b) = T \) and \( j(c) = T \), and \( g(a, b, c) = F \) otherwise. By the definition of the truth table for \( \land \), it follows that \( g(a, b, c) = k(a, b) \land j(c) = \land ((k(a, b), j(c))) \).

Now the function that takes \((a, b, c) \rightarrow (a, b)\) is the function \((\pi_1, \pi_2) : A \times B \times C \rightarrow A \times B\), and the function that takes \((a, b, c) \rightarrow c\) is the function \(\pi_3 : A \times B \times C \rightarrow C\). Putting all these maps together we obtain: \( g = \land \circ (k \circ (\pi_1, \pi_2), j \circ \pi_3)! \) This description of \( g \) is purely in the language of category theory, and can therefore be generalized to an arbitrary topos.

Thus, let \( T \) be any topos, \( x \) a variable of type \( A \), \( z \) a variable of type \( B \), \( y \) a variable of type \( C \), and \( f : A \rightarrow B \), \( h : C \rightarrow C \) arrows of \( T \). Let \( k : A \times B \rightarrow \Omega \) be the interpretation of \( z = f(x) \), and let \( j : C \rightarrow \Omega \) be the interpretation of \( b(y) = y \). Then the interpretation of the formula \( z = f(x) \land b(y) = y \) is defined to be the arrow:

\[
g = \land \circ (k \circ (\pi_1, \pi_2), j \circ \pi_3) : A \times B \times C \rightarrow \Omega.
\]

As a final example, consider the formula \( z = f(x) \lor b(z) = w \) where \( x \) is of type \( A \), \( z \) of type \( B \), and \( w \) of type \( A \). The interpretation will be an arrow \( g : A \times A \times B \rightarrow \Omega \). The first \( A \) is for \( x \), the second \( A \) is for \( w \), and \( B \) is for \( z \). We build up \( g \) from the interpretations of the component atomic formulas \( z = f(x) \) and \( b(z) = w \), together with the “or” arrow \( \lor : \Omega \times \Omega \rightarrow \Omega \).

Suppose the interpretation of \( z = f(x) \) is the arrow \( k : A \times B \rightarrow \Omega \), and the interpretation of \( b(z) = w \) is the arrow \( n : A \times B \rightarrow \Omega \). Then the arrow \( g : A \times A \times B \rightarrow \Omega \) is defined by \( g = \lor \circ m \), where \( m : A \times A \times B \rightarrow \Omega \times \Omega \) is defined by \( m = (k \circ (\pi_1, \pi_2), n \circ (\pi_1, \pi_3)) \) and \( \lor : \Omega \times \Omega \rightarrow \Omega \) is the “or” arrow.
The general situation is the following. If the formula $\phi$ is built up from atomic formulas using the propositional connectives, and if $\phi$ contains the variables $x_1, \ldots, x_n$ of respective types $A_1, \ldots, A_n$, then the interpretation of $\phi$ will be an arrow from $A_1 \times A_2 \times \cdots \times A_n$ to $\Omega$.

To complete our description of the internal interpretation of the Mitchell-Bénabou language, we must treat the quantifiers $\exists$ (there exists) and $\forall$ (for all).

To begin, we recall the distinction between free variables and bound variables for formulas containing quantifiers: Free variables are those for which one is “free” to substitute values. This means that if $\phi(x_1, \ldots, x_n)$ is a formula containing free variables $x_1, \ldots, x_n$, of respective types $A_1, A_2, \ldots, A_n$, then the interpretation of $\phi$ should be an arrow from $A_1 \times A_2 \times \cdots \times A_n$ to $\Omega$. In case the formula contains no free variables, then its interpretation will be an arrow $1 \to \Omega$. (In the case of the category of sets, this will be the “true” arrow if the formula is true, and the “false” arrow otherwise.)

We can now describe the internal interpretation of $\forall$ and $\exists$. Suppose $\phi(x, y)$ is a formula containing two free variables $x, y$, with $x$ of type $A$ and $y$ of type $B$. Then the interpretation of $\phi(x, y)$ will be an arrow $g : A \times B \to \Omega$. We construct the interpretation of $(\exists x)\phi(x, y)$ as follows.

Let $C : A \times B$ be the subobject of $A \times B$ classified by the arrow $g : A \times B \to \Omega$. Let $\pi_2 : A \times B \to B$, and consider the “existential quantification along $\pi_2$” functor $\exists_{\pi_2} : P(A \times B) \to P(B)$ described in Section 2.5. If we apply this functor to $C \to A \times B$, we obtain a subobject $D \to B$ of $B$. We define the interpretation of $(\exists x)\phi(x, y)$ to be the classifying map $h : B \to \Omega$ of the subobject $D \to B$.

Let us see what this all means in the category of sets. The arrow $g : A \times B \to \Omega$ will be the classifying map of the subset $C \subset A \times B$ consisting of all pairs $(a, b)$ such that $\phi(a, b)$ is true. The functor $\exists_{\pi_2}$ will take $C$ to the subset $D$ of $B$ consisting of all points $r$ of $B$ such that there exists some point $c$ in $C$ such that $\pi_2(c) = r$. This means $D$ consists of all points $r$ of $B$ such that there exists some point $a$ in $A$ such that $(a, r)$ is in $C$, that is, $\phi(a, r)$ is true. Thus $D$ is precisely the subset of $B$ for which $(\exists x)\phi(x, y)$ is true. The categorical construction works!
In a similar way, the internal interpretation of the formula \((\forall x)\phi(x, y)\) can be constructed using the “universal quantification along \(\pi_2\)” functor \(\forall_{\pi_2} : \mathcal{P}(A \times B) \rightarrow \mathcal{P}(B)\).

The above constructions generalize to formulas \(\phi(x, y_1, \ldots, y_n)\) containing any number of free variables, in a straightforward way.

We have described now all the rules necessary to construct the internal interpretation of any formula of the Mitchell-Bénabou language. The internal interpretation is still something quite formal; it doesn’t really tell us what a formula means; it just associates with any formula a certain arrow of the topos. This is, however, enough to provide a criterion of “truth” for formulas of the Mitchell-Bénabou language. This criterion of truth is called “internal validity,” as we shall now describe.

We shall begin by reviewing the concept of validity for formulas in the ordinary context of algebra in the category of sets. We recall that a formula is said to be valid if it is true for all possible values of the free variables in the formula. For example, if \(x, y\) are variables of type \(\mathbb{Z}\), then the formula \(y = x + 1\) is not a valid formula, because there are values for \(x, y\) for which the formula is false, for example, \(x = 3, y = 8\). The formula \(x + y = y + x\) is, however, valid; it is true for all possible values of \(x, y\).

If we have a valid formula, such as \(x + y = y + x\), then the interpretation function \(g : \mathbb{Z} \times \mathbb{Z} \rightarrow \Omega\) takes every element \((a, b)\) of \(\mathbb{Z} \times \mathbb{Z}\) to the true element \(T\) of \(\Omega\), that is, \(g((a, b)) = T\). This property of \(g\) can be expressed in the language of arrows in the following way. The interpretation arrow \(g : \mathbb{Z} \times \mathbb{Z} \rightarrow \Omega\) can be “factored” \(g = t \circ h\), where \(h\) is the (unique) arrow \(\mathbb{Z} \times \mathbb{Z} \rightarrow 1\), and \(t\) is the “true” arrow \(t : 1 \rightarrow \Omega\); that is,

\[
g((a, b)) = t(h((a, b))) = t(P) = T.
\]

We say that \(g\) factors through the “true” arrow \(t : 1 \rightarrow \Omega\). This is taken to be the criterion for internal validity of a formula of the Mitchell-Bénabou language.

**Definition.** A formula of the Mitchell-Bénabou language is internally valid if its internal interpretation \(g\) factors through the true arrow \(t : 1 \rightarrow \Omega\).
For example, the formula \( g = f(x) \land h(x) = w \), considered above, would be internally valid if its interpretation \( g : A \times A \times B \to \Omega \) could be expressed in the form \( g = t \circ h \), where \( h : A \times A \times B \to 1 \), and \( t : 1 \to \Omega \) is the true arrow.

At this point we have developed a criterion of “truth” for formulas of the Mitchell-Bénabou language, even though we still don’t know what the language means. This notion of truth can be shown to be governed by the laws of intuitionistic logic in the following sense. If the formulas \( \phi_1, \phi_2, \ldots, \phi_n \) are internally valid in \( T \), and if the formula \( \psi \) can be inferred from these formulas using the rules of inference of intuitionistic logic, then \( \psi \) also will be internally valid in \( T \). This should not be surprising, since the internal propositional calculus of the topos was used to define the internal interpretation of the Mitchell-Bénabou language.

Having defined the notion of internal validity, one can explore the relationship between the “internal” description of the topos in its own set-theoretic language, and the “external” description of the topos in the language of category theory. These two descriptions of the topos turn out to be in perfect harmony with one another.

For example, suppose we have three arrows \( f : A \to B \), \( g : B \to C \), \( h : A \to C \), and \( x \) is a variable of type \( A \). Then the formula \( g(f(x)) = h(x) \) is internally valid if and only if the formula \( h = g \circ f \) is externally valid.

As a second example, suppose \( f : A \to B \), and \( x, y \) are two variables of type \( A \). Then the formula \( f(x) = f(y) \Rightarrow x = y \) is internally valid if and only if “\( f \) is a monic arrow” is externally valid; that is, if and only if \( f \circ g = f \circ h \) implies \( g = h \) is externally valid. The formula \( f(x) = f(y) \Rightarrow x = y \) is the set-theoretic assertion that “the function \( f \) is one-to-one.” Thus the category theoretic statement “\( f \) is a monic arrow” is externally true if and only if the set-theoretic statement “the function \( f \) is one-to-one” is internally true.

As a final example, suppose \( f : A \to B \), \( x \) is of type \( A \), and \( y \) is of type \( B \). Then the formula \( \forall y \exists x (f(x) = y) \) is internally valid if and only if “\( f \) is an epic arrow” is externally valid. The formula \( \forall y \exists x (f(x) = y) \) is the set-theoretic assertion that “the function \( f \) is onto.” Thus the category-theoretic statement “\( f \) is an epic arrow” is externally true if and only if the set-theoretic statement “the function \( f \) is onto” is internally true.
At this point in our development we can begin to appreciate the way topos theory has provided a new integration of set theory and category theory. A topos can be described in two different languages, an “internal” set-theoretic language and an “external” category-theoretic language. Topos theory demonstrates that these two descriptions are in perfect harmony and shows how each language can be translated into the other.

The correspondence between the internal and external descriptions of a topos makes it possible to develop the theory of a topos internally, in the Mitchell-Bénabou language, and then interpret the theorems as statements about the categorical structure of the topos. This provides a powerful tool for the development of topos theory. When one develops topos theory internally in this way, one reasons about the objects just as if they were sets, and the arrows just as if they were functions. One must be alert, however, to use only principles of logical inference that are intuitionistically valid, because the internal logic of a topos will in general be intuitionistic, and not classical.

In the following section we shall see how certain set theory “axioms” are always internally valid in a topos, and how a topos thereby provides a natural model for intuitionistic set theory.

3.7 Internal Set Theory
Set theory, as a foundational theory of mathematics, is developed axiomatically, starting from the Zermelo-Fraenkel axioms. One of the motivations for the development of topos theory was the search for a category-theoretic foundation for set theory. In this section we shall see how certain set theory axioms will be satisfied in any topos. This will make possible the axiomatic development of set theory in the internal language of the topos.

The internal set-theory of a topos is developed in the Mitchell-Bénabou language, which differs from the symbolic language of Zermelo-Fraenkel set theory in several fundamental respects. The most significant difference is that the Mitchell-Bénabou language is a typed language, having a different type of term for each object of the topos. This will necessitate some difference in the way the axioms are formulated. Nevertheless, one can formulate axioms expressing the essential content of most of the Zermelo-Fraenkel axioms, and on this basis
develop mathematical theories internally within the topos. This we shall now consider.

The first of the Zermelo-Fraenkel axioms is the axiom of extensionality. The axiom of extensionality asserts that the sets $x$, $y$ are equal if and only if they contain precisely the same elements. The symbolic expression of this axiom in Zermelo-Fraenkel set theory is

$$x = y \iff \forall z (z \in x \iff z \in y)$$

$x = y$ if and only if for every $z$, $z \in x$ if and only if $z \in y$.

To express this axiom in the Mitchell-Bénabou language, we just need to specify the types of the variables. If the variable $z$ is of type $A$, then the variables $x$ and $y$ must be of type $P(A)$. One can show then that this formula of the Mitchell-Bénabou language is always internally valid.

We consider next the axiom of pairs. This asserts that for any $x$ and $y$ one can always form the pair set $z = \{x, y\}$. The symbolic expression of this axiom is the following:

$$\exists z \forall w (w \in z \iff (w = x \lor w = y))$$

There exists a $z$ such that for all $w$, $w \in z$ if and only if $w = x$ or $w = y$.

This axiom expresses a constructive principle that allows one to create new sets from given sets, in this case the set $z = \{x, y\}$ from the sets $x$ and $y$. As in the case of the axiom of extensionality, we can express this principle in the Mitchell-Bénabou language provided we use variables of appropriate type. If $x$ is of type $A$, then $w$ and $y$ must also be of type $A$ and $z$ must be of type $P(A)$. It can be shown then that the formula: $\exists z \forall w (w \in z \iff (w = x \lor w = y))$ is internally valid. One can in fact do something more; one can find a specific term $\tau$ of the Mitchell-Bénabou language designating the set $z = \{x, y\}$. That is, one can find a term $\tau$ with the property that the formula $w \in \tau \iff (w = x \lor w = y)$ is internally valid. We shall describe now how this is done. The argument is somewhat subtle, but it is a very powerful technique that can be generalized to construct terms representing sets characterized by any property expressible in the Mitchell-Bénabou language. We shall examine
the construction first in the particular case of the pair set \( z = \{x, y\} \), and
then we shall consider the general case and several of its applications.

We wish to construct a term \( \tau \) of type \( P(A) \) such that the formula
\( w \in \tau \iff (w = x \lor w = y) \) is internally valid, where \( x, y \), and \( w \) are of
type \( A \). We begin by considering the formula \( \phi \): \( w = x \lor w = y \). The
interpretation of the formula \( \phi \) is an arrow \( f : A \times A \times A \to \Omega \), where
the first \( A \) is associated with \( x \), the second with \( y \), and the third with \( w \). Now the triple product \( A \times A \times A \) is isomorphic to \( (A \times A) \times A \); the
isomorphism is given by \( g = (\pi_1 \circ \pi_1, \pi_2 \circ \pi_1, \pi_2) : (A \times A) \times A \to A \times A \times A \). (In the category of sets, this is simply the function mapping \( ((a, b), c) \) to \( (a, b, c) \).) Let \( h = f \circ g : (A \times A) \times A \to \Omega \). (In the category of sets, \( h \) is
the function that maps \( ((a, b), c) \) to \( (a, b, c) \).) Because of the adjointness
of the functors \(- \times A\) and \(-^A\), the arrow \( h \) corresponds to a unique arrow
\( k : A \times A \to \Omega^A \). But \( A \) is the object \( P(A) \); hence \( k \) is an arrow from \( A \times A \)
to \( P(A) \). Now \( (x, y) \) is a term of type \( A \times A \), and hence \( k((x, y)) \) is a term
of type \( P(A) \). This is the required term \( \tau \)!

Let us see why this \( \tau \) works in the category of sets. By the defi-
nition of the adjunction between \(- \times A\) and \(-^A\), the function \( k : A \times A \to \Omega^A \) takes the ordered pair \( (a, b) \in A \times A \) to the function
\( m : A \to \Omega^A \) defined by \( m(c) = h((a, b), c) = f(a, b, c) \). But by the definition
of the interpretation of \( \phi \), \( f(a, b, c) = T \) if \( c = a \) or \( c = b \), and \( f(a, b, c) = F \)
otherwise. This means that \( m \) is the classifying function for the subset
\( \{a, b\} \subseteq A \). When we identify the power set \( P(A) \) with the hom-set \( \Omega^A \),
the subset \( \{a, b\} \subseteq A \) is thus identified with the function \( m \). The func-
tion \( k : A \times A \to P(A) \) thus maps the ordered pair \( (a, b) \) to the set \{a, b\};
this means that the term \( k(x, y) \) designates the set \{x, y\}.

In any topos, the term \( k(x, y) \), constructed as above, will represent
the pair \{x, y\} in the sense that the formula \( w \in k(x, y) \iff (w = x \lor w = y) \)
will always be internally valid.

The above construction generalizes to any set characterized by some
property expressible in the Mitchell-Bénabou language. The general
situation is the following.

Suppose \( A \) is an object of the topos. We wish to construct the “sub-
set” of \( A \) consisting of “all \( w \) in \( A \) such that \( \phi(y_1, \ldots, y_n, w) \) is true,”
where \( \phi(y_1, \ldots, y_n, w) \) is a formula of the Mitchell-Bénabou language
containing free variables, \( y_1, y_2, \ldots, y_n, w \). (Here, \( w \) is of type \( A \), and
y_1, \ldots, y_n \text{ are of respective types } B_1, \ldots, B_n.) \text{ One can always construct a term } \tau \text{ of type } P(A) \text{ representing the desired subset, that is, a term } \tau \text{ such that the formula } w \in \tau \iff (y_1, y_2, \ldots, y_n, w) \text{ is internally valid. The construction is the following.}

Let } f: B_1 \times B_2 \times \cdots \times B_n \times A \to \Omega \text{ be the interpretation of } \phi. \text{ Let } g \text{ be the isomorphism}

\[
g = (\pi_1 \circ \pi_1, \pi_2 \circ \pi_1, \ldots, \pi_n \circ \pi_1, \pi_2): \quad (B_1 \times B_2 \times \cdots \times B_n) \times A \to B_1 \times B_2 \times \cdots \times B_n \times A.
\]

Set } h = f \circ g: (B_1, \ldots, B_n) \times A \to \Omega. \text{ Then by adjointness of } - \times A \text{ and } -^A \text{ as above, } h \text{ corresponds to an arrow } k: B_1 \times B_2 \times \cdots \times B_n \to \Omega^A. \text{ The term } k(y_1, y_2, \ldots, y_n) \text{ is the required term } \tau.

Following are several other examples of this technique of term construction.

**Binary Unions.** Suppose we wish to construct a term } \tau \text{ representing the union of two sets } y_1 \cup y_2. \text{ The term } \tau \text{ is required to have the property that the formula } w \in \tau \iff (w \in y_1 \lor w \in y_2) \text{ is internally valid. If } w \text{ is of type } A, \text{ then } y_1 \text{ and } y_2 \text{ must be of type } P(A), \text{ and } \tau \text{ also will then have type } P(A). \text{ Here the formula } \phi(y_1, y_2, w) \text{ is the formula } w \in y_1 \lor w \in y_2.

**Binary Intersections.** Suppose we wish to construct a term } \tau \text{ representing the intersection of two sets } y_1 \cap y_2. \text{ The term } \tau \text{ is required to have the property that the formula } w \in \tau \iff (w \in y_1 \land w \in y_2) \text{ is internally valid. If } w \text{ is of type } A, \text{ then } y_1 \text{ and } y_2 \text{ must be of type } P(A), \text{ and } \tau \text{ also will then have type } P(A). \text{ Here the formula } \phi(y_1, y_2, w) \text{ is the formula } w \in y_1 \land w \in y_2.

**Arbitrary unions.** One of the Zermelo-Fraenkel axioms asserts that for any set } y, \text{ one can form the union } \cup y \text{ of all the sets in } y, \text{ that is, the set consisting of all possible elements of elements of } y. \text{ If the term } \tau \text{ is to designate the union set } \cup y, \text{ then the formula } w \in \tau \iff \exists x (w \in x \land x \in y) \text{ must be internally valid. If } w \text{ is of type } A, \text{ then } x \text{ must be of type } P(A), \text{ and } y \text{ of type } P(P(A)). \text{ The term } \tau \text{ will then be of type } P(A). \text{ Here the formula } \phi \text{ is the formula } \exists x (w \in x \land x \in y). \text{ (We have used } x \text{ and } y \text{ rather than } y_1 \text{ and } y_2.)
Arbitrary intersections. One can similarly construct arbitrary intersections. The intersection set $\cap y$ is the set of all $w$ that are elements of every element of $y$. If the term $\tau$ is to designate the intersection set $\cap y$, then the formula $w \in \tau \iff \forall x(x \in y \iff w \in x)$ must be internally valid. If $w$ is of type $A$, then $x$ must be of type $P(A)$, and $y$ of type $P(P(A))$. The term $\tau$ will then be of type $P(A)$. In this case, $\phi$ is the formula $\forall x(x \in y \iff w \in x)$.

Power-set operation. As a final example we consider the power-set construction. One of the Zermelo-Fraenkel axioms is the power-set axiom: For every $y$ there exists a set $P(y)$ consisting of all possible subsets of $y$. In a sense, a topos has an “external” power set operation that takes each object $A$ to the power-object $P(A)$. The power-set construction can be internalized as follows. We wish to construct a term $\tau$ representing the power-set $P(y)$, where $y$ is a variable of the Mitchell-Bénabou language. This means that the formula $w \in \tau \iff (w \subseteq y)$ must be internally valid. Now the formula $w \subseteq y$ is an abbreviation for the formula $\forall x(x \in w \Rightarrow x \in y)$; thus the expanded expression of the formula characterizing $\tau$ is: $w \in \tau \iff \forall x(x \in w \Rightarrow x \in y)$. Thus if $x$ is of type $A$, then $y$ and $w$ must be of type $P(A)$, and $\tau$ will then be of type $P(P(A))$. Here $\phi$ is the formula: $\forall x(x \in w \Rightarrow x \in y)$.

In general, for every formula $\phi(y_1, \ldots, y_n, w)$ of the Mitchell-Bénabou language one can construct a term $\tau$ and introduce the set-theory axiom: $w \in \tau \iff \phi(y_1, \ldots, y_n, w)$. These “axioms” can all be shown to be internally valid using a category-theoretic argument. The totality of these axioms constitutes the axiom scheme of comprehension.

The axiom of extensionality, together with the axiom scheme of comprehension, completes the axiomatization of set theory for an arbitrary topos.

These axioms we have described so far, however, are not yet adequate to develop very much of the body of mathematics. What is lacking is an “axiom of infinity” that provides us with an infinite set.

Infinite sets lie at the heart of modern mathematics. The standard Zermelo-Fraenkel axioms of set theory include an axiom of infinity asserting the existence of an infinite set containing all the natural numbers. From the other axioms alone, there is no way to infer the existence of an infinite set. In fact, the hereditarily finite sets (all sets generated
from the null set in a finite number of steps) satisfy all the other axioms, so some additional axiom is required.

The category consisting of all hereditarily finite sets is easily seen to be a topos. This means that a topos, in general, need not contain an “infinite set.” If we wish to have an infinite set in the internal set theory of a topos, we must impose some additional requirement on the topos.

The appropriate requirement is that the topos contain a natural number object, an object that represents internally the infinite set of natural numbers \( N = \{0, 1, 2, 3, \ldots \} \). To motivate the categorical definition of a natural number object, we shall first review the process of recursive definition of a function.

A recursive definition is a self-referral kind of definition, in which a function \( f: N \to A \) is defined in terms of itself. The recursive definition utilizes the successor function \( s: N \to N \) to sequentially unfold all the values of the function \( f \).

The successor function \( s \) is the function that takes each natural number to its “successor”: \( s(0) = 1, s(1) = 2, s(2) = 3 \), and so on. A typical example of a recursive definition is the definition of addition for natural numbers in terms of the successor function.

The recursive definition of addition is given by the following two formulas:

\[
\begin{align*}
(i) \quad m + 0 &= m \\
(ii) \quad m + s(n) &= s(m + n).
\end{align*}
\]

From these two formulas, one can sequentially unfold all possible sums of natural numbers. For example:

\[
\begin{align*}
3 + 0 &= 3 \quad \text{(by i)} \\
3 + 1 &= 3 + s(0) \\
&= s(3 + 0) \quad \text{(by ii)} \\
&= s(3) = 4 \\
3 + 2 &= 3 + s(1) \\
&= s(3 + 1) \quad \text{(by ii)} \\
&= s(4) = 5.
\end{align*}
\]
The self-referral aspect of the definition is found in (ii), in which “+,” the operation being defined, occurs on both sides of the equals sign, and hence is being defined in terms of itself. This “curving back” process provides a “loop” for sequentially unfolding all the values of the operation “+.” This is reminiscent of the verse of the Bhagavad Gita, “Curving back on myself I create again and again” (9.8). This verse gives expression to the mechanics through which creation sequentially unfolds from the self-interacting dynamics of the Samhita. A discussion of the role of recursion in the foundations of set theory and its relationship to Maharishi Vedic Science in Weinless’s “Self-Referral in the Foundation of Mathematics” (Volume 5, Part 1 of this series, 2011).

Once addition has been defined, one can define multiplication recursively by the formulas:

(i) \( m \cdot 0 = 0 \)
(ii) \( m \cdot s(n) = m \cdot n + m. \)

A final example is the definition of powers of an element \( a \) of a ring \( R \):

(i) \( a^0 = 1 \) ( = the identity element of \( R \))
(ii) \( a^n = a^n \cdot a. \)

In this example, we are defining a function \( f : N \to R, n \to a^n \), by a pair of formulas:

(i) \( f(0) = 1 \)
(ii) \( f(s(n)) = f(n) \cdot a. \)

These examples generalize to the following situation. Let \( X \) be any set, \( g : X \to X \) a function from \( X \) to itself, and let \( a \) be any element of \( X \). Then we can define a function \( f : N \to X \) recursively by the formulas:

(i) \( f(0) = a \)
(ii) \( f(s(n)) = g(f(n)). \)

It is easy to see that these two formulas sequentially unfold all the values of \( f \):
$f(0) = a,$
$f(1) = f(s(0)) = g(f(0)) = g(a),$
$f(2) = f(s(1)) = g(f(1)) = g(g(a)),$
$f(3) = g(g(g(a))),$ and so on.

We can translate this recursive definition into the language of arrows as follows. We replace the element 0 of $N$ by an arrow $0 : 1 \to N$ (where 1 is the terminal object), and we replace the element $a$ of $X$ by an arrow $a : 1 \to X$, to obtain diagram (3.4).

The triangle says $f \circ 0 = a$, that is, $f(0) = a$, and the square says $f \circ s = g \circ f$, that is, $f(s(n)) = g(f(n))$ for all $n$.

We are thus led to the following definition of a natural number object in a topos.

Definition. A natural number object $(N, s, 0)$ consists of an object $N$, an arrow $s : N \to N$, and an arrow $0 : 1 \to N$, with the following universal property: for every triple $(X, g, a)$ consisting of an object $X$, an arrow $g : X \to X$, and an arrow $a : 1 \to X$, there exists a unique arrow $f : N \to X$ such that the diagram (3.4) commutes.

It is not hard to prove that, from the set-theoretic point of view, the existence of a natural number object is equivalent to the axiom of infinity. More precisely, one can prove from $ZF – Infinity$ (the $ZF$ set theory axioms without the axiom of infinity) that there is an infinite set if and only if there is a natural number object.

If we work within any topos that contains a natural number object $N$, we can, in the usual way, construct from $N$ an integer object $Z$ and a rational number object $Q$, having all the familiar properties. Using the power-set operation, one can then construct a real number object $R$. It is at this point that a significant departure from classical mathematics occurs.
In classical analysis, there are several different ways to construct the reals $\mathbb{R}$ from the rationals $\mathbb{Q}$, for example, Cauchy sequences and Dedekind cuts. These different constructions are classically equivalent in the sense that they give rise to isomorphic sets of real numbers. It is necessary, however, to use classical logic to show that these constructions are equivalent; in a non-boolean topos, these constructions will in general give rise to nonisomorphic real number objects. This means that the “Cauchy” reals will not be the same as the “Dedekind” reals. Thus the internal mathematics developed in a non-boolean topos will differ significantly from classical mathematics.

There is one further axiom of Zermelo-Fraenkel set theory we must consider: the axiom of choice. This is a very powerful principle of set theory that is applied in all fields of classical mathematics. The axiom of choice describes the formation of a set on the basis of an infinity of simultaneous choices. Its precise formulation is the following.

**Axiom of Choice.** Let $S$ be a set, each of whose elements is a non-empty set, and suppose that whenever $A, B$ are two distinct elements of $S$ then $A \cap B = \emptyset$ (that is, the elements of $S$ are pairwise disjoint). Then there exists a set $T$ containing precisely one element from each element $A$ of $S$.

The set $T$ “chooses” one element from each element of $S$. If $S$ is an infinite set, then the formation of $T$ involves in general an infinity of choices, and these choices must be completed to form a set.

The axiom of choice thus describes the functioning of a level of intelligence that can make an infinity of choices simultaneously. This capability can be attributed to the unmanifest field of intelligence at the basis of creation, a field of intelligence described in Maharishi Vedic Science as cosmic intelligence. A discussion of the axiom of choice as a description of cosmic intelligence is contained in Weinless (2011).

The axiom of choice is an example of a nonconstructive principle of mathematics. It allows one to infer the existence of sets that cannot be concretely exhibited or “constructed.” A striking example is provided by the Banach-Tarski paradox, through which the axiom of choice is applied to decompose a sphere into four pieces that can be reassembled to form two spheres the same size as the original. These four pieces are
non-measurable sets and cannot be concretely displayed, but their existence can nevertheless be inferred from the axiom of choice.

The whole school of intuitionistic mathematics in fact arose as a reaction to such nonconstructive methods, giving rise to a formulation of mathematics based upon a different structure of logic. Topos theory provides a framework in which one can explore the relationship between the axiom of choice and the internal logic of a topos. It is easy to express the axiom of choice in categorical language. If we form the union $V = \bigcup S$ of all the sets in $S$ in our formulation of the axiom of choice, then we have an epic function $f$ from $V$ to $S$ that takes each element of $V$ to the element of $S$ that contains it as an element. The set $T$ is then described by a function $g$ from $S$ to $V$ that takes each element $A$ of $S$ to the corresponding element of $T$. The relation between $f$ and $g$ is simply $g \circ f = 1_V$.

We are thus led to the following formulation of the axiom of choice for a topos.

**Definition.** A topos satisfies the *axiom of choice* if every epic arrow $f : A \to B$ has a right inverse, that is, there exists an arrow $g : B \to A$ such that $g \circ f = 1_A$.

It can be shown that if a topos satisfies the axiom of choice, then its internal logic must be boolean. This means that the double negation law $\neg\neg P = P$, as well as the law of the excluded middle, $P \lor \neg P = T$, are consequences of the validity of the axiom of choice in a topos. Thus the “nonconstructive” nature of the axiom of choice is not compatible with the “constructive” internal logic of a non-boolean topos.

### 3.8 Sheaf Semantics

We have discussed the Mitchell-Bénabou language at some length, but there remains one major question to be addressed: What is the meaning of this language? The internal interpretation of the language provides a formal criterion of truth, but it doesn’t tell us what the language really means. What is required is a semantics for the language, whereby the objects can be understood as “real” sets, and the arrows as “real” functions. The required semantics is provided by an adaptation of the Kripke semantics for intuitionistic logic, called the *Kripke-Joyal*
semantics or sheaf semantics. We shall begin this section by describing the formulation of this semantics for sheaves and shall then consider the general formulation for an arbitrary topos.

Let us consider specifically the example of sheaves over $X = (0, 1)$ discussed in Section 3.2. The mathematical objects here are the different possible sheaves. Each sheaf represents a “set” and each sheaf morphism a “function.” To apply the Kripke semantics, we must somehow interpret these sheaves and morphisms as a Kripke model, presenting ordinary sets and functions in different stages of knowing. What is immediately suggested is to take as stages of knowing the different “stages of evaluation” considered in Section 3.2, that is, the different possible points $r \in (0, 1)$. At any such point $r$, the stalk $A_r$ of a sheaf $A$ is an ordinary set, and a morphism $f : A \to B$ is represented by an ordinary function $f_r : A_r \to B_r$. The concept of “stages of evaluation” that we introduced to describe “variable” sets is obviously very close in spirit to the concept of “stages of knowing” in the Kripke semantics. However, some modification is required to make this approach work. The reason is that the Kripke semantics requires a partial ordering of the stages of knowing, such that the elements at a given stage have a natural interpretation as elements at any “later” stage. While it is true that the points $r \in (0, 1)$ have a natural ordering, there in general is no necessary correspondence between elements of the stalks at two distinct points $r$ and $s$.

There is, however, a correspondence between elements at $r$ and elements at points sufficiently close to $r$. This suggests that the proper formulation of stages of knowing should somehow reflect the continuous topological structure of the line. The appropriate formulation, developed by Joyal, is to take as stages of knowing all possible open subsets of $(0,1)$. The partial ordering is given by the inclusion relation $\supseteq$, so that, if $U \supseteq V$, we treat $V$ as a later stage of knowing than $U$.

We have seen that the open sets $U \subseteq X$ correspond to the subobjects of 1 in the topos $\text{Shv}(X)$; the subobject corresponding to the open set $U$ we designated $S^U$. We can thus identify the stages of knowing with the subobjects of 1.

Now, it is easily seen that if $U$ and $V$ are open sets, then if $U \subseteq V$ there exists a unique arrow $f : S^U \to S^V$; otherwise there exists no such arrow $f$. This means that the partial ordering of the stages of knowing
can likewise be described in categorical terms: if $B$ and $C$ are subobjects of 1, then the stage of knowing $C \leq$ the stage of knowing $B$ if and only if there exists an arrow $f: B \to C$.

Suppose $U$ is an open set and $A$ is a sheaf. The elements of $A$ in stage of knowing $U$ are taken to be the sections of $A$ over $U$, that is, the different possible arrows $f: S^U \to A$. This is the appropriate generalization of the concept of the stalk at a point to an arbitrary open set. If we think of the stages of knowing as identified with the subobjects of 1, then the elements of the sheaf $A$ in stage of knowing $B$ are then simply the different possible arrows $f: B \to A$.

If $A$ is an object of the topos, and $B$ is a subobject of 1, we shall refer to the elements of $A$ in stage $B$ as $B$-elements of $A$. Suppose now $C$ is a later stage than $B$. To have a Kripke model, we must know how elements of $A$ at stage $B$ are to be interpreted as elements of $A$ at stage $C$; that is, we must have a mapping from $B$-elements of $A$ to $C$-elements of $A$. The appropriate mapping is simply the following: If $f: B \to A$ is a $B$-element of $A$, and if $g: C \to B$ is the unique arrow from $C$ to $B$, then $f \circ g: C \to A$ is the corresponding $C$-element of $A$. Thus the mapping from $B$-elements to $C$-elements is simply given by composition with $g$.

In geometrical terms this means the following. Suppose $U$ and $V$ are open sets with $V \subseteq U$, and $A$ is a sheaf. An element of $A$ at stage $U$ is a section of $A$ over $U$. The corresponding element in the later stage $V$ is the section of $A$ over $V$ obtained by restricting the $U$ section to $V$, that is, by considering only that part of the $U$ section lying above the subset $V$. One consequence of this correspondence is that distinct elements in stage $U$ can sometimes correspond to the same element in a later stage $V$. We see an example in the sheaf $S_7$ of Figure 3.1, in which the two global sections in stage $(0, 1)$ correspond to the same section $C$ in the later stage $(0.5, 1)$.

Having described the meaning of the objects of the topos $\mathbf{Shv}(X)$ as sets in a Kripke model, we describe now the meaning of the arrows as functions. Suppose $f: A \to B$ is an arrow of the topos, and $Y$ is a subobject of 1. Then the $Y$-elements of $A$ are all possible arrows from $Y$ to $A$, and the $Y$-elements of $B$ are all possible arrows from $Y$ to $B$. In stage $Y$, $f$ is interpreted as the function from $Y$-elements of $A$ to $Y$-elements of $B$ that takes each arrow $g: Y \to A$ to the arrow $f \circ g: Y \to B$. That is, the function $f$ is just composition by the arrow $f$; that is, $f(g) = f \circ g$. In
geometrical terms, the function \( f \) in stage \( U \) describes the way the morphism of sheaves \( f : A \to B \) maps \( U \)-sections of \( A \) to \( U \)-sections of \( B \).

We next describe the interpretation of the ordered-pair notation \((-,-)\) of the Mitchell-Bénabou language. Suppose \( g : Y \to A \) is a \( Y \)-element of \( A \), and \( h : Y \to B \) is a \( Y \)-element of \( B \). Then we must interpret the ordered pair notation \((g, h)\) as designating an element of \( A \times B \). The appropriate interpretation is simply the arrow \((g, h) : Y \to A \times B\) derived from the universal property of the direct product \( A \times B \).

This completes the interpretation of terms of the Mitchell-Bénabou language as designating “real” elements. We must next consider atomic formulas. There are two types of atomic formulas: equations and membership relations. The interpretation of equations is simple. If \( f \) and \( g \) are \( Y \)-elements of \( A \), then we interpret the formula \( f = g \) to be true if and only if the arrows \( f \) and \( g \) are the same.

To interpret the membership relation, we need a criterion to determine when a formula of the form \( f \in g \) is true, when \( f : Y \to A \) is a \( Y \)-element of \( A \), and \( g : Y \to P(A) \) is a \( Y \)-element of \( P(A) \). Consider the evaluation arrow \( ev : A \times \Omega^A \to \Omega \). The composition \( ev \circ (f, g) : Y \to \Omega \) is then a \( Y \)-element of \( \Omega \). Now the “true” element of \( \Omega \), in stage \( Y \), is just the arrow \( t \circ h : Y \to \Omega \), where \( h \) is the unique arrow from \( Y \) to \( 1 \) and \( t \) is the “true” arrow. We interpret the formula \( f \in g \) to be true if and only if these two \( Y \)-elements of \( \Omega \) are equal: \( ev \circ (f, g) = t \circ h \).

This completes the description of the elementary “facts” at each stage of knowing in the Kripke model. We next give the rules for interpreting formulas of the Mitchell-Bénabou language in each stage of knowing.

Let \( \phi(x_1, \ldots, x_n) \) be a formula containing free variables \( x_1, \ldots, x_n \), of respective types \( A_1, \ldots, A_n \), and let \( a_1, \ldots, a_n \) be \( Y \)-elements of \( A_1, \ldots, A_n \). The rules will tell us when a formula of the form \( \phi(a_1, \ldots, a_n) \) holds in the stage of knowing \( Y \). We shall use the notation “\( \models_Y \phi(a_1, \ldots, a_n) \)” to designate that “\( \phi(a_1, \ldots, a_n) \) holds at stage \( Y \).” Following are the six rules for the semantics.

1. **\( \wedge \) rule.** \( \models_Y \phi(a_1, \ldots, a_n) \land \psi(b_1, \ldots, b_m) \) if and only if \( \models_Y \phi(a_1, \ldots, a_n) \) and \( \models_Y \psi(b_1, \ldots, b_m) \).
2. **\( \lor \) rule.** \( \models_Y \phi(a_1, \ldots, a_n) \lor \psi(b_1, \ldots, b_m) \) if and only if there exist stages \( W \) and \( Z \), with \( Y = W \cup Z \), such that \( \models_W \phi(a_1', \ldots, a_n') \) and \( \models_Z \psi(b_1', \ldots, b_m') \). Here the “primed” elements \( a_1', \ldots, a_n' \) are the
$W$-elements corresponding to the respective $Y$-elements $a_1, \ldots, a_n$, and similarly $b'_1, \ldots, b'_m$ are the $Z$-elements corresponding to $b_1, \ldots, b_m$.

(3) $\Rightarrow$ rule. $\models_Y \phi(a_1, \ldots, a_n) \Rightarrow \psi(b_1, \ldots, b_m)$ if and only if for every stage $Z$ such that $Y \leq Z$, if $\models_Z \phi(a_1', \ldots, a'_n)$ then $\models_Z \psi(b'_1, \ldots, b'_m)$. Here $a_1', \ldots, a'_n, b'_1, \ldots, b'_m$ designate the $Z$-elements corresponding to the respective $Y$-elements $a_1, \ldots, a_n, b_1, \ldots, b_m$.

(4) $\neg$ rule. $\models_Y \neg\phi(a_1, \ldots, a_n)$ if and only if for no stage $Z$ for which $Y \leq Z$ does one have $\models_Z \phi(a_1', \ldots, a'_n)$.

(5) $\forall$ rule. $\models_Y \forall x \phi(x, a_1, \ldots, a_n)$ if and only if for every stage $Z$ for which $Y \leq Z$ and every $Z$-element $b$ of $A$ one has $\models_Z \phi(b, a_1', \ldots, a'_n)$.

(6) $\exists$ rule. $\models_Y \exists x \phi(x, a_1, \ldots, a_n)$ if and only if there exist an $m$ and stages $Z_1, \ldots, Z_m$ such that $Y = Z_1 \cup Z_2 \cup \cdots \cup Z_m$ and such that, for each $Z_i$, there is a $Z_i$-element $b_i$ of $A$ such that $\models_{Z_i} \phi(b_i, a_1', \ldots, a'_n)$. Here $a_1', \ldots, a'_n$ designate the $Z_i$-elements corresponding to $a_1, \ldots, a_n$.

We note that the above rules agree with the rules for the Kripke semantics given in Section 3.4, with the exception of the “or” rule (2) and the “exists” rule (6). The modification of these two rules has its basis in the notion of “local truth” for sheaf semantics. The idea is the following:

Suppose $\phi$ is a formula and $Y$ is a stage of knowing. Suppose for every point $P$ of $Y$ there is some open set $U$ containing $P$ such that $\phi$ holds in $U$. We say that $\phi$ holds locally in $Y$. The concept of local truth in sheaf semantics is that if a formula $\phi$ holds locally in some stage of knowing $Y$, then it must hold in $Y$. This requirement gives rise to rules (2) and (6). In the case of rule (2), if $\phi$ holds in stage $W$ and $\psi$ holds in stage $Z$, then certainly $\phi \lor \psi$ will hold in both $W$ and $Z$. If $Y = W \cup Z$, then every point $P \in Y$ will lie in either $W$ or $Z$ and hence $\phi \lor \psi$ will hold locally in $Y$. Therefore the formula $\phi \lor \psi$ should hold in $Y$. The justification for rule (6) is similar.

The formulation of sheaf semantics given above can be generalized to an arbitrary topos. In the case of the category of sheaves $\text{Shv}(X)$, the stages of knowing are taken to be all open subsets of $X$; we have
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seen that these correspond to all subobjects of the terminal object 1 of $\text{Shv}(X)$. In the most general formulation of the Kripke semantics for an arbitrary topos, the stages of knowing are taken to be all possible objects of the topos. This means that the stages of knowing are the same as the objects of knowledge, the “sets,” giving rise to a self-referral structure of knowledge.

What are the elements of a set $A$ in stage $X$? In the case of the topos $\text{Shv}(X)$ considered above, the $U$-elements of $A$ were just the sections of the sheaf $A$ over $U$, which corresponded to the different possible arrows from the sheaf $SU$ to $A$. The generalization to an arbitrary topos is to define the $X$-elements of $A$ to consist of all possible arrows from $X$ to $A$. In this way each object $A$ of the topos becomes a “real” set in each stage of knowing, having well-defined elements. The arrows of the topos likewise become interpreted as “real” functions in each stage of knowing; if $f$ is an arrow from $A$ to $B$, and if $g : X \to A$ is an $X$-element of $A$, then $f$ takes $g$ to the $X$-element of $B$ given by $f \circ g$.

The ordinary formulation of the Kripke semantics requires that the stages of knowing be partially ordered. For a topos in general, the objects of the topos are not partially ordered, so some modification of the semantics is required. This is achieved by replacing the partial ordering of the stages of knowing by the concept of a “passage” from one stage of knowing to another. The idea is that a passage from a stage of knowing $X$ to a stage of knowing $Y$ tells one how the elements of a set $A$ at stage $X$ become interpreted as elements of $A$ at the “later” stage $Y$. The appropriate generalization of the semantics for sheaves is to define the passages from stage $X$ to stage $Y$ to consist of all possible arrows $f$ from $Y$ to $X$. If $g : X \to A$ is an $X$-element of $A$, then the passage $f$ takes $g$ to the $Y$-element of $A$ given by $g \circ f : Y \to A$.

In the general formulation of sheaf semantics for a topos, we see that the objects have a two-fold significance and the arrows a three-fold significance. The objects represent the sets, that is, the objects of knowledge, and at the same time, the stages of knowing, that is, the values of the knower. The arrows represent simultaneously the following:

(a) the functions from one set to another, that is, the transformations between the objects of knowledge;
(b) the passages from one stage of knowing to another, that is, the transformations between the values of the knower;
(c) the elements of a set in a given stage of knowing, that is, the knower-known relationship.

The rules (1)–(6) for sheaf semantics generalize to an arbitrary topos in a straightforward way. The only subtlety involves (2) and (6), which are formulated for sheaves in terms of unions of stages of knowing. To formulate these rules for an arbitrary topos, it is necessary to find an appropriate categorical formulation that generalizes the concept of a union of open sets. The correct concept, in this context, is that of a jointly epic family of arrows.

Definition. The arrows \( f_1 : X_1 \to A, f_2 : X_2 \to A, \ldots, f_n : X_n \to A \) are jointly epic if the following condition is satisfied: whenever \( g : A \to B \) and \( h : A \to B \) satisfy \( g \circ f_1 = h \circ f_1, g \circ f_2 = h \circ f_2, \ldots, g \circ f_n = h \circ f_n \), then \( g = h \).

The relationship of jointly epic families of arrows to unions of sets is the following. Consider the category \( \mathsf{Shv}(X) \). Suppose \( U \) and \( V \) are open subsets of \( X \), and let \( S^U, S^V \) be the corresponding sheaves. We have seen that if \( U \subseteq V \) then there is a unique arrow \( f \) from \( S^U \) to \( S^V \); if \( U \nsubseteq V \) then there is no arrow from \( S^U \) to \( S^V \). Suppose now \( V_1, V_2, \ldots, V_n \) are open subsets of \( U \). Then it is straightforward to verify that \( U = V_1 \cup V_2 \cup \cdots \cup V_n \) if and only if the arrows \( f_1 : S^{V_1} \to S^U, f_2 : S^{V_2} \to S^U, \ldots, f_n : S^{V_n} \to S^U \) are jointly epic.

We can now formulate the rules (1)–(6) for the sheaf semantics for an arbitrary topos.

(1) \( \land \text{ rule.} \) \( \models_Y \phi(a_1, \ldots, a_n) \land \psi(b_1, \ldots, b_m) \) if and only if \( \models_Y \phi(a_1, \ldots, a_n) \) and \( \models_Y \psi(b_1, \ldots, b_m) \).

(2) \( \lor \text{ rule.} \) \( \models_Y \phi(a_1, \ldots, a_n) \lor \psi(b_1, \ldots, b_m) \) if and only if there exist two passages, \( h : Y \to X \) and \( k : Z \to X \), from \( X \) to stages of knowing \( Y, Z \), such that \( h, k \) are jointly epic and such that \( \models_Y \phi(a_1 \circ h, \ldots, a_n \circ h) \) and \( \models_Z \psi(b_1 \circ k, \ldots, b_m \circ k) \).

(3) \( \Rightarrow \text{ rule.} \) \( \models_Y \phi(a_1, \ldots, a_n) \Rightarrow \psi(b_1, \ldots, b_m) \) if and only if for every passage \( h : Y \to X \) from \( X \) to a stage of knowing \( Y \), we have \( \models_Y \phi(a_1 \circ h, \ldots, a_n \circ h) \) implies \( \models_Y \psi(b_1 \circ k, \ldots, b_m \circ k) \).
(4) \( \neg \) rule. \( \models_{Y} \neg \phi(a_1, \ldots, a_n) \) if and only if, for no passage \( h : Y \to X \) from \( X \) to a stage of knowing \( Y \), do we have \( \models_{Y} \phi(a_1 \circ h, \ldots, a_n \circ h) \).

(5) \( \forall \) rule. \( \models_{X} \forall x \phi(x, a_1, \ldots, a_n) \) (where \( x \) is a variable of type \( B \)) if and only if for every passage \( h : Y \to X \) from \( X \) to a stage of knowing \( Y \), and every \( Y \)-element of \( B, b : Y \to B \), we have \( \models_{Y} \phi(b, a_1 \circ h, \ldots, a_n \circ h) \).

(6) \( \exists \) rule. \( \models_{X} \exists x \phi(x, a_1, \ldots, a_n) \) if and only if there exist an \( m \) and passages \( b_1 : Y_1 \to X, \ldots, b_m : Y_m \to X \), from \( X \) to stages of knowing \( Y_1, \ldots, Y_m \), such that \( b_1, \ldots, b_m \) are jointly epic and such that, for \( i = 1, 2, \ldots, m \), there exists a \( Y_i \)-element \( b_i : Y_i \to B \) of \( B \) such that \( \models_{Y} \phi(b_i, a_1 \circ b_1, \ldots, a_n \circ b_1) \).

A passage \( X \to Y \) is generally spoken of as a passage from \( Y \) to the later stage of knowing \( X \). The rules for the Kripke-Joyal semantics tell us that to determine whether a formula \( F \) holds at stage \( Y \), we must in general know whether its constituent formulas hold in “later” stages \( X \). This suggests that a passage \( X \to Y \) be thought of rather as a “memory” of stage \( Y \) in stage \( X \).

Having defined the notion of “holding at stage \( X \),” it is simple to define the notion of Kripke-Joyal validity.

**Definition:** Let \( \phi(x_1, \ldots, x_n) \) be a formula of the Mitchell-Bénabou language, having free variables \( x_1, \ldots, x_n \) of types \( A_1, \ldots, A_n \), respectively. Then \( \phi \) is Kripke-Joyal valid if for every stage of knowing \( X \) and each \( X \)-element \( a_i \) of \( A_i \), we have \( \models_{X} \phi(a_1, \ldots, a_n) \).

Thus, a formula is Kripke-Joyal valid if it holds in all stages of knowing for all elements at that stage. The notion of Kripke-Joyal validity thus corresponds to the familiar notion of validity of a formula, which considers a formula to be valid if it is true for all substitutions of elements for the free variables in the formula. The only difference is that for the Kripke-Joyal semantics, the sets have different elements in different stages of knowing, so one must consider all possible substitutions in all possible stages of knowing. If one considers only sentences (that is, formulas containing no free variables), then the Kripke-Joyal valid sentences are precisely the sentences that hold in the “terminal” stage of
knowing, 1, which contains the “memory” of all other stages of knowing.

We have thus, finally, arrived at two different notions of validity for formulas of the Mitchell-Bénabou language, the notion of internal validity introduced in Section 3.6, and the notion of Kripke-Joyal validity, defined in terms of the actual “elements” of the objects. It is not difficult to show that these two notions of validity in fact coincide.

**Theorem.** A formula $\phi$ of the Mitchell-Bénabou language is Kripke-Joyal valid if and only if it is internally valid.

Sheaf semantics thus brings out the meaning of the internal interpretation of the Mitchell-Bénabou language in terms of actual elements. In this we find the completion of one major theme in the categorical approach to the foundations of mathematics: starting from the language of arrows and recovering the “elements” of set theory.

The set-theoretic foundational viewpoint sees everything in terms of the principle of existence, as expressed in the membership relation, $\in$. The universe of sets is structured in layers of existence, one inside another. The set-theoretic viewpoint sees all values of dynamism and transformation—functions, operations, and so on—in terms of the membership relation; everything is a set. A function, the reader will recall, is viewed as just a set of ordered pairs. Thus the set-theoretic viewpoint sees all values of transformation and relationship in terms of the principle of existence; dynamism is seen to be structured in the silent fabric of pure existence.

Category theory presents a complementary viewpoint, which sees everything in terms of the principle of dynamism. It describes all values of relationship in terms of the composition relation of arrows, the sequential combination of two values of transformation to yield a third. For this viewpoint to be complete, it must be able to describe the membership relation of set theory in terms of the principle of dynamism expressed in the language of arrows. This is achieved by sheaf semantics, which sees the “elements” of a set to be just arrows. With this insight, the self-referral structure of knowledge within a topos is fully brought to light.
The viewpoints of set theory, category theory, and topos theory have a correspondence in the Vedic literature in three of the Upangas: Yoga, Karma Mimansa, and Vedanta. The viewpoint of set theory is analogous to the viewpoint of Patanjali’s Yoga Sutras, which sees the silent value of pure existence at the basis of all expressions of creation. The viewpoint of category theory corresponds to the viewpoint of Jaimini’s Karma Mimansa, which sees everything in terms of the principle of dynamism.

The grand synthesis of these two viewpoints is provided by topos theory, which simultaneously sees each value in terms of the other, uniting the two viewpoints into a single holistic viewpoint. This corresponds to the theme of Vyasa’s Vedanta, presenting the ultimate synthesis of all viewpoints in the holistic awakening of awareness—Brahman Consciousness.

Topos theory not only unifies the viewpoints of set theory and category theory, but it further unifies classical mathematics with intuitionistic mathematics. Topos theory sees the differences between classical and intuitionistic mathematics to just express the differences in the internal logic of different toposes. The holistic vision of topos theory sees not a single universe of sets, but an infinite diversity of toposes, each of which is a complete mathematical universe in its own right, having its own internal logic and its own self-referral structure of knowledge. Topos theory sees as the ultimate holistic expression of mathematics not a single mathematical universe, but the category of all toposes, which synthesizes all these diverse possibilities into a single transcendental wholeness.

3.9 The Classifying Topos
We have seen that every topos presents a generalized set-theory universe, governed typically by intuitionistic logic rather than the familiar two-valued classical logic. Toposes thereby provide natural models for higher-order intuitionistic theories. We recall the distinction between first-order and higher-order theories. In a first-order theory, the variables are restricted to range over elements of a specific set, or several specific sets. For example, in the first-order theory of commutative rings, one has only variables that can range over the elements of a ring. In a higher-order theory, one has second-order variables that can range
over sets of elements, third-order variables that can range over sets of elements, and so on. We have seen how the internal language of a topos contains variables of all orders; if \( A \) is any object, then there are variables of type \( A \) that range over the elements of \( A \), variables of type \( P(A) \) that range over arbitrary subsets of \( A \), and so on. Toposes thus provide natural models for higher-order intuitionistic logic, and the main application of topos theory in fact has been to construct models of specific higher-order intuitionistic theories.

These applications utilize a very general and powerful technique for constructing a topos that provides a model for a given theory. This technique is called the topos-theoretic method of forcing, and is based upon the construction of the classifying topos of a geometric theory. (As the reader may recall, Paul Cohen developed the technique of forcing in the context of set theory, which was used by him to establish the independence of the continuum hypothesis and by others to establish a large number of other independence results. It was observed later that techniques involving the classifying topos, which were being developed by topos theorists, provided a natural and unexpected generalization of Cohen’s method. See Scedrov, 1984.) In this section we shall introduce the concept of the classifying topos by considering a simple algebraic example, the theory of commutative rings.

The construction of the classifying topos will be seen to provide an interesting mathematical commentary on the relationship between pure knowledge and organizing power, as described in Maharishi Vedic Science. Pure knowledge refers to the self-referral structure of knowledge in which consciousness knows itself; the knower, known, and process of knowing are in a unified, undivided state. Maharishi has explained that the dynamism inherent in this self-referral structure of knowledge gives rise to the sequential emergence of the richas of Rik Veda, the expressions of pure knowledge. The richas present the hierarchical structure of the laws of nature that eternally exists at the unmanifest basis of creation. In relation to their source in the self-interacting dynamics of intelligence, the richas provide a sequential elaboration and commentary on the totality of knowledge expressed in the first syllable of Rik Veda, AK. This is the theme of Maharishi’s Aparusheya Bhashya, or uncreated commentary, of Rik Veda.
In the theme of the *Apaurusheya Bhashya*, the sequential emergence of the richas of the Veda continues with the emergence of creation itself from the structure of the laws of nature. The *Richas* of the Veda thus have a creative aspect through which they bring forth the diverse expressions of the laws of nature in phenomenal creation. Maharishi uses the term “organizing power” to describe this creative potential latent within the structure of pure knowledge, the richas of the Veda. The expression of the organizing power of a particular law of nature is found in those phenomena that are governed by that law.

Maharishi speaks of the *Richas* of the Veda as the expressions of the language of nature. This language has a meaning, but this level of meaning differs from the ordinary level of meaning of language that is confined to intellectual understanding. The meaning of the language of nature is the organizing power latent within it, and this meaning spontaneously manifests itself in the diverse phenomena of creation.

Ultimately, Maharishi explains, all of creation is just the expression of the organizing power contained in the structure of pure knowledge. In the Vedic literature, the Brahmanas present the mechanics of transformation through which the organizing power contained in the structure of pure knowledge, the *richas* of the Veda, becomes concretely unfolded.

In mathematics, the role of pure knowledge is played by the symbolic structure of the language of mathematics, the expressions of the abstract knowledge. The structure of pure knowledge is expressed in the logical structure of the theories of mathematics, the specific rules of logical transformation that sequentially unfold the symbolic expressions of knowledge, the theorems of a mathematical theory. The organizing power of the theory is expressed in the models of the theory, the specific mathematical structures that satisfy the axioms of the theory. In the case of the theory of commutative rings, for example, the models are all the different possible commutative rings.

The general relationship between theories and their models is studied in the area of mathematics called *model theory*; model theory is thus that area of mathematics devoted to the study of the relationship between pure knowledge and organizing power, corresponding to the theme of the Brahmanas in Vedic Science. Lawvere introduced a new approach to the study of this relationship between theories and models.
based upon category theory; this approach is called functorial semantics. The idea of functorial semantics is to represent the logical structure of a theory by a logical category $L$, in such a way that all possible models of the theory in some category $K$ correspond to the different possible functors $F$ from $L$ to $K$. The theme of functorial semantics is thus to use functors to describe the transformation of pure knowledge into the diverse expressions of its organizing power. Topos theory has played a central role in elaborating this description of the relationship between knowledge and organizing power, as we shall consider now in the context of the theory of commutative rings.

We shall begin by describing the logical category associated with the theory of commutative rings. This is the category $\mathbf{FPCR}^{\text{op}}$, the opposite category of the category of finitely presented commutative rings.

A finitely presented commutative ring is a ring presented in the following way. One begins with a free ring $\mathbb{Z}[X_1, \ldots, X_n]$ generated by a finite number of indeterminates $X_1, \ldots, X_n$. One then specifies a finite number of polynomials $g_1, \ldots, g_m$ in the indeterminates $X_1, \ldots, X_n$. Let $I$ designate the set of all polynomials of the form $c_1 g_1 + c_2 g_2 + \cdots + c_m g_m$, where the coefficients $c_1, c_2, \ldots, c_m$ are arbitrary polynomials in $\mathbb{Z}[X_1, \ldots, X_n]$. It is easily checked that $I$ is the smallest ideal in $\mathbb{Z}[X_1, \ldots, X_n]$ that contains each of the polynomials $g_1, \ldots, g_m$. The ideal $I$ is called the ideal generated by $g_1, \ldots, g_m$. The quotient ring $\mathbb{Z}[X_1, \ldots, X_n] / I$ is called a finitely presented commutative ring; this quotient ring is also designated $\mathbb{Z}[X_1, \ldots, X_n] / (g_1, \ldots, g_m)$.

Consider now the category $\mathbf{FPCR}$ whose objects are all finitely presented commutative rings and whose arrows are ring homomorphisms. The category we wish to study is the opposite category, $\mathbf{FPCR}^{\text{op}}$. We pause to define the notion of an opposite category.

Definition. Let $K$ be a category. The opposite category $K^{\text{op}}$ is constructed as follows. (1) The objects of $K^{\text{op}}$ are the same as the objects of $K$. (2) For each arrow $f : A \to B$ of $K$, one introduces an opposite arrow $f^{\text{op}} : B \to A$ of $K^{\text{op}}$. (3) If $f^{\text{op}} : B \to A$ and $g^{\text{op}} : C \to B$, one defines the composition $f^{\text{op}} \circ g^{\text{op}} : C \to A$ to be $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$. (Since $f : A \to B$ and $g : B \to C$, it follows that $g \circ f : A \to C$ and therefore $(g \circ f)^{\text{op}} : C \to A$.)

The category $K^{\text{op}}$ is just the mirror image of $K$, for which the directions of all arrows are reversed. Because of the reversal of arrows, cones
in $K$ correspond to cocones in $K^{\text{op}}$ and vice versa; thus limits in $K$ correspond to colimits in $K^{\text{op}}$.

The trivial ring is the terminal object in the category $\text{FPCR}$; it is therefore the initial object in the category $\text{FPCR}^{\text{op}}$. The ring of integers $\mathbb{Z}$ is the initial object in the category $\text{FPCR}$; it is therefore the terminal object in the category $\text{FPCR}^{\text{op}}$. We saw in Section 2.3 that $\mathbb{Z}[X] \cong \mathbb{Z}[X_1, X_2]$ in the category of commutative rings; the same relationship therefore holds in the subcategory, $\text{FPCR}$. It follows that in $\text{FPCR}^{\text{op}}$ the “dual” relationship holds: $\mathbb{Z}[X] \times \mathbb{Z}[X] = \mathbb{Z}[X_1, X_2]$.

The free ring $\mathbb{Z}[X]$ plays a very special role in the category $\text{FPCR}^{\text{op}}$: it can be made, in a natural way, into an internal commutative ring in this category. Addition and multiplication are defined as follows:

1. **Addition.** We must define an addition arrow $+: \mathbb{Z}[X] \times \mathbb{Z}[X] \to \mathbb{Z}[X]$. Since $\mathbb{Z}[X] \times \mathbb{Z}[X] = \mathbb{Z}[X_1, X_2]$ in $\text{FPCR}^{\text{op}}$, we require an arrow $+: \mathbb{Z}[X_1, X_2] \to \mathbb{Z}[X]$ in $\text{FPCR}^{\text{op}}$. Let $f: \mathbb{Z}[X] \to \mathbb{Z}[X_1, X_2]$ be the ring homomorphism that takes the indeterminate $X$ to the polynomial $X_1 + X_2$. We define the addition arrow $+$ to be $f^{\text{op}}: \mathbb{Z}[X_1, X_2] \to \mathbb{Z}[X]$.

2. **Multiplication.** The multiplication arrow $\cdot: \mathbb{Z}[X] \times \mathbb{Z}[X] \to \mathbb{Z}[X]$ is defined like the addition arrow, where in this case $f: \mathbb{Z}[X] \to \mathbb{Z}[X_1, X_2]$ is the ring homomorphism that takes $X$ to $X_1 \cdot X_2$ and the multiplication arrow is defined to be $f^{\text{op}}$.

3. **Minus.** The minus arrow $-: \mathbb{Z}[X] \to \mathbb{Z}[X]$ is defined again as $f^{\text{op}}$, where $f: \mathbb{Z}[X] \to \mathbb{Z}[X]$ is the ring homomorphism that takes $X$ to $-X$.

4. **Identity.** It remains to define the identity elements $0$ and $1$. These are defined by arrows $0: 1 \to \mathbb{Z}[X]$ and $1: 1 \to \mathbb{Z}[X]$, where $1$ designates the terminal object of the category (see Section 2.3). Since the terminal object $1$ of $\text{FPCR}^{\text{op}}$ is $\mathbb{Z}$, we require arrows $0: \mathbb{Z} \to \mathbb{Z}[X]$ and $1: \mathbb{Z} \to \mathbb{Z}[X]$ in $\text{FPCR}^{\text{op}}$. We define $0 = f^{\text{op}}$, where $f: \mathbb{Z}[X] \to \mathbb{Z}$ is the ring homomorphism that takes $X$ to $0 \in \mathbb{Z}$, and we define $1 = g^{\text{op}}$, where $g: \mathbb{Z}[X] \to \mathbb{Z}$ is the ring homomorphism that takes $X$ to $1 \in \mathbb{Z}$.

It is straightforward to verify that the above definitions of $+, \cdot, -, 0,$ and $1$ make $\mathbb{Z}[X]$ into an internal commutative ring in $\text{FPCR}^{\text{op}}$. This
ring has a very special property; it can be transformed into any internal ring in any category by application of a suitable functor to \( \text{FPCR}^{\text{op}} \). Let us see why this is so.

Let \( K \) be a category and let \( R \) be an internal ring in \( K \). We shall assume that the category \( K \) contains all finite limits. We wish to construct a functor \( F : \text{FPCR}^{\text{op}} \to K \) such that \( F \) takes the internal ring \( \mathbb{Z}[X] \) to \( R \). The required functor \( F \) is constructed as follows.

Let \( A \) be an object of \( \text{FPCR}^{\text{op}} \). Then \( A = \mathbb{Z}[X_1, \ldots, X_n] / (g_1, \ldots, g_m) \), where \( g_1, \ldots, g_m \) are polynomials in \( X_1, \ldots, X_n \). The idea is to define \( F(A) \) to be the subobject of \( R^n = R \times \cdots \times R \) consisting of all \( n \)-tuples \( (x_1, \ldots, x_n) \) of “elements” of \( R \) such that \( g_1(x_1, \ldots, x_n) = 0, \ldots, g_m(x_1, \ldots, x_n) = 0 \). Now in general \( R \) is just an object in a category \( K \), so it is not meaningful to talk about the “elements” of \( R \). Nevertheless, the construction of the appropriate subobject of \( R^n \) can always be carried out categorically, without the use of explicit elements, using pullbacks and equalizers, in conjunction with the arrows \(+ : R \times R \to R\), \(- : R \times R \to R\), \(0 : R \times R \to R\), and \(1 : R \times R \to R\), which define the internal ring structure on \( R \). We shall not, however, give the details.

It remains to define the functor \( F \) on arrows. Suppose \( f^{\text{op}} : A \to B \) is an arrow of \( \text{FPCR}^{\text{op}} \), where \( A \) is as above and \( B = \mathbb{Z}[Y_1, \ldots, Y] / (h_1, \ldots, h) \). Then \( F(A) \) and \( F(B) \) as constructed above are subobjects \( S \) of \( R^n \) and \( T \) of \( R^r \) respectively. Since \( f^{\text{op}} : A \to B \), it follows that \( f : B \to A \) is a ring homomorphism from \( B \) to \( A \). It can be shown that every homomorphism \( f : B \to A \) comes from a homomorphism \( f' : \mathbb{Z}[Y_1, \ldots, Y] \to \mathbb{Z}[X_1, \ldots, X] \). Specifying a homomorphism from \( \mathbb{Z}[Y_1, \ldots, Y] \) to \( \mathbb{Z}[X_1, \ldots, X] \) amounts to specifying \( r \) polynomials \( l_1, \ldots, l_r \) in the indeterminates \( X_1, \ldots, X_n \); \( l_i \) is the polynomial \( Y_1 \) maps to, \( l_2 \) is the polynomial \( Y_2 \) maps to, \ldots, \( l_r \) is the polynomial \( Y_r \) maps to. These \( r \) polynomials \( l_1, \ldots, l_r \) will define a function \( k \) from \( R^n \) to \( R^r \), namely the function that takes the \( n \)-tuple \( (x_1, \ldots, x_n) \) to the \( r \)-tuple \( (l_1(x_1, \ldots, x_n), \ldots, l_r(x_1, \ldots, x_n)) \). The function \( k \) can be constructed categorically as an arrow \( k : R^n \to R^r \). Moreover, the constraint that the \( r \) polynomials \( l_1, \ldots, l_r \) must satisfy for the homomorphism \( f' \) to induce a homomorphism of the quotient rings \( f : B \to A \) is precisely the condition that guarantees that the function \( k \) maps the subobject \( S \) of \( R^n \) to
the subobject $T$ of $R^r$! (The constraint on the polynomials $l_1, \ldots, l_r$ is that each of the $s$ polynomials $h_1(l_1(X_1, \ldots, X_n), \ldots, l_r(X_1, \ldots, X_n)), \ldots, h_1(l_1(X_1, \ldots, X_n), \ldots, l_r(X_1, \ldots, X_n))$ is required to lie in the ideal generated by $g_1, \ldots, g_m$.) The functor $F$ is defined to take the arrow $f^{op}: A \to B$ to the arrow from $S$ to $T$ obtained by restricting the function $k$ to $S$.

Once the functor $F$ is constructed, it remains to show that it takes the internal ring structure of $\mathbb{Z}[X]$ to the internal ring structure of $R$. This means that the functor $F$ take the arrows defining $+, \cdot, -, 0$, and $1$ for $\mathbb{Z}[X]$ in $\text{FPCR}^{op}$ to the corresponding arrows for $R$ in $K$. This follows in a straightforward way from the definition of the functor $F$.

It furthermore can be shown that the functor $F$ is left-exact. A functor $G: K \to K'$ is said to be left-exact if it takes universal cones in $K$ to universal cones in $K'$; this means that the functor $G$ preserves limits. For example, $G$ will preserve terminal objects (if $1$ is a terminal object in $K$, then $G(1)$ will be a terminal object in $K'$). Likewise, $G$ will preserve direct products: $G(A \times B) = G(A) \times G(B)$ for all $A$ and $B$.

Thus if $R$ is any internal ring in any category $K$, there will always exist a left-exact functor $F: \text{FPCR}^{op} \to K$ that takes the internal ring $\mathbb{Z}[X]$ to $R$. Conversely, if $F: \text{FPCR}^{op} \to K$ is any left-exact functor, then $F$ will take the internal ring $\mathbb{Z}[X]$ to an internal ring in $K$. This is because $F$ will take the diagrams characterizing the internal commutative ring structure of $\mathbb{Z}[X]$ to diagrams characterizing an internal commutative ring in $K$.

It can be shown, furthermore, that if $R$ is an internal ring in $K$, then the left-exact functor $F: \text{FPCR}^{op} \to K$ that takes the ring $\mathbb{Z}[X]$ to $R$ is unique (up to a natural isomorphism). This is because the values of the functor $F$ are determined for the objects $\mathbb{Z}[X], \mathbb{Z}[X] \times \mathbb{Z}[X], 1$, and the arrows $+, \cdot, -, 0$, and $1$, and all the objects and arrows of $\text{FPCR}^{op}$ can be generated from these using products, pullbacks, and equalizers. Because $F$ is required to be left-exact, it must preserve these constructions; by the uniqueness of universal constructions (up to isomorphism), the values of the functor $F$ are thus uniquely determined up to isomorphism.

The category $\text{FPCR}^{op}$ thus satisfies the requirement of functorial semantics for providing a logical category representing the structure of the theory of commutative rings: the models of the theory, in any cate-
category \( K \), correspond precisely to the different possible left-exact functors \( F \) from \( \text{FPCR}^{\text{op}} \) to \( K \). We can think of the logical category \( \text{FPCR}^{\text{op}} \) as presenting the structure of “pure knowledge” of the theory of commutative rings. We use the expression “pure knowledge” in this context as describing the syntactical structure of the theory, that is, the symbolic structure of the expressions of knowledge. In the case of the category \( \text{FPCR}^{\text{op}} \), the entire structure of the category can be described syntactically. The objects can be described by simply specifying a finite set of polynomials; the object \( \mathbb{Z}[X_1, \ldots, X_n]/(g_1, \ldots, g_m) \) can be described by simply specifying the \( m \) polynomials \( g_1, \ldots, g_m \) in the indeterminates \( X_1, \ldots, X_n \). We saw that the arrows could be described by specifying a finite list of polynomials \( l_1, \ldots, l_r \) subject to certain constraints; these constraints can also be formulated syntactically in terms of provability of certain relationships of polynomials in the theory of commutative rings. The category \( \text{FPCR}^{\text{op}} \) can thus be described completely in terms of the syntactical structure of the theory of commutative rings, the level of pure knowledge.

The models of the theory are the diverse expressions of the organizing power of the theory. The transformation of pure knowledge into the diverse expressions of its organizing power is characterized functorially; it is the different possible functors, applied to the logical category \( \text{FPCR}^{\text{op}} \), that yield all the diverse expressions of the organizing power of the theory of commutative rings.

We have seen further that the category \( \text{FPCR}^{\text{op}} \) contains within itself a generic commutative ring, \( \mathbb{Z}[X] \). The commutative ring \( \mathbb{Z}[X] \) can be transformed into any commutative ring, in any category, by application of a suitable functor. The generic ring \( \mathbb{Z}[X] \), being a commutative ring, is a model of the theory; it is an expression of the organizing power contained in the structure of pure knowledge of the theory. This generic ring \( \mathbb{Z}[X] \), however, is located in fact within the logical category \( \text{FPCR}^{\text{op}} \), the expression of pure knowledge.

The category \( \text{FPCR}^{\text{op}} \), in which the generic ring \( \mathbb{Z}[X] \) lives, does not, however, have a very rich categorical structure. We shall now see how the generic ring \( \mathbb{Z}[X] \) can be embedded in a much richer category, in fact in a topos. This topos will be the classifying topos \( S \) of the theory of commutative rings. The topos \( S \) will be the category of presheaves.
over $\mathbf{FPCR}^{op}$, and the embedding of $\mathbf{FPCR}^{op}$ in $\mathcal{S}$ will be the Yoneda embedding. We pause to define this new terminology.

If $K$ is any category, then the category of presheaves over $K$ is the category $\mathbf{Set}^{K^{op}}$ of all functors $F : K^{op} \to \mathbf{Set}$ from the opposite category $K^{op}$ of $K$ to the category of sets. This functor category is always a topos.

The Yoneda embedding is the functor $G$ from $K$ to the topos $\mathbf{Set}^{K^{op}}$ defined as follows.

(i) If $A$ is an object of $K$, then $G(A)$ is the functor $F : K^{op} \to \mathbf{Set}$ defined in the following way for objects and arrows:

(a) If $B$ is an object of $K^{op}$, then $F(B) = K(B, A)$; that is, $F$ takes the object $B$ of $K^{op}$ to the hom-set $K(B, A)$, an object of the category of sets.

(b) If $f^{op} : B \to C$ is an arrow of $K^{op}$ (so that $f : C \to B$ is an arrow of $K$), then $F(f^{op})$ is the function from $F(B) = K(B, A)$ to $F(C) = K(C, A)$ that takes any arrow $g : B \to A$ to the arrow $g \circ f : C \to A$.

(ii) If $f : A \to B$ is an arrow of $K$, then $G(f)$ is the natural transformation from $G(A)$ to $G(B)$ defined as follows: For every $C$ in $K^{op}$, $G(f)(C)$ must be an arrow from $G(A)(C)$ to $G(B)(C)$. Since $G(A)(C) = K(C, A)$, and $G(B)(C) = K(C, B)$, $G(f)$ must be a function from $K(C, A)$ to $K(C, B)$. This function is defined as follows: If $g$ is an element of $K(C, A)$, then $g$ is an arrow $g : C \to A$, and we define $G(f)(g) = f \circ g : C \to B$, which is an element of $K(C, B)$.

The Yoneda lemma states that the functor $G$ described above is (I) faithful and (II) full. Condition (I) means that $G$ takes distinct objects of $K$ to distinct objects of $\mathcal{S}$ and distinct arrows of $K$ to distinct arrows of $\mathcal{S}$. Condition (II) means that if $A$ and $B$ are elements of $K$, then every arrow $h : G(A) \to G(B)$ in $\mathcal{S}$ can be expressed in the form $h = G(f)$ for some arrow $f : A \to B$ in $K$.

The above two properties tell us that the Yoneda functor $G$ locates an exact replica of the category $K$ within the topos $\mathcal{S}$. The topos $\mathcal{S} = \mathbf{Set}^{K^{op}}$ is a special type of topos called a Grothendieck topos. The most general way of constructing a Grothendieck Topos is to start with a category $K$, put a Grothendieck topology on the category $K^{op}$, and consider all functors from $K^{op}$ to $\mathbf{Set}$ that are continuous with respect to
this topology. The category of all these continuous functors will be a Grothendieck topos. Almost all the important examples of toposes are in fact Grothendieck toposes.

To describe the universal property of the Yoneda embedding, we must first define the appropriate concept of morphism of toposes. The proper concept of morphism, in this context, is called a geometric morphism.

Definition. Let $S$ and $T$ be toposes. A geometric morphism $f: S \rightarrow T$ consists of a pair of functors $f_*: T \rightarrow S$ and $f^*: S \rightarrow T$ such that $f^*$ is the left adjoint of $f_*$ and such that $f^*$ is left-exact. The functor $f_*$ is called the direct image of $f$, and $f^*$ is called the inverse image of $f$.

We can now describe the universal property of the Yoneda embedding $G: K \rightarrow \text{Set}^{K^{\text{op}}}$.

Universal Property of Yoneda Embedding. Suppose $K$ is any category, let $S$ be the category of presheaves $\text{Set}^{K^{\text{op}}}$, and let $G$ be the Yoneda embedding $G: K \rightarrow S$. Then (i) $G$ is a left-exact functor from $K$ to the Grothendieck topos $S$, and (ii) If $S'$ is any Grothendieck topos and $F: K \rightarrow S'$ is any left exact functor from $K$ to $S'$, then there exists a unique geometric morphism $f: S' \rightarrow S$ from $S'$ to $S$ such that $F = f^* \circ G$; that is, the diagram (3.5) commutes.

Let us see what this tells us for the case when $K = \text{FPCR}^{\text{op}}$, the logical category for the theory of commutative rings. Since the Yoneda embedding $G: \text{FPCR}^{\text{op}} \rightarrow S$ is left-exact, $G$ takes the generic ring $\mathbb{Z}[X]$ in $\text{FPCR}^{\text{op}}$ to a commutative ring $R_0$ in the topos $S$. Now suppose $R$ is any commutative ring in any topos $S'$; we know then that there exists a unique left-exact functor $F: \text{FPCR}^{\text{op}} \rightarrow S'$ such that $F$ takes the generic ring $\mathbb{Z}[X]$ to $R$. The universal property of the Yoneda embedding then tells us that there exists a unique geometrical mor-
phism \( f: S \to S' \) such that \( F = f^* \circ G \). This means that \( F \) takes the ring \( R_0 \) in \( S \) to the ring \( R \) in \( S' \).

The ring \( R_0 \) in \( S \) is thus a generic ring for Grothendieck toposes: for any ring \( R \) in any Grothendieck topos \( S' \) there exists a unique geometric morphism \( f: S \to S' \) such that \( f^*(R_0) = R \). We call \( S \) the classifying topos for the theory of commutative rings, and we call the internal ring \( R_0 \) in \( S \) the generic commutative ring.

What is \( R_0 \)? Since \((\text{FPCR}^{\text{op}})^{\text{op}} = \text{FPCR}\), The topos of presheaves over \( \text{FPCR}^{\text{op}} \) is \( S = \text{Set}^{\text{FPCR}} \), the category of all functors \( \text{FPCR} \to \text{Set} \). The Yoneda embedding \( G: \text{FPCR}^{\text{op}} \to S \) takes each object \( A \) of \( \text{FPCR}^{\text{op}} \) to the hom-functor \( \text{FPCR}^{\text{op}}(-, A) \)—the functor that takes each object \( B \) of \( \text{FPCR} \) to the hom-set \( \text{FPCR}^{\text{op}}(B, A) \).

Now \( R_0 = G(\mathbb{Z}[X]) \); thus \( R_0 = \text{FPCR}^{\text{op}}(-, \mathbb{Z}[X]) \). This functor takes each object \( B \) of \( \text{FPCR} \) to \( \text{FPCR}^{\text{op}}(B, \mathbb{Z}[X]) \), that is, the set of all arrows \( B \to \mathbb{Z}[X] \) in \( \text{FPCR}^{\text{op}} \). But by the definition of the opposite category, these correspond precisely to the arrows \( \mathbb{Z}[X] \to B \) in \( \text{FPCR} \). Thus \( R_0 \) corresponds to the functor from \( \text{FPCR} \to \text{Set} \) that takes each finitely presented commutative ring \( B \) to the hom-set \( \text{FPCR}(\mathbb{Z}[X], B) \), that is, the set of all ring homomorphisms from \( \mathbb{Z}[X] \) to \( B \). But by the universal property of the free commutative ring \( \mathbb{Z}[X] \), there exists a unique homomorphism from \( \mathbb{Z}[X] \to B \) that takes \( X \) to each element of \( B \). Thus the hom-set \( \text{FPCR}(\mathbb{Z}[X], B) \) is in one-to-one correspondence with the set of elements of \( B \), that is, the underlying set of \( B \). This means that \( R_0 \) corresponds in fact to the forgetful functor \( \text{FPCR} \to \text{Set} \). Thus the generic commutative ring turns out to be none other than the forgetful functor!

So once again the forgetful functor has turned up, this time in the unexpected role of the generic commutative ring. The forgetful functor, which does nothing but “forget,” is thus found to possess the dynamic structure of a commutative ring. We shall see shortly, in fact, that the forgetful functor, in its role as the generic commutative ring, embodies the infinite organizing power of ring theory!

We have at this point succeeded in finding a generic commutative ring \( R_0 \), living in a topos \( S \), a much richer environment than the category \( \text{FPCR}^{\text{op}} \). We shall consider now the implications of the richness of the environment.
The category $\mathbf{FPCR}^{op}$ contains finite limits, so that equational algebraic formulas such as $x + y = z$, $x^2 + z = 1$, and so on, can be interpreted internally, as well as the conjunctions of such formulas, for example, $x + y = z \land x^2 + z = 1$. But this is pretty much the limit of the ability to interpret symbolic language with variables internally within $\mathbf{FPCR}^{op}$. We have seen, however, how in a topos one can interpret the rich structure of the Mitchell-Bénabou language, containing not only the logical constructs $\land$, $\lor$, $\neg$, $\Rightarrow$, $\exists$, $\forall$, but also the membership relation $\in$ of set theory. The generic ring $R_0$, since it lives in a topos, can thus be described internally using this much richer language.

Now suppose we have a geometric morphism $f : \mathcal{S}' \to \mathcal{S}$. Then $f^*$ takes $R_0$ to a commutative ring $R$ in $\mathcal{S}'$. We can ask, which of the structural properties of $R_0$, describable in the Mitchell-Bénabou language, will be preserved by $f^*$, and therefore will be reflected in $R$? The answer is that, in general, not all properties will be preserved; properties, however, that are expressible in coherent logic will always be preserved. We take a moment to explain this new type of logic.

A coherent formula is a formula built up from atomic formulas, plus the logical constants $T$ and $F$, by using disjunction ($\lor$) applied to arbitrary sets of formulas, conjunction ($\land$) applied to finite sets of formulas, and $\forall$. It is required further that a coherent formula contain only a finite number of free variables. We note that in creating a coherent formula one is not permitted to use $\neg$, $\exists$, or $\Rightarrow$. One is permitted, however, to make infinitary disjunctions: $P_1 \lor P_2 \lor P_3 \lor \ldots$, provided there are only a finite number of free variables altogether. Thus the permissible-formulas transcend the finite limitations of ordinary symbolic mathematical language.

A coherent sequent is an implication of the form $F \Rightarrow G$, where $F$ and $G$ are coherent formulas.

For a Grothendieck topos it can be shown that the internal interpretation of the Mitchell-Bénabou language can be extended to infinitary coherent formulas, and this in turn gives a criterion for logical validity of coherent sequents: $F(x_1, \ldots, x_n) \Rightarrow G(x_1, \ldots, x_n)$ is internally valid if the subobject of $A_1 \times \cdots \times A_n$ defined by the internal interpretation of $F$ is a subobject of that subobject defined by the internal interpretation of $G$. (Here $x_1$ is of type $A_1$, $x_n$ of type $A_n$.)
Now suppose we have two toposes $S$ and $S'$, and $f$ is a geometric morphism from $S'$ to $S$. The inverse image $f^*$ of $f$ is then a functor $f^*: S \to S'$. The functor $f^*$ takes objects of $S$ to objects of $S'$, and arrows of $S$ to arrows between the corresponding objects of $S'$. We recall that for any topos, the objects define the \textit{types} of the Mitchell-Bénabou language, and the arrows correspond to the \textit{function symbols} of the Mitchell-Bénabou language. The functor $f^*$ thus maps the types and function symbols of the internal language of $S$ to the types and function symbols of the internal language of $S'$. This means that $f^*$ provides a translation of the internal language of $S$ into the internal language of $S'$. This translation between the internal languages of toposes \textit{always} preserves the validity of coherent sequents: if $F \Rightarrow G$ is internally valid in $S$, then it remains valid in $S'$.

Let us see what this means for the classifying topos of ring theory $S$. Let $R_0$ be the generic commutative ring, and suppose $R_0$ has some property expressed by the coherent sequent $F \Rightarrow G$ (so $F \Rightarrow G$ is internally valid in $T$). If now $R$ is any commutative ring in any topos $S'$, we know there is a geometric morphism $f$ whose inverse image functor $f^*: S \to S'$ takes $R_0$ to $R$. The functor $f^*$ translates the sequent $F \Rightarrow G$ to the corresponding sequent in the language of $S'$, asserting the same property about $R$; because $f^*$ preserves the validity of coherent sequents, it follows that this sequent is internally valid in $S'$; that is, $R$ also has the same property expressed by this sequent. This means that every property of the generic commutative ring $R_0$ that can be expressed by a coherent sequent must be true for all possible commutative rings in all possible toposes.

The generic commutative ring $R_0$ thus can speak for all possible commutative rings. The generic ring $R_0$ contains the infinite organizing power of the theory of commutative rings in seed form. If we can gain knowledge of its properties, then we know that these properties, so long as they are coherent, are valid for all possible commutative rings.

A generic ring, capable of speaking for all possible rings, cannot be located in the “classical” universe of sets. One must go to a non-classical topos governed by non-boolean internal logic to find such a holistic expression of the organizing power of ring theory, in which all the classical possibilities are simultaneously lively.
The account we have given above for the theory of commutative rings generalizes to any theory whose axioms can be expressed by coherent sequents. Such a theory is called a geometric theory. Every geometric theory $T$ has a classifying topos $S$, a Grothendieck topos containing a generic $T$-model $V_0$. $V_0$ is an internal model of the theory $T$ in the topos $S$, and has the following universal property: if $V$ is any model of the theory $T$ in any topos $S'$, then there exists a unique geometric morphism $f: S' \to S$ such that $f^*$ maps the generic $T$-model $V_0$ to $V$. The correspondence between $T$-models in $S$ and geometric morphisms $S' \to S$ is in fact an equivalence of categories: the morphisms between $T$-models in $S'$ correspond to natural transformations between the corresponding geometric morphisms. Thus the category of $T$-models in $S'$ is equivalent to the category of geometric morphisms $S' \to S$.

Not only does every geometric theory have a classifying topos, but every Grothendieck topos is in fact the classifying topos of a geometric theory (for which it is also the generic model!). Every Grothendieck topos $S$ has its own canonical theory $T$, expressed in its own internal language. The canonical theory $T$ is just the description of the structure of the topos $S$ in its own Mitchell-Bénabou language. The sequents of $T$ are thus internally valid in $S$, and hence $S$ is a model of $T$. The topos $S$ turns out in fact to be the classifying topos of its canonical theory: If $V$ is any model of the theory $T$ in any Grothendieck topos $S'$, then there is a unique geometric morphism $f: S' \to S$ such that $f^*$ takes $S$ to $V$.

From this we see, for example, that the theory of commutative rings and the canonical theory of its classifying topos $T$ are essentially equivalent theories; the models of either theory, in any category $T'$, correspond simply to the different geometric morphisms $f: T' \to T$, and the morphisms between two models in $T'$ correspond to the natural transformations between the corresponding geometric morphisms. Thus, finding the classifying topos for a given theory, such as the theory of commutative rings, amounts to finding a level of mathematical reality whose own natural canonical theory is equivalent to the theory we start with.

We can think of the canonical theory of a topos as the “story of the topos,” told in its own language, to itself. Geometric morphisms provide an interpretation of this canonical theory in other toposes; they are the way in which the topos tells its own story to other toposes! The
total organizing power of the canonical theory is found in its generic model, the topos itself. Thus when the interpretation of the canonical theory is self-referral, when the topos tells its own story to itself, then the totality of organizing power is found fully lively in the structure of the knowledge.

This situation has a striking parallel in the description of the language of nature, the Veda, in Maharishi Vedic Science. The Richas of the Veda present the self-referral level of language in which nature tells its own story to itself. This is the unified level of natural law in which knowledge and organizing power are in an undivided state; the infinite organizing power of the knowledge is fully lively in the structure of the knowledge itself.

Furthermore, Maharishi Vedic Science tells us that the synthesis of all knowledge can be located within the self-referral structure of knowledge, the Veda. This has a mathematical parallel in the way in which every geometric theory is equivalent to the canonical theory of its classifying topos, the latter theory presenting the self-referral structure of mathematical knowledge. When we construct the classifying topos of a given theory, we are just finding a level of mathematical reality which, when it tells its own story in its own language to itself, gives expression to a theory equivalent to the given one. It is on this self-referral level alone that the infinite organizing power of the original theory can be captured in its completeness.

We have seen that the construction of the classifying topos of a geometric theory provides a mathematical description of the transformation of the structure of pure knowledge into the expression of its infinite organizing power. This construction lies at the heart of the topos-theoretic method of forcing, a generalization of the set-theoretic method of forcing discussed in detail in Weinless (2011). The set-theoretic method is applied to construct mathematical universes displaying desired properties; these universes are governed by classical logic. The topos-theoretic technique is applied to create intuitionistic universes displaying desired properties. These universes, governed by intuitionistic logic, typically display extraordinary properties that are not even consistent under classical logic, so that no classical models for these properties can exist. The essence of the topos-theoretic technique of forcing is to find suitable geometric axioms to characterize the
intended intuitionistic reality and then construct the classifying topos of the geometric theory. In this way, the knowledge of the mechanics of transformation of knowledge into the expression of its organizing power is practically applied to create mathematical universes embodying virtually any desired property.

This theme has a parallel in Maharishi Vedic Science in the technology of the *yagyas*. The *yagyas* are specific, prescribed procedures through which the infinite organizing power of natural law contained in the structure of pure knowledge is utilized to bring fulfillment to any intention. The *yagyas* have their basis in the Brahmanas, which detail the mechanics of transformation through which the organizing power latent in the structure of pure knowledge becomes concretely expressed.

The topos-theoretic method of forcing has been applied to construct models for virtually every formulation of intuitionistic mathematics of interest. In this way, topos theory has made possible a reconciliation of the contradictory viewpoints of classical and intuitionistic mathematics. These two approaches are no longer seen as competing approaches, where each invalidates the other. Topos theory reconciles these viewpoints by providing a way to understand intuitionistic mathematics from a classical perspective; an intuitionistic theory is just the development of mathematics within a topos, utilizing the internal logic and internal language of the topos. We have seen how this ability to “understand” intuitionistic mathematics has its basis in a description of the structure of knowledge that takes into account both the knower and the known; this is the Kripke semantics. We have seen further how topos theory provides the full blossoming of the Kripke semantics, in describing the self-referral structure of knowledge in which the values of knower and known are one and the same. It is this description of the self-referral structure of knowledge that has made possible the synthesis and unification of classical and intuitionistic mathematics.

When we examine the role of topos theory in the foundations of mathematics, what is most striking is the value of unification and synthesis of complementary, and even contradictory, values. We have discussed the unification of the contradictory approaches to reality of classical mathematics and intuitionism. We find also the unification of the complementary approaches of set theory and category theory, as discussed in the preceding sections. A topos is a category that is at the
same time a kind of set-theoretic universe; it can be described equivalently in two different languages, the language of category theory and its set-theoretic Mitchell-Bénabou language. Topos theory shows how these two languages can be translated into one another and provide essentially equivalent descriptions of the same underlying reality, viewed from two different perspectives.

Topos theory thus provides the grand synthesis of the foundational viewpoints of set theory, category theory, and intuitionistic mathematics. We have seen how this synthesis has been achieved on the basis of the self-referral structure of knowledge provided by sheaf semantics. The role of topos theory in the foundations of mathematics thereby provides a striking parallel to the role of Vedanta in Maharishi Vedic Science, as discussed in Section 3.8. It is Vedanta that locates the grand synthesis of all approaches to knowledge in the self-referral structure of knowledge of the Samhita.

In this introductory account we have been able to give just a glimpse of the richness of topos theory. We do hope this has been enough to give the reader some appreciation of an exciting new direction in the foundations of mathematics and its connections to the holistic science of consciousness, Maharishi Vedic Science.

References


Part V

Appendices
Modern Science and *Vedic Science*:
An Introduction

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ABOUT THE AUTHOR

Kenneth Chandler holds a Ph.D. in Philosophy from the University of Texas at Austin. He served as Head of the Department of the Science of Creative Intelligence at Maharishi International University (today, Maharishi University of Management). Dr. Chandler continues his research into consciousness and is currently at work on a book on descriptions of the experience of transcending and pure consciousness in the mainstream classics of philosophy, science, religion, and the arts. It will be a three-volume set covering the Vedic tradition to the present.
(The following article served as the Introduction to the inaugural issue of the journal *Modern Science and Vedic Science*.)

**Modern Science and Vedic Science: An Introduction**

This journal (*Modern Science and Vedic Science*) provides a forum for research on the forefront of mankind’s expanding knowledge of the universe. It is devoted to exploration of the unified field of all the laws of nature through the combined approaches of modern science and ancient Vedic science, as brought to light by Maharishi Mahesh Yogi. The identification of the unified field by modern physics is only the first glimpse of a new area of investigation that underlies all disciplines of knowledge, and which can be explored not only through objective science but through a new technology of consciousness developed by Maharishi.

The unified field is now beginning to be understood through modern physics as the unified source of the entire universe, as a unified state of all the laws of nature from which all force and matter fields sequentially emerge according to exact dynamical principles. As each science and each academic discipline progresses to uncover its own most basic laws and foundational principles, each is beginning to discover that the roots of these laws and principles can be traced to the unified field.

This journal recognizes a new method of gaining knowledge of the unified field that combines the approach of the modern sciences with that of the most ancient of sciences, the ancient tradition of Vedic science. Many thousands of years ago, the seers of the Himalayas discovered, through exploration of their silent levels of awareness, a unified field where all the laws of nature are found together in a state of wholeness. This unity of nature was directly experienced to be a self-referral state of consciousness which is unbounded, all-pervading, unchanging, and the self-sufficient source of all existing things. They experienced and gave expression to the self-interacting dynamics through which this unified field sequentially gives rise to the diversity of all laws of nature. That experience is expressed in the ancient Vedic literature.
In our own time, Maharishi has brought to light the knowledge of this ancient science and integrated it with the modern sciences in such a way that Vedic science and modern science are now seen as complementary methods of gaining knowledge of the same reality—the unified field of all the laws of nature. The knowledge of this ancient science that Maharishi has brought to light is known as Maharishi Vedic Science.

Maharishi Vedic Science is to be understood, first of all, as a reliable method of gaining knowledge, as a science in the most complete sense of the term. It relies upon experience as the sole basis of knowledge, not experience gained through the senses only, but experience gained when the mind, becoming completely quiet, is identified with the unified field. This method, examined in relation to the modern sciences, proves to be an effective means of exploring the unified field of all the laws of nature. On the basis of this method, complete knowledge of the unified field becomes possible. It is possible to know the unified field both subjectively on the level of direct experience through exploration of consciousness and objectively through the investigative methods of modern science. Maharishi Vedic Science gives complete knowledge of consciousness, or the knower, complete knowledge of the object known, and complete knowledge of the process of knowing. In knowing the unified field, all three—knower, known, and process of knowing—are united in a single unified state of knowledge in which the three are one and the same.

Maharishi has developed and made available a technology for the systematic exploration of the unified field. This technology is a means by which anyone can gain access to the unified field and explore it through experience of the simplest and most unified state of consciousness. As this domain of experience becomes universally accessible, the unified field becomes available as a direct experience that is a basis for universal knowledge. The technology for gaining access to the unified field is called the Transcendental Meditation technique and its advanced programs, and the science based on this experience, which links modern science and Maharishi Vedic Science in a single unified body of knowledge, is called the Science of Creative Intelligence.

Maharishi is deeply committed to applying the knowledge and technology of the unified field for the practical benefit of life. He has
developed programs to apply this knowledge to every major area of human concern, including the fields of health, education, rehabilitation, and world peace. These applications of Maharishi’s technologies of consciousness have laid it open to empirical verification and demonstrated its practical benefit to mankind. Hundreds of scientific studies have already established its usefulness. From these results, it is clear that Maharishi’s technologies of consciousness are far more beneficial than technologies based on present day empirical science; they promise to reduce and even eliminate war, terrorism, crime, ill health, and all forms of human suffering.

These technologies, which are the applied value of Maharishi Vedic Science, represent a great advance in methods for gaining knowledge. Past science was based on a limited range of knowledge gained through the senses. This new technology opens to mankind a domain of experience of a deeper and more far-reaching import. It places within our grasp a new source of discovery of laws of nature that far exceeds the methods of modern science, yet remains complementary to these methods.

Modern science and Maharishi Vedic Science, explored together, constitute a radically new frontier of knowledge in the contemporary world, opening out vistas of what it is possible for mankind to know and to achieve, which extend far beyond present conceptions, and which demand a re-evaluation of current paradigms of reality and a reassessment of old conceptions of the sources and limits of human knowledge.

This introductory essay will provide a preliminary understanding of what the unified field is, what Maharishi Vedic Science is, and how Maharishi Vedic Science and modern science are related. It also defines fundamental concepts and terminology that will be frequently used in this journal and surveys the practical applications of this new technology. We begin with a description of the unified field as understood in modern science.

**The Unified Field of Modern Science**

Within the last few years, modern theoretical physics has identified and mathematically described a unified field at the basis of all observable states of physical nature. Einstein’s hope of finding a unified field theory to unite the electromagnetic, gravitational, and other known
force fields has now been virtually realized in the form of unified quantum field theories. Instead of having several irreducible and distinct force fields, physics can now mathematically derive all four known force fields from a single supersymmetric field located at the Planck scale (10^{-33} \text{cm} or 10^{-43} \text{sec.}), the most fundamental time-distance scale in nature. This field constitutes an unbounded continuum of non-changing unity pervading the entire universe. All matter and energy in the universe are now understood to be just excitations of this one, all-pervading field.

Physics now has the capacity to describe accurately the sequence by which the unified field of natural law systematically gives rise, through its own self-interacting dynamics, to the diverse force and matter fields that constitute the universe. With a precision almost undreamed of a few years ago, the modern science of cosmology can now account for the exact sequence of dynamical symmetry breaking by which the unified field, the singularity at the moment of cosmogenesis, sequentially gave rise to the diverse force fields and matter fields. It is now possible to determine the time and sequence in which each force and matter field decoupled from the unified field, often to within a precision of minute fractions of a second. This gives us a clear understanding of how all aspects of the physical universe emerge from the unified field of natural law.

Mathematics, physiology, and other sciences have also located a unified source and basis of all the laws of nature in their respective disciplines. In mathematics, the foundational area of set theory provides an account of the sequential emergence of all of mathematics out of the single concept of a set and the relationship of set membership. The iterative mechanics of set formation at the foundation of set theory directly present the mechanics of an underlying unified field of intelligence that is self-sufficient, self-referral, and infinitely dynamic in its nature. Investigations into the foundations of set theory are ultimately investigations of this unified field of intelligence from which all diversity of the discipline emerge in a rigorous and sequential fashion. In physiology, it is the DNA molecule that contains, either explicitly or implicitly, the information specifying all structures and functions of the individual physiology. In this sense, therefore, it is DNA that unifies the discipline by serving as a unified source to which the diversity of physiological functioning can be traced.
Each of the modern sciences may indeed be said to have glimpsed a unified state of complete knowledge in which all laws of nature are contained in seed form. Each has gained some knowledge of how the unified field of natural law sequentially unfolds into the diverse expressions of natural law constituting its field of study. Modern science is now discovering and exploring the fundamental unity of all laws of nature.

**Maharishi Vedic Science**

Maharishi Vedic Science is based upon the ancient Vedic tradition of gaining knowledge through exploration of consciousness, developed by the great masters in the Himalayas who first expressed this knowledge and passed it on over many thousands of years in what is now the oldest continuous tradition of knowledge in existence. Maharishi’s work in founding Maharishi Vedic Science is very much steeped in that ancient tradition, but his work is also very much imbued with the spirit of modern science and shares its commitment to direct experience and empirical testing as the foundation and criterion of all knowledge. For this reason, and other reasons to be considered below, it is also appropriately called a science. The name “Maharishi Vedic Science” thus indicates both the ancient traditional origins of this body of knowledge and the modern commitment to experience, system, testability, and the demand that knowledge be useful in improving the quality of human life.

The founders of the ancient Vedic tradition discovered the capability of the human mind to settle into a state of deep silence while remaining awake, and therein to experience a completely unified, simple, and unbounded state of awareness, called pure consciousness, which is quite distinct from our ordinary waking, sleeping, or dreaming states of consciousness. In that deep silence, they discovered the capability of the mind to become identified with a boundless, all-pervading, unified field that is experienced as an eternal continuum underlying all existence. They gave expression to the self-sufficient, infinitely dynamic, self-interacting qualities of this unified state of awareness; and they articulated the dynamics by which it sequentially gives rise, through its own self-interacting dynamics, to the field of space-time geometry, and subsequently to all the distinct forms and phenomena that constitute the universe. They perceived the fine fabric of activity, as Maharishi explains it, through which this unity of pure consciousness, in the pro-
cess of knowing itself, gives rise sequentially to the diversity of natural law and ultimately to the whole of nature.

This experience was not, Maharishi asserts, on the level of thinking, or theoretical conjecture, or imagination, but on the level of direct experience, which is more vivid, distinct, clear, and orderly than sensory experience, perhaps much in the same way that Newton or Einstein, when they discovered the laws of universal gravitation or special relativity, enjoyed a vivid experience of sudden understanding or a kind of direct “insight” into these laws. The experience of the unified field of all the laws of nature appears to be a direct experience of this sort, except that it includes all laws of nature at one time as a unified totality at the basis of all existence—an experience obviously far outside the range of average waking state experience.

The ancient Vedic literature, as Maharishi interprets it, expresses, in the sequence of its flow and the structure of its organization, the sequence of the unfoldment of the diversity of all laws of nature out of the unified field of natural law. The Veda is thus to be understood as the sequential flow of this process of the oneness of pure consciousness giving rise to diversity; and Maharishi Vedic Science is to be understood as a body of knowledge based on the direct experience of the sequential unfoldment of the unified field into the diversity of nature. It is an account, according to Maharishi, of the origin of the universe from the unified field of natural law, an account that is open to verification through direct experience, and is thus to be understood as a systematic science.

These ancient seers of the Vedic tradition developed techniques to refine the human physiology so that it can produce this level of experience, techniques that were passed on over many generations, but were eventually lost. Maharishi’s revival and reinterpretation of ancient Vedic science is based on his revival of these techniques which have now been made widely accessible through the training of thousands of teachers of the Transcendental Meditation program. He has thus provided a reliable method of access to this field of direct experience where the oneness of pure consciousness gives rise to the diversity of the laws of nature; and he has also developed applications of this technology that render it open to experimental testing. These applications will be considered below.
Maharishi describes the experience of this unified field of consciousness as an experience of a completely unchanging, unbounded unity of consciousness, silently awake within itself. Gaining intimate familiarity with the silence of pure consciousness, Maharishi holds, one gains the ability to experience within that silence an eternal “fabric” or “blueprint” of all laws of nature that govern the universe, existing at the unmanifest basis of all existence. This unmanifest basis of life, where all laws of nature eternally reside in a collected unity, is experienced as the fabric of the silent field of consciousness itself, which is not in space and time, but lies at the unmanifest basis of all manifest activity in space and time. Through Maharishi’s work, this experience comes to be understood (as we see below) as a normal state of consciousness that arises in the natural course of human development.

Glimpses of this universal domain of experience, where all possibilities reside together in an eternally unified state, have been reported in almost every culture and historical epoch, from Plato to Plotinus and Augustine, and from Leibniz to Hegel and Whitehead. Scientists like Kepler, Descartes, Cantor, and Einstein also appear to have written of it and seemingly drew their insights into the laws of nature from this experience. Descartes (1908) writes, for example, of an experience that he had as a young man of “penetrating to the very heart of the kingdom of knowledge” and there comprehending all the sciences, not in sequence, but “all at once.” Scientists and writers from many traditions have described this experience of unity, which confirms that it is completely universal, and not a product of a particular cultural tradition. Just as the Vedic tradition has been misunderstood, however, so have those descriptions of consciousness found in these different cultural traditions; for without a technique that makes the experience systematically accessible to everyone, the understanding that this is a universal experience of the most fundamental level of nature’s activity has been obscured, and has not before now emerged into the light of universal science.

According to Maharishi Vedic Science, it is not only possible to gain direct experience of the unity of natural law at the basis of the manifest universe, but one can also directly experience the unity of nature sequentially giving rise to the diversity of natural law through its own self-interacting dynamics. Maharishi’s most recent research has
centered on delving deeply into the analysis of these self-interacting dynamics of consciousness.

**The Self-Interacting Dynamics of Consciousness**

When one gains the capability, through practice of the Transcendental Meditation technique, of remaining awake while becoming perfectly settled and still, one gains the ability to experience a completely simple, unified, undifferentiated, self-referral state of pure consciousness, which is called Saṁhitā in the Vedic literature, in which knower, known, and process of knowing are one and the same. Consciousness is simply awake to itself, knowing its own nature as simple, unified pure consciousness. Yet in knowing itself, the state of pure consciousness creates an intellectually conceived distinction between itself as knower, itself as known, and itself as process of knowing. In Vedic literature, this is reflected in the distinction between Rishi (knower), Devatā (process of knowing), and Chhandas (object of knowledge). According to Maharishi, from the various interactions and transformations of these three intellectually conceived values in the unified state of pure consciousness, all diverse forms of knowledge, all diverse laws of nature, and ultimately all diversity in material nature itself sequentially emerge.

The conscious mind, awake at this totally settled and still level of awareness, can witness the mechanics by which this diversification of the many out of the unity of pure consciousness takes place. The mechanics of Rishi, Devatā, and Chhandas transforming themselves into Saṁhitā, Saṁhitā transforming itself into Rishi, Devatā, and Chhandas, and Rishi, Devatā, and Chhandas transforming themselves into each other are the mechanics by which the unity of pure consciousness gives rise to the diversity of natural law. These mechanics are expressed in the sequential unfoldment of Vedic literature. These are the self-interacting dynamics of consciousness knowing itself, which, Maharishi asserts, sequentially give rise to all diversity in nature.

Maharishi (1986) describes this self-referral state of consciousness as the basis of all creative processes in nature:

This self-referral state of consciousness is that one element in nature on the ground of which the infinite variety of creation is continuously emerging, growing, and dissolving. The whole field of change emerges from this field of non-change, from this self-referral, immortal state of
consciousness. The interaction of the different intellectually conceived components of this unified self-referral state of consciousness is that all-powerful activity at the most elementary level of nature. That activity is responsible for the innumerable varieties of life in the world, the innumerable streams of intelligence in creation. (pp. 25–26)

The Structure of Maharishi Vedic Science

One of Maharishi’s most important contributions to Vedic scholarship has been his discovery of the *Aparausheya Bhashyā*, the “uncreated commentary” of the Rk Veda, which brings to light the dynamics by which the Veda emerges sequentially from the self-interacting dynamics of consciousness. According to Maharishi’s analysis, the Veda unfolds through its own commentary on itself, through the sequential unfoldment, in different-sized packets of knowledge, of its own knowledge of itself. All knowledge of the Veda is contained implicitly even in the first syllable “Ak” of the Rk Veda, and each subsequent expression of knowledge elaborates the meaning inherent in that packet of knowledge through an expanded commentary. The phonology of that syllable, as analyzed by Maharishi, expresses the self-interacting dynamics of consciousness knowing itself. As pure consciousness interacts with itself, at every stage of creation a new level of wholeness emerges to express the same self-interacting dynamics of Rishi, Devatā, and Chhandas.

Thus the body of Vedic literature reflects, in its very organization and structure, the sequential emergence of all structures of natural law from the unity of pure consciousness. Each unit of Vedic literature—Rk Veda, Sāma Veda, Yajur-Veda, Atharva Veda, Upanishad, Āraṇyakas, Brāhmaṇa, Vedāṅga, Upāṅga, Itihās, Purāṇ, Smṛiti, and Upaveda—expresses one aspect or level of the process. As Maharishi (1986) describes it:

> The whole of Vedic literature is beautifully organized in its sequential development to present complete knowledge of the reality at the unmanifest basis of creation and complete knowledge of all of its manifest values. (p. 28)

Veda, Maharishi asserts, is the self-interaction of consciousness that ultimately gives rise to the diversity of nature. The diversity of creation sequentially unfolding from the unity of consciousness is the result of
distinctions being created within the wholeness of consciousness, as consciousness knows itself. Thus from the perspective of Maharishi Vedic Science, the entire universe is just an expression of consciousness moving within itself. All activity in nature is just activity within the unchanging continuum of the wholeness of consciousness.

Through the texts of ancient Vedic science, as interpreted by Maharishi, we possess a rich account of the emergence of diversity out of the unity of natural law. On the basis of this account, it becomes feasible to compare the Vedic description of the origin of the universe with that of the modern sciences.

**Modern Science and Maharishi Vedic Science**

When Maharishi heard from major scientists of the recent advances of unified field theory in physics, he asserted that modern science had glimpsed the unified field described in ancient Vedic science. “The knowledge of the unified field,” he said (1986, p. 29), “has been discovered by modern science during just the last few years, but the complete knowledge of the unified field has always been available in the Vedic literature.” Modern science, he proposed, had now arrived at the edge of comprehending, through unified quantum field theories, what Vedic science had described on the basis of exploration of the least excited state of consciousness since ancient times: that all diversity in nature sequentially emerges from a unified source through a precise self-interacting dynamics. Modern experimental science and Maharishi Vedic Science could now be seen as two diverse yet mutually complementary approaches to knowing the same underlying reality—one through the empirical method, the other through the exploration of the least excited state of consciousness. Through Maharishi’s inspiration, this has become a major research program that has engaged the attention of many scientists and that has yielded very rich results.

Over the past decade, Maharishi has participated in numerous symposia with major scientists on the theme of exploring modern science and Vedic science to discover detailed structural similarities in their descriptions of the unified field. These symposia have attracted eminent unified field theorists, mathematicians, and physiologists, including a number of Nobel laureates, as well as many of the most highly recognized Pandits of the Vedic tradition. Out of these interactions has come
a meeting of two traditions, East and West, on the ground of their common theme: the investigation of the unified field. Those who have followed these symposia have recognized a deep and impressive structure of knowledge common to both traditions. Both identify a boundless, all-pervading field underlying all states of matter and energy in the universe; both locate it on the most fundamental time-distance scale of nature; both assign to it the same properties of self-sufficiency, self-interaction, infinite dynamism, unboundedness, and unity, among many other common attributes; both identify a threefold structure at the basis of all nature; and both describe a dynamics by which the diversity of nature sequentially emerges from this unified field according to precise laws. The result of these symposia has been that many scientists, following Maharishi’s lead, now feel confident to assert that the unified field described by physics and the unified field of consciousness described by Vedic science are one and the same.

In the first issue of *Modern Science and Vedic Science*, the lead article by John Hagelin explored many of the deep connections between contemporary unified field theory in physics and Maharishi Vedic Science from the standpoint of an active field theorist. His work brought these two diverse methods of inquiry into close relation, drawing upon both the latest developments of unified field theories and the direct experience of the unified field.

Dr. Hagelin presented evidence for Maharishi’s assertion that the unified field of consciousness and the unified field of physics are the same. His main empirical evidence for this new paradigm was drawn from experimental research in the social sciences on the “Maharishi Effect”—the measurable effects on society resulting from the practice of the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying. As further evidence for the identity of consciousness and the unified field, he cited deep parallels between the descriptions of the unified field found in physics and Maharishi Vedic Science. These strikingly similar descriptions support the conclusion that modern science and Maharishi Vedic Science are two complementary methods of approach to the same underlying unity of nature.
The New Paradigm of the Unity of Nature

It is a common belief that the unified field of physics is an objective reality of nature and that consciousness is a subjective experience, and that the two belong, consequently, to different categories of existence. According to this understanding, one is purely material, the other is purely mental, and the two cannot, therefore, be equated.

Through the experience of pure consciousness described in Maharishi Vedic Science, that unified level of intelligence is experienced, not as a mere subjective and localized phenomenon of thought or sensation, but as a non-changing, unbounded field of Being, pervading all forms and phenomena in the universe on a non-active, silent, unmanifest level. Objective and subjective aspects of nature are seen as but two manifest modes of this unified field at the unmanifest basis of existence. A thorough examination of the nature of the unified field in physics and the descriptions of unbounded consciousness brought to light by Maharishi support the thesis that they are but two complementary modes of apprehending a single underlying reality.

The view of nature as consisting of billiard-ball-type objects, each separate, discrete, and isolated from the other, belongs to the old classical Newtonian view of the world. Quantum field theory in modern physics no longer views nature in this way, but provides a new understanding in which the primary reality is that of quantum fields. All forms of matter and energy are understood to be excitations of these underlying fields. In the last year and a half, the apparently different fields of gravity, electromagnetism, and the weak and strong interactions have been theoretically unified as different levels of expression of one single underlying field. All forms and phenomena in the universe are just modes of vibratory excitation of this one all-pervading unified field.

Today, the success of modern physics in unifying our understanding of physical nature is mirrored in the success of Maharishi Vedic Science in unifying our understanding of consciousness. When the unbounded level of pure consciousness is gained as a direct experience, all activity in nature is experienced as an excited state of that one all-pervading field. Since quantum field theory also describes all activity in the universe as excitations of one underlying field, the simplest interpretation is that there is a single unified field which can be known both
through direct experience and through the objective sciences. In this new understanding of the unity of nature, mind and matter cease to be viewed as ultimately different and come to be seen as expressions of a deeper unity of unbounded consciousness.

The unity of nature is not merely a hypothetical unity, nor a unity of intellectual understanding or interpretation. It is a unity of direct experience that has been described in almost every tradition and every historical epoch. Maharishi Vedic Science only brings to light what has been the experience of many of the greatest minds throughout history. What is radically new is that Maharishi has provided a systematic and reliable method by which anyone can gain access to this level of experience. This method of access is the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying.

**The Transcendental Meditation and TM-Sidhi Programs, including Yogic Flying**

The Transcendental Meditation and TM-Sidhi programs, including Yogic Flying, have been introduced by Maharishi as an effective means for opening the unified field to all as a direct experience. In this way, the unified field becomes universally accessible to systematic exploration.

The key component of these programs is the Transcendental Meditation technique, which provides a systematic procedure by which the mind is allowed to settle naturally into a state of restful alertness, the self-referral state of pure consciousness, in which the mind is completely silent and yet awake. In this way, the state of pure consciousness, which has been the subject of philosophical speculation throughout the centuries, can now be investigated on the basis of direct experience. Maharishi’s immensely important contribution to the clarification and elucidation of this experience of pure consciousness will be a theme for analysis in future issues of this journal.

This quiet, still level of consciousness has rarely been experienced in the past because no systematic and effective technique has been available for providing that experience. The Transcendental Meditation technique is a simple, natural, and effortless procedure for allowing the awareness to settle into a state of deep silence while remaining awake. It has proved to be uniquely effective in making this level of experience widely accessible. Through the deep rest gained during the
practice of the technique, balance is systematically created on all levels of physiological functioning, and the nervous system is habituated to a more settled, coherent, and alert style of functioning. In time, a state of completely integrated functioning is gained, in which pure consciousness is spontaneously and permanently maintained. Once this state is established, the silent, self-referral field of awareness is always present as a stable, non-changing ground underlying all changing states of awareness. This integrated state of consciousness, Maharishi holds, is the basis of all excellence in life and provides the foundation for the further development of higher states of consciousness through the practice of the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying.

**Maharishi’s Programs for the Development of Higher States of Consciousness**

The ultimate purpose of all aspects of the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying, and Vedic Science is the development of consciousness, the unfoldment of the full human potential to live life in enlightenment. Enlightenment is that fully developed state of life in which one enjoys complete knowledge and lives in total fulfillment. In this state, one lives in harmony with all the laws of nature, enjoying the full support of natural law to achieve any desire without making mistakes.

Maharishi has identified a specific sequence of higher states of consciousness, each distinct from waking, dreaming, and sleeping, which, he asserts, arise in the normal full course of human development. Each state of consciousness unfolds on the basis of a concrete shift in the mode of the individual’s neurophysiological functioning. These states can be distinguished from waking, dreaming, and sleeping on the basis of their distinct physiological correlates. The higher states of consciousness that arise in this developmental sequence are, Maharishi asserts, a source of greater joy, knowledge, and fulfillment than ordinary waking state life.

The attainment of these higher states of consciousness is the basis for fully understanding and applying the theoretical assertions of Maharishi Vedic Science. Maharishi Vedic Science is just the exposition of the full range of direct experience that unfolds during the course of the natural
development of human consciousness. These states of consciousness are universal stages of human development accessible to everyone through the practice of Maharishi’s technologies of consciousness. What before was shrouded in the veil of mysticism is now scientifically understood as a normal, natural stage of human life available to anyone.

An article in the first issue of *Modern Science and Vedic Science*, by Dr. Charles Alexander and others (1987) examined the empirical evidence, drawn from behavioral and neurophysiological research, for the existence of these higher stages of human development. This article unfolded the scientific basis for understanding and verifying higher states of consciousness from the standpoint of a developmental psychologist, and laid the basis for a new paradigm of human development.

**Research on the Relation between Modern Science and Maharishi Vedic Science**

Each individual nervous system, when refined through Maharishi’s technologies of consciousness, is an instrument through which the silent field of pure unbounded consciousness becomes accessible as a field of inquiry. Since the unified field is all-pervading and everywhere the same, a nervous system finely enough attuned in its functioning can gain the ability, according to Maharishi, to experience and identify itself with that unbounded, undifferentiated, and unified field underlying all activity in nature. By taking one’s awareness from the gross level of sensory objects to perception of finer levels of activity, one gains the ability to experience that level of nature’s functioning at which the unity of pure consciousness gives rise to diversity. Gaining this unified state of consciousness is the means by which anyone can experience and confirm the structure of knowledge and reality described in Maharishi Vedic Science. This is partly what makes Maharishi Vedic Science a precise, verifiable science: All theoretical structures of the science can be verified through a reliable, systematic, effective technology. Other foundational aspects of this science will be considered below.

Maharishi’s technologies of consciousness become, in the modern world, a method for the investigation of the unified field and the most refined level of nature’s activity through direct experience. Modern physics, through its objective method of inquiry, has glimpsed a unified field underlying all of nature, but physics has reached a fundamental
impasse in its ability to experimentally investigate the unified field, because the energies required to probe these finer scales exceed those attainable by any conceivable particle accelerator technology. When physics can go no further, Maharishi’s technologies of consciousness, facilitate inquiry beyond the limitations of the objective approach by providing an effective means of exploring the unified field on the level of direct experience.

This exploration of the unified field through the subjective experience of consciousness is a well-structured program of research. It is guided by the knowledge of Maharishi Vedic Science set forth by Maharishi in conjunction with the modern sciences. When descriptions of the unified field from the standpoint of modern science, of Maharishi Vedic Science, and of direct experience coalesce, the three together provide a basis for complete knowledge. This program of research is based on Maharishi’s exposition of the Vedic literature as a complete and detailed expression of the unified field.

According to Maharishi’s exposition of the Veda, the sequential emergence of the diverse laws of nature from the unified field can be directly experienced in the field of consciousness as a sequence of sounds; these are presented in the sequential emergence of phonological structures of the Vedic texts. Veda is just the structure of the self-interacting dynamics through which the unified field gives rise to the diverse expressions of natural law. Fundamental theoretical concepts in physics and other disciplines, insofar as they are valid descriptions of nature, should therefore correspond to different aspects of Vedic literature that describe these realities from the standpoint of direct experience.

The basic program of research of modern science and Maharishi Vedic Science, as conceived by Maharishi, thus has three major goals: (1) to develop an integrated structure of knowledge by fathoming the depth of correspondence between the principles of modern science and Vedic Science; (2) to provide, from Maharishi Vedic Science, a foundation in direct experience for the most profound theoretical concepts of modern science; and (3) to resolve the impasse faced by the objective approach of modern science through the addition of the subjective approach of Maharishi Vedic Science, which provides complete knowledge of nature on the basis of the complete development of the knower.
In another issue of *Modern Science and Vedic Science* [see Vol. 5, Pt. 1 of this series], Dr. M.H. Weinless (1987) explored set theory and other foundational areas of modern mathematics in relation to Maharishi Vedic Science. In a proposed issue, Drs. R.K. Wallace, D.S. Pasco, and J.B. Fagan (1988) explore the fundamental relationship between Maharishi Vedic Science and the foundational areas of modern physiology, such as molecular biology. Their paper also discusses the extent to which fundamental principles of Maharishi Vedic Science can be used to further investigation of DNA structure and function.

The discovery of deep structures of knowledge and principles common to Maharishi Vedic Science and modern science represents such a profound contribution to our understanding of nature that this journal was founded to foster continued scholarly investigation of the interrelations between these complementary methods of gaining knowledge. Knowledge gained by direct experience of the fine fabrics of nature’s activity, and knowledge gained by the experimental methods of modern science coalesce in a new integrated method of inquiry that offers both the fundamental principles of modern science and the expressions of direct experience in Maharishi Vedic Science as two facets of one reality of nature’s functioning.

Maharishi (1986) sums up the relation between Maharishi Vedic Science, modern science, and his technologies of consciousness:

Maharishi Vedic Science is applied through the Technology of the Unified Field. We speak of the unified field in connection with Maharishi Vedic Science because of the similarity of what has been discovered by physics and what exists in the self-referral state of human consciousness. The Technology of the Unified Field [That is, Transcendental Meditation and TM-Sidhi programs, including Yogic Flying—Eds.] is a purely scientific procedure for the total development of the human psyche, the total development of the race. This is a time when objective, science-based progress in the world is being enriched by the possibility of total development of human life on earth, and this is the reason why we anticipate the creation of a unified field-based civilization. (p. 35)

On the basis of the universal availability of this domain of experience, an empirical science of consciousness becomes possible for the first time.
The Science of Creative Intelligence: Foundations of a New Science of Consciousness

The unified science that links the objective method of modern science and the subjective method of Maharishi Vedic Science, while preserving the integrity of each, is called the Science of Creative Intelligence (SCI). Maharishi himself has laid the foundations of this new science by showing, first, how a precise subjective science of consciousness is established on the basis of the direct experience of consciousness in its pure form; and second, how the experimental method can be used to test empirically the assertions of the subjective science. Through Maharishi’s work, for the first time in history, the full potential of human consciousness can be investigated both through direct experience and through the objective methods of modern science. The foundations of this new science linking the subjective and objective method will now be considered.

Experiential Foundations

Prior to Maharishi’s work, the term consciousness was considered too vague and indefinite to be allowed into scientific discussion. It was excluded from science as a metaphysical term because consciousness was not objectively observable, and therefore apparently not amenable to scientific investigation. Through Maharishi’s work, the concept of consciousness has been given a precise, well-defined meaning on the basis of direct experience, and its relation to the objective framework of science has been precisely specified.

The experience of pure consciousness, available to anyone through regular practice of the Transcendental Meditation technique, is a basis for precise experiential knowledge of consciousness in its simplest, most fundamental, and most unified state. Even though consciousness can never be an object of experience, when the conscious mind becomes completely settled in a wakeful state, it experiences its own nature as pure wakefulness, pure consciousness, without any activity or objective content. Through the repeatable, systematic experience of this silent but wakeful state of mind, the concept of pure consciousness, which has been subject to conjecture and debate throughout the centuries, is now available to direct experience.
Having laid the basis for introducing consciousness into science as a precise concept, it remained for Maharishi to develop a program of applied research to test theoretical predictions of Maharishi Vedic Science. Identifying consciousness with the unified field provides a precise understanding of where consciousness is located in the framework of the sciences. To create an empirical science of consciousness, however, it was also necessary to account for how consciousness could be investigated through experimental research.

**Empirical Foundations**

Maharishi’s work has laid the foundation for an experimental investigation of consciousness. He has led the way in drawing out predictions of Vedic science that are open to testing, translating discussions of consciousness, derived from experience of higher states of consciousness, into predictions of experimentally observable phenomena. Three examples will illustrate this principle.

Pure consciousness, as was noted above, is experienced during the practice of the Transcendental Meditation technique as a state of pure restful alertness. This purely subjective experience does not, however, establish objectively whether it is in fact a state of deep rest and alertness, or only seems to be. If a person is in a deep state of rest and alertness, Maharishi has asserted, then physiological evidence of deep rest and alertness should be observable. Reduced levels of oxygen consumption, reduced breath rate, and other measures of more refined physiological activity would be predicted. Patterns of EEG coherence in the alpha range, indicative of restful alertness, should also be observed. Early pioneering research by Dr. R.K. Wallace (1986) found that these changes do indeed occur. In this way, statements about the subjective experience of consciousness were translated into empirically verifiable assertions. The basis of this correlation between consciousness and physiology is a principle, fundamental to Maharishi’s thinking, that for every state of consciousness there is a corresponding state of physiological functioning. The range of physiological correlates of the experience of pure consciousness is a subject of continuing research.

Consider a second example. Pure consciousness is understood in Maharishi Vedic Science as a clear and settled state of awareness. Anyone who gains this state is said to have a mind like a placid lake, unrippled
by waves, and thus able to reflect the world in a precise, non-agitated manner. Maharishi drew from this several predictions. One is that a person growing in the ability to experience pure consciousness would experience more stable and orderly physiological functioning. This can be translated into the testable prediction that subjects regularly practicing the Transcendental Meditation program display increased stability of the autonomic nervous system. Another prediction is that the practice of the Transcendental Meditation program will produce greater perceptual clarity and greater orderliness of thinking. Translated into specific terms, this leads to the prediction that practicing the Transcendental Meditation program will produce measurable increases on such scales as auditory discrimination, brain wave coherence, and problem solving ability. Research has been designed, carried out, and reported in the literature which measures the growth of these parameters in groups practicing the Transcendental Meditation program by comparison to control groups, thus providing objective verification of the predicted correlates of the subjective experience of pure consciousness.

A third example of how assertions of Maharishi Vedic Science can be translated into testable form is found in the sociological experiments on the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying. The hypothesis is that a group of people practicing this technology in one place, by bringing their awareness to the level of perfect orderliness in the unified field, will enliven qualities of harmony and orderliness in collective consciousness, thus producing measurable positive changes in the quality of societal life. Many experiments have been designed by Maharishi and carried out, demonstrating the power of this technology to produce significant changes in the level of coherence, positivity, balance, and stability in society, even on a global scale. (See Experimental Research, below.) The results of these experiments strongly support Maharishi’s assertion that consciousness is identical with the unified field.

Experimental Research
Over 600 hundred experimental studies in the areas of physiology, psychology, and sociology provide substantial confirmation of many basic assertions of Maharishi Vedic Science in the arena of empirical science. Many of these studies, now published in major scientific jour-
nals throughout the world, have been collected in the volumes called *Scientific Research on the Transcendental Meditation Programme: Collected Papers, Vols. 1–6* (1977–1991). This research provides experimental validation of the efficacy of the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying. Because this research—from over 600 scientific studies at over 300 universities and research institutions in 33 countries, published in more than 100 scientific journals—is too extensive to summarize here, the reader is referred to the *Collected Papers* for articles cited in this and other professional journals. Overall, this research probably represents the most concerted, well-designed research program on a potential means to benefit mankind ever conceived. Its present standing is that, taken together as a body of research, it is one of the most impressive confirmations of a theory of human potential ever executed.

Although it is beyond the scope of this introduction to go into the details of this research, it is worthwhile to mention some of the broad categories of scientific investigation that have evolved to guide the research program of the Science of Creative Intelligence. The main areas of research include studies on the individual and society. Research on benefits to the individual may be further subdivided into studies of physiological changes (both during and after the practice); cognitive, psychological, and behavioral changes; benefits to health and social behavior; and benefits to athletic performance, performance in business, and academic performance. Research on social benefits through collective practice may be further grouped into research on families, city populations, national populations, and global population. These research studies fall into the categories of crime prevention, accident prevention, benefits to economy, health, violence reduction, and world peace.

On the basis of this research, basic assertions of Maharishi Vedic Science become verifiable through empirical science. There is, moreover, a unity of theory underlying these diverse predictions and tests. These studies, taken as a whole, constitute a coherent research program that tests the prediction that repeated experience of the unified field results in greater orderliness, coherence, and positivity, in both individual and social life. Research on these changes not only tests fundamental theory, but demonstrates the practical benefits of this new
technology. Maharishi’s technologies of consciousness become open to experimental testing precisely because they have significant practical applications in improving every area of human life.

**Practical Applications**

of the *Transcendental Meditation* and *TM-Sidhi* Programs,
including *Yogic Flying*

Maharishi has frequently asserted that the purpose of Maharishi Vedic Science is to benefit life, not merely to give knowledge for its own sake. Knowledge, he holds, is for action, action for achievement, and achievement for fulfillment. The ultimate purpose of Maharishi Vedic Science and its applied technology is, therefore, to bring human life to fulfillment.

Maharishi’s technologies of consciousness bring fulfillment to individual life by unfolding the full potential of consciousness. When higher states of consciousness are realized, Maharishi emphasized, life is lived in “twenty-four-hour bliss.” Gaining contact with the unified field, one enjoys spontaneous right action, lives life in total accord with all the laws of nature, and accomplishes any life-supporting desire. Violations of natural law cease, and all suffering, which is caused by violation of natural law, comes to an end. Life is lived free from mistakes, in inner and outer fulfillment. Such is the fundamental purpose of the technologies Maharishi has created.

**Perfect Health**

Maharishi’s technologies of consciousness have important practical applications in the area of health. According to Maharishi, sickness arises from imbalance. Perfect health means wholeness, balance on all levels of life. When individual life is established in the unified field of all the laws of nature, all actions are spontaneously in accord with natural law. In terms of physiological functioning, this means perfect integration and balance, from the biochemical and molecular levels to the macroscopic, organismic levels.

Maharishi Ayurveda is an integral part of Maharishi Vedic Science. It is a revitalized form of the ancient ayurvedic science of life and health, restored to its original purity and effectiveness by Maharishi.
According to Maharishi, the cornerstone of Ayurveda is the development of consciousness. Perfect health in mind, body, and behavior is the result of perfect balance in consciousness and physiology. This develops through the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying, when the mind identifies itself with the unified field, the field of perfect balance and wholeness.

Maharishi Ayurveda combines Maharishi’s technologies of consciousness with specific procedures to treat and prevent illness and promote longevity. Maharishi Ayurveda Medical Centers have been established in many countries to eliminate the basis of sickness, create perfect health, and reverse the aging process. Over the last fifteen years, research into the effects of Maharishi’s technologies of consciousness, on health have been carried out at research institutions all over the world, and Maharishi’s recent emphasis on Ayurveda provides many new research opportunities for investigating the applications of Vedic Science in the area of health.

Maharishi’s technologies of consciousness also include technologies to accomplish specific goals of individual and social life. The TM-Sidhi program has been founded by Maharishi to utilize the knowledge and the organizing power of the unified field for improving achievements in every area of human endeavor.

**Unfolding Full Human Potential through the Transcendental Meditation and TM-Sidhi programs**

When one gains the level of experience of the self-interacting dynamics of consciousness, Maharishi holds, one gains command over all the laws of nature. Stationed at the source of all the laws of nature, at the “central switchboard” of nature’s activity, human consciousness can command all the laws of nature to create any desirable effect in the material world. Maharishi has brought forth a program for gaining mastery over all the laws of nature, based on the formulations found in the ancient *Yoga Sūtras* of Patanjali, one of the principal books of Vedic literature. This is the TM-Sidhi program, in which the mind gains the ability to function from the level of the self-interacting dynamics of the unified field. Once established in pure self-referral awareness through the practice of the Transcendental Meditation program, an individual
gains the ability to draw upon the organizing power of the unified field to accomplish anything. Since the unified field is the source of all existence, its organizing power is infinite, and one who functions from this level has unlimited organizing ability. Established in that unified field of all possibilities on the unmanifest level of existence before consciousness assumes the form of matter, all possibilities open to one’s awareness and one can govern the expressions of the unified field as it transforms itself into matter. As Maharishi (1986) expresses it:

In this program, human awareness identifies itself with that most powerful level of nature’s functioning and starts to function from there. The purpose of the TM-Sidhi program is to consciously create activity from that level from where nature performs. (p. 74)

Through the practice of the TM-Sidhi program, Maharishi predicts, it will become possible to achieve levels of body-mind coordination hitherto deemed impossible. It will be possible, he asserts, to realize the ancient dream of flying through the air, and to develop highly enhanced powers of hearing, seeing, and intuition that extend the senses far beyond the limits currently conceived to be possible. In the Yogic Flying technique, which Maharishi developed from the Yoga Sūtras, the silent state of self-referral consciousness is integrated most fully with outer activity as the body lifts in spontaneous hops, generating inner bliss and maximum coherence in brain functioning. Other Vedic texts describe the ability to move through the air at will as a result of perfection of this Yogic Flying technique. By activating laws of nature that are now hidden to ordinary methods of scientific investigation, the TM-Sidhi program provides a research methodology to explore what is possible for mankind to achieve on the basis of functioning from that level where the conscious mind has become identified with the unified field. This is the basis of a technological revolution more powerful and beneficial to life than any conceived through empirical science.

The Maharishi Effect
The TM-Sidhi program, when practiced in groups, is even more powerful than the TM-Sidhi program practiced alone. The collective practice of the TM-Sidhi program can produce an influence that affects the entire world in measurable ways. This global influence of coherence
generated through the group practice of the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying, has been called the “Maharishi Effect.”

As early as 1960, Maharishi predicted that when individuals practice the Transcendental Meditation and TM-Sidhi programs in sufficiently large groups, a measurable increase in orderliness, coherence, and positive trends would be observed in society. By enlivening the life-supporting and evolutionary qualities of the unified field, such as perfect orderliness, infinite dynamism, and self-sufficiency, Maharishi held, these qualities would be enlivened in collective consciousness and this would have positive, measurable effects on a wide social scale.

Over the years, social scientists developed formulas for predicting the size of the group necessary to create a “phase transition” in society to a measurably higher quality of life. These formulas, calculated on the basis of analogous phase transitions, from disorder to orderliness, studied in physics, came out to be approximately one percent of a population practicing the Transcendental Meditation program, and a much smaller percentage, on the order of the square root of one percent, practicing the TM-Sidhi program.

Since 1978, many experimental studies have been performed to measure the effect of large groups practicing the TM-Sidhi program. Experimental confirmation of the principle has been the consistent result. The Maharishi Effect is now as well documented as any principle of modern social science. In creating this technology, Maharishi has provided an effective method of social change that operates from the silent, harmonizing level of the unified field to produce a transformation in the quality of collective consciousness, thereby effortlessly creating coherence on a global scale. Maharishi (1986) describes how this effect is produced:

The transcendental level of nature’s functioning is the level of infinite correlation. When the group awareness is brought in attunement with that level, then a very intensified influence of coherence radiates, and a great richness is created. Infinite correlation is a quality of the transcendental level of nature’s functioning from where orderliness governs the universe. (p. 75)

D. W. Orme-Johnson and M. C. Dillbeck (1987) have summarized the empirical research on the Maharishi Effect. They surveyed
experimental studies documenting the sociological improvements resulting from the group practice of the TM-Sidhi program. Based on these results Maharishi asserts that the collective practice of the TM-Sidhi program in groups of 8000 (the square root of one percent of the world’s population) would produce coherence in the collective consciousness of the entire world. Statistically significant reductions in crime, accidents, fatalities, and disease, and other positive benefits on a global scale observed during experimental periods have established this as an effective means of changing collective consciousness and thereby changing the quality of life in the world—simply by enlivening the source of order and coherence at the basis of nature, from the level of the unified field.

**Maharishi’s Program to Create World Peace**

The most dramatic application of the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying, is Maharishi’s program to create world peace through the creation of a permanent group of 8000 collectively practicing Maharishi’s technologies of consciousness. These technologies are a basis for eliminating negativity and destructive tendencies throughout the world. Large groups of experts in the TM-Sidhi program, creating coherence, during experimental periods, have provided ample opportunity for scientific research. During these experimental periods, conflict and violence have been reduced in war-torn areas and negative trends have been reversed. Over thirty studies have established the efficacy of this technology to eliminate conflict and promote life-supporting, positive trends throughout the world.

Maharishi clearly lays out the basis of his program to create world peace. Stress, he holds, is the basic cause of all negativity, violence, terrorism, and national and international conflicts. Stress generated by the violation of natural law causes strained trends and tendencies in the environment. Through the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying, human intelligence can be identified with the unified field, and violations of natural law will cease. “Reinforcement of evolutionary power in world consciousness is the only effective way,” Maharishi holds, “to neutralize all kinds of negative
trends in the world and maintain world consciousness on a high level of purity” (Maharishi’s *Program to Create World Peace*, 1986, p. 7).

The global applications of this new science and technology are almost beyond present levels of imagination. Yet scientific research has found measurable reductions in levels of violence, crime, and other indications of negativity during the practice of the TM-Sidhi program in sufficiently large groups during experimental trial periods. Here for the first time in history is a scientific basis for creating world peace, ending terrorism, and reducing the negative trends of society.

On the basis of these studies, Maharishi holds that world peace can be guaranteed now, within a few years, through the establishment of groups of 8000; he holds that perfect health and unlimited longevity can be achieved for individual life, and that balance, coherence and health in society can be established in our generation. War, crime, poverty, and all problems that bring unhappiness to the family of man can be entirely eliminated. Life, he holds, can be lived in absolute abundance and fulfillment. Maharishi has called upon every significant individual in the world to act now to adopt this program for world peace by creating groups of 8000 collectively practicing the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying, to establish world peace and guarantee its perpetuation.

The practical benefits that Maharishi foresees through these new technologies are far greater than those achieved by the technology based on present science. As science has investigated deeper levels of nature, from microbes to molecules to atoms, new technologies have emerged which apply the knowledge in areas such as medicine and nuclear power. In drawing upon the deepest and most powerful level of natural law, the level of the unified field, Maharishi Vedic Science lays the basis for much more powerful technologies still. Where modern medicine has been able to eliminate some diseases by drawing upon microscopic levels, Maharishi Vedic Science lays the basis for the elimination of all disease, and more importantly, for the creation of perfect health and reversal of aging. While modern science has produced nuclear technology but no technology for peaceful resolution of conflict, Maharishi Vedic Science draws upon the infinite organizing power of the unified field at the basis of nature to create social harmony.
and world peace while preserving cultural integrity and stimulating prosperity and progress.

Maharishi’s Technologies of Consciousness as a New Method of Gaining Knowledge

The bold assertions about what is practically possible through the application of Maharishi’s technologies of consciousness must be understood in the context of the new method of gaining knowledge that Maharishi has founded. The history of science testifies that as new methods of gaining knowledge of deeper and more unified levels of natural law become available, more powerful and useful technologies become available. Maharishi’s technologies of consciousness are based on the deepest and most unified level of knowledge of nature. It should not be surprising, therefore, that this technology provides a radically new source of organizing power to fulfill the highest goals of mankind.

These technologies of consciousness offer a fundamentally new approach to knowledge that has not been available before. In asserting that it is possible for one individual to know all the laws of nature and the entirety of the universe within his or her own consciousness, Maharishi is well aware that he is introducing an account of human potential that goes well beyond the concept of the limits of knowledge that has dominated in the scientific era. This new paradigm of knowledge must be examined in a new light.

It is a widespread belief in the modern age that the only valid method of gaining knowledge is by moving outward through the senses, that is, through the methods of the empirical sciences. It is, however, only the historical failure of subjective approaches that has led to this belief. It cannot be thought that the senses are the only way of gaining knowledge, and those who cling to the belief that it is, only allow old habits to stand in the way of exploring new possible sources of knowledge.

Subjective approaches to knowledge in the past failed to bear fruit because they failed to provide an effective and reliable method of access to an invariant and universal domain of direct experience. They thus failed to establish independent standards of knowledge, they failed to produce methods of distinguishing truth from error, they failed to produce consensus even among those practicing the same method, and
they failed to produce practical technological benefits through the practice of the method.

Maharishi’s technologies of consciousness are different from subjective approaches in the past, and must therefore be considered on separate grounds. They provide an effective, reliable method of opening the mind to an invariant and universal level of nature which is everywhere, and yet not ordinarily open to experience because the mind usually functions on more active levels. By providing a technology to make this non-active level of nature available as a direct experience, Maharishi has made this domain available to all as a new field of inquiry; and, where there is a new source of experience of something universal, unchanging, and objectively verifiable, a new source of knowledge is available.

The Science of Creative Intelligence gives a new account of how complete knowledge is possible. When the mind becomes completely settled and still, according to this account, it gains the ability to perceive on the most refined levels of nature’s functioning—the all-pervading unified field where all laws reside in a collective totality. It not only experiences this unified field, it becomes identified with it; it is the unified field and thus knows the unified field as its own universal Self. On this level of knowledge, there is no separation of knower from the known. Nothing lies outside the range of the knower. All laws of nature and everything in the universe can be known as intimately as one’s own Self. Mind and body cease to be seen as separate realities. Maharishi (1986) says:

In reality our self-referral state of consciousness is the unified field—not an object of knowledge as a rose is when we say, “I see that rose.” The unified field is not an object in this way; it is the subject itself. The unified field is a self-referral state of awareness that knows itself, and in knowing itself is the knower and the known, both together. (p. 96)

On this account, there is no distinction between the knower and the reality that it knows. Since it is the Self that knows itself, there is nothing ultimately outside the consciousness of the knower, and there are therefore no limits on what can be known. [This unbounded value of the Self is written with an uppercase “S” to distinguish it from the ordinary, localized self we typically experience.] If true, this account of knowledge provides a fundamentally new source of discovery of the
laws of nature, like the empirical sciences, in that it relies on experience as a source of knowledge, but distinct from these sciences in that it draws upon a wider range of experience. As a new source of discovery, it extends the power of scientific investigation; yet it remains within the scope of empirical science by being subject to procedures of objective verification.

**Maharishi University of Management**

Maharishi University of Management, formerly Maharishi International University, was founded by Maharishi in 1971, based on the principles of the Science of Creative Intelligence. One of the major functions of this University is to show how each discipline and each level of natural law arises from the unified field of pure consciousness. The specialty of Maharishi University of Management is the knowledge of the unified field of pure consciousness from the standpoint of each academic discipline. At Maharishi University of Management, each modern discipline traces the diversity of laws back to a unified source in the unified field of pure consciousness and shows how the diversity of laws emerge from this unified field through the self-interacting dynamics of consciousness. Just as physics and mathematics have discovered increasingly unified levels of natural law at the basis of their discipline, thus tracing the diversity of its laws to their source in the unified field, so every academic discipline can ultimately show how its laws derive sequentially from the unified field. This project of unification of knowledge, a long sought goal throughout Western intellectual history, is now being systematically pursued and completed at Maharishi University of Management.

This enterprise includes developing charts to show how each modern discipline arises from the unified field of pure consciousness. For each discipline, a Unified Field Chart has been constructed to show how the discipline sequentially emerges from the unified field through the self-interacting dynamics of knower, known, and process of knowing. These Unified Field Charts constitute a major unification of knowledge, showing at a glance how all the diversity of knowledge emerges from a unified source.

Since the unified field is understood as a field of consciousness, and consciousness is the most fundamental level of each student’s own Self,
the study of the unified field at Maharishi University of Management constitutes a method of systematically relating all knowledge to the student’s Self. The success of Maharishi University of Management’s Consciousness-Based education is due in part to this program of relating all knowledge to the unified field and the unified field to the Self. Because all students and faculty at Maharishi University of Management collectively practice the Transcendental Meditation technique, regularly gaining the direct experience of the unified field of pure consciousness, this unified field increasingly becomes a living reality. This unified field ceases to be an abstract concept and becomes as intimate as the Self. The experience of faculty and students has been that learning and inquiry is joyful and most fulfilling in this environment of Consciousness-Based education.

[The reader is referred to other issues of the journal *Modern Science and Vedic Science* as well as to other volumes in this book series *Consciousness-Based Education: A Foundation for Teaching and Learning in the Academic Disciplines* for articles illustrating how Maharishi Vedic Science is transforming our understanding of modern academic disciplines. —Eds.]

**Maharishi’s Work in Historical Perspective: An Appreciation**

Maharishi has created a major watershed in world intellectual history. He has laid the foundation for a fundamental change both in intellectual history and in the history of technology and civilization itself. His work has created a new paradigm of the unity of human knowledge, and, we may expect, will unify the sciences and humanities in a more integrated way than ever before. He has, moreover, brought to an end the old notion that man is born to suffer and that life is a struggle. The practical programs he has founded provide a scientifically validated basis for reducing and even eliminating crime, war, terrorism, poverty, and other problems that beset mankind; more importantly, his discoveries make it possible to live life in the fulfillment of pure knowledge and permanent bliss consciousness and to achieve the highest goals of human endeavor. He has laid the basis for a new civilization, founded on new principles of complete, reliable, useful, fulfilling knowledge—
the knowledge of the unified field of pure consciousness as the perfectly orderly, unified source of nature.

Maharishi is unique in the world today. He has not offered conjectures and hypotheses about reality and human potential, nor does he set himself up as a final authority on matters of knowledge when he speaks rather of experience as the ultimate basis of knowledge. The experience of which he has spoken is derived from a new source, from the level of fully developed human life gained when one's awareness is open to the unified field of pure consciousness. Maharishi's life is an example of that which he taught. Unlike those whose teaching is based solely on the personal authority of the individual, Maharishi has founded universities, sciences, technologies, and other institutions based on universal principles through which any individual can gain the direct experience of the fully unfolded nature of life and validate the truth of what is described in the science. Because of this, Maharishi is held in highest esteem by millions of people around the world.

Maharishi has provided the means of unfolding the dormant creative genius within everyone, and he has established institutions through which the knowledge of how to unfold this potential will be perpetuated generation after generation. He has, moreover, used this knowledge to found programs to create perfect health, progress, prosperity, and permanent peace for the world—programs to end suffering and allow life to be lived in spontaneous accord with natural law. These institutions are not just ideals, but functioning institutions whose practical achievements are now well documented and available for all to examine.

Everyone now has the ability, with the availability of the Transcendental Meditation and TM-Sidhi programs, including Yogic Flying, to engage in this great experiment of identifying one's awareness with the total potential of natural law and to spontaneously live in accord with all the laws of nature while established in the awareness of the unified field of pure consciousness. The experience of approximately three million people who have learned the Transcendental Meditation technique testifies to its practicality and its effortlessness and ease of practice. Experimental studies have shown that its benefits are real and concrete. On this basis, Maharishi has foreseen the creation of a new era of civilization—Heaven on Earth—in which life will be lived
in fullness and abundance without suffering. Maharishi’s work eliminates the very basis of stress and suffering and lays the ground for a new civilization, a unified field-based, ideal civilization that draws on the infinite organizing power of the unified field of pure consciousness to bring human life to fulfillment.

References
Wallace, R. K., Orme-Johnson, D. W., & Dillbeck, M. C. (Eds.).


Kenneth Chandler’s “Modern Science Vedic Science: An Introduction,” here revised/updated, was originally published in Modern Science and Vedic Science, 1(2), pp. v-xxvi. It is reprinted with permission of the publisher.
Electronic Resources and Publications

LINKS

Education

Maharishi University of Management: www.mum.edu
Maharishi School of the Age of Enlightenment:
   www.maharishischooliowa.org
Maharishi’s Consciousness-Based Education: www.CBEprograms.org
International Foundation of Consciousness-Based Education: CBEfoundation@ifcbe.org
David Lynch Foundation for Consciousness-Based Education and
World Peace: www.davidlynchfoundation.org

Transcendental Meditation Program

Maharishi’s Technologies of Consciousness: www.tm.org
Maharishi Channel: www.maharishichannel.org
Maharishi Channel Archives:
   www.maharishichannel.in/archives/gfc-archive.html
Maharishi Lectures and Interviews (film clips): www.tm.org/maharishi
Invincible America Assembly: www.invincibleamerica.org
Global Country of World Peace: www.globalcountry.org
Global Good News Site: www.globalgoodnews.com
Fortune Creating Homes: www.FortuneCreatingHomes.com
Sthapathy Veda: www.sthapathyaveda.com

Research

Center for Brain, Consciousness, and Cognition: www.drfredtravis.com
Truth about TM: http://www.truthabouttm.org/truth/TMResearch/
   TMResearchPublications/PublishedResearch/index.cfm
PHONES NUMBERS

1-888-LEARN TM (1-888-532-7686)
Maharishi University of Management (1-641-472-7000)

PUBLICATIONS

These publications are available from Maharishi University of Management Press: http://mumpress.com and at the MUM Bookstore.

Books by Maharishi Mahesh Yogi

Science of Being and Art of Living
Bhagavad-Gita: A New Translation and Commentary, Chapters 1–6
Celebrating Perfection of Education
Celebrating Perfection in Administration
Vedic Knowledge for Everyone
Inaugurating Maharishi Vedic University

Consciousness-Based Books Imprint from MUM Press

The series Consciousness-Based Education: A Foundation for Teaching and Learning in the Academic Disciplines contains 12 volumes, available in 2011.

Maharishi Vedic Science Education
Physiology and Health Physics
Mathematics Literature
Art Management
Government Computer Science
Sustainable Living World Peace

Each volume includes a paper introducing the Consciousness-Based understanding of the discipline and a Unified Field Chart that conceptually maps all branches of the discipline, illustrating how the discipline emerges from the field of pure consciousness, the Self of every individual. These charts connect the “parts” of knowledge to the “wholeness” of knowledge and the wholeness of knowledge to the Self of the student.
Subsequent papers show how a Consciousness-Based approach may be applied in various branches of the discipline; these papers include occasional examples of student work. Each volume ends with an appendix describing Maharishi Vedic Science and Technologies of Consciousness in detail.